Holonomic gates in pseudo-Hermitian quantum systems

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The time-dependent pseudo-Hermitian formulation of quantum mechanics allows one to study open system dynamics in analogy to Hermitian quantum systems. In this setting, we show that the notion of holonomic quantum computation can equally be formulated for pseudo-Hermitian systems. Starting from a degenerate pseudo-Hermitian Hamiltonian we show that, in the adiabatic limit, a non-Abelian geometric phase emerges which realizes a pseudounitary quantum gate. We illustrate our findings by studying a pseudo-Hermitian gain-loss system which can be written in the form of a tripod Hamiltonian by using the biorthogonal representation. It is shown that this system allows for arbitrary pseudo-$U(2)$ transformations acting on the dark subspace of the system.

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I. INTRODUCTION

In the standard formulation of quantum mechanics (QM), observables are associated with Hermitian operators. This Hermiticity condition ensures that the spectrum of the observable is real valued, thus making a physical interpretation possible. It was first shown by Bender and Boettcher [1] that also non-Hermitian systems, obeying $\mathcal{PT}$ symmetry (parity-time-reversal symmetry), can show real spectra. This observation revived serious investigations into unconventional quantum mechanics. In particular, pseudo-Hermitian QM [2] (and the related biorthogonal QM [3]) have received special attention. This theory investigates pseudo-Hermitian systems, in which the Hamiltonian of the quantum system is non-Hermitian but can still be associated with a Hermitian counterpart. Such peculiar behavior leads to a whole class of new Hamiltonians that could reveal interesting new physics.

In this paper, we are particularly interested in the paradigm of holonomic quantum computation (HQC) [4,5], which is based on the emergence of a (non-Abelian) geometric phase (holonomy) during a cyclic time evolution of a quantum system [6,7]. Corresponding to a holonomy there is a non-Abelian gauge field mediating the computation in the form of a parallel transport. These types of gauge fields are realized in systems where the demand for degeneracy can be satisfied. Examples of such systems are cold atomic samples [8] and artificial atoms in superconducting circuits [9]. Recently, the implementation of such gauge fields was realized in systems of coupled waveguides [10]. Another successful scheme utilized the spin-orbit coupling of polarized light in asymmetric microcavities [11].

Holonomic quantum computing is a purely geometric approach to quantum computational problems. Unitary gates are implemented by generating a suitable holonomy from a Hamiltonian system. The transformation that a quantum state undergoes is the shadow (horizontal lift) of a loop in a parameter space (manifold) $\mathscr{M}$. In this context, the question of computational universality can be understood as the capability of generating a set of closed paths such that the holonomy spans up the entire unitary group [4]. Universality is typically reached only in a subspace of the whole Hilbert space $\mathscr{H}$, the so-called quantum code $\mathscr{C}$. The most common choice is to take $\mathscr{C}$ as the ground state of the Hamiltonian. This results in a type of ground-state computation in the lowest-energy eigenvalue manifold [5]. The elements of the (quantum) code $\mathscr{C}$ are called the (quantum) code words, as the gates act on them and, in that way, perform the computation.

Holonomy groups often appear in the context of gauge theories. This stems from their intrinsic connection to gauge fields, which can be elucidated by studying the theory of fiber and vector bundles [12,13]. Physical implementations of holonomic gates were considered in non-linear Kerr media [14], superconducting quantum dots [15], and quantum electrodynamical circuits [16], but to the best of our knowledge only for Hermitian systems. This broad range of possible implementations, together with the fault tolerance of HQC [17], make it desirable to generalize the concept of holonomic gates beyond Hermitian QM.

It has been pointed out that, in order to generate a non-Abelian geometric phase (holonomy), the Hermiticity of the Hamiltonian is not a necessary condition [18]. Indeed, an explicit calculation of an Abelian geometric phase for a $\mathcal{PT}$-symmetric system has been provided in Ref. [19]. However, because degeneracy plays such a crucial role in the theory of HQC, we will extend the theory from Refs. [19,20] to the non-Abelian case. With this, one is in principle able to implement quantum computational gates by means of pseudo-Hermitian systems. The conservation of the norm of quantum states is of utmost importance and will be discussed in this paper, referring to time-dependent models for pseudo-Hermitian QM. The occurrence of new physical effects from these types of holonomic gates is deeply connected to the question of measurable consequences of the underlying Hilbert space metric [21]. The idea of a pseudo-Hermitian representation of geometric phases could also be of interest in the theory of

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open quantum systems. The latter subject showed, by studying lossy systems, deep relations to pseudo-Hermitian and $\mathcal{PT}$-symmetric QM.

This paper is organized as follows. In Sec. II we briefly review the dynamics of pseudo-Hermitian systems [13,22], and emphasize the change of the Hilbert space metric associated with a pseudo-Hermitian quantum system [23]. In this framework, we will show in Sec. III that it is possible to derive a non-Abelian gauge field arising from an adiabatic mapping onto a degenerate subspace of the system, by extending the ideas of Ref. [20] to the degenerate case. Section IV contains additional remarks and theoretical considerations on the construction of pseudo-Hermitian Hamiltonians from a gain-loss system or a biorthogonal basis. Following that, we will discuss the example of a degenerate interaction Hamiltonian, the Hermitian analog of which can be found in the area of light-matter coupling. The gauge field is explicitly calculated and properties of the system are discussed in detail in Sec. V. Finally, we summarize our results with some concluding remarks in Sec. VI. In Appendix A, we derive the transformation law for the gauge field. A more sophisticated treatment of the geometry of pseudo-Hermitian quantum systems involves Grassmann and Stiefel manifolds, which can be found in Appendix B.

II. DYNAMICS OF PSEUDO-HERMITIAN SYSTEMS

We begin by briefly recalling the time-dependent dynamics of pseudo-Hermitian quantum systems, following mainly Refs. [13,20,22]. We consider a time-dependent N-dimensional ($N < \infty$) pseudo-Hermitian Hamiltonian $H(t) \neq H^\dagger(t)$, that is, $\mathscr{H} \cong \mathbb{C}^N$. The generalization to infinite-dimensional systems might be well possible, but is of marginal interest for HQC. Such a pseudo-Hermitian system can be viewed as being Hermitian with respect to a similarity transformation

$$H^\dagger(t) = \eta(t)H(t)\eta^{-1}(t), \tag{1}$$

where $\eta(t)$ (we sometimes suppress the time argument for brevity) is a Hermitian and positive definite operator, often referred to as the Hilbert-space metric [23,24]. The latter induces a new inner product

$$\langle\varphi, \psi\rangle_\eta = \langle\varphi|\,\eta\,|\psi\rangle, \tag{2}$$

for all vectors φ and ψ in the new Hilbert space $\mathscr{H}_{\eta(t)}$. Note that Hermitian operators in $\mathscr{H}$ do not have to be Hermitian in $\mathscr{H}_{\eta(t)}$.

A different point of view can be taken by investigating the eigenvalue problem of H [3]. For a Hermitian operator over $\mathscr{H}_{\eta(t)}$, all its eigenvalues are real and its instantaneous eigenstates

$$\begin{aligned} H(t)\,|\phi_n(t)\rangle &= E_n\,|\phi_n(t)\rangle, \\ H^\dagger(t)\,|\tilde{\phi}_n(t)\rangle &= E_n\,|\tilde{\phi}_n(t)\rangle \end{aligned} \tag{3}$$

form a biorthogonal basis $\{|\phi_n\rangle, |\tilde{\phi}_n\rangle\}$ with $\langle\tilde{\phi}_n|\phi_m\rangle = \delta_{nm}$ [13]. Combining Eqs. (1)–(3), we find that $|\tilde{\phi}_n\rangle = \eta\,|\phi_n\rangle$.

The time evolution $U : \mathscr{H} \to \mathscr{H}$ of a quantum system differs from conventional QM in that U is no longer unitary, $U^\dagger U \neq \mathbb{1}$. However, as it was shown in Ref. [22], a generalized unitarity condition can be established. For any two physical states $|\Phi(t)\rangle = U(t, t_0)\,|\Phi(t_0)\rangle$ and $|\Psi(t)\rangle = U(t, t_0)\,|\Psi(t_0)\rangle$ in $\mathscr{H}$ one demands that

$$\frac{d}{dt}\langle\tilde{\Phi}|\Psi\rangle = \frac{d}{dt}\langle\Phi|\,\eta\,|\Psi\rangle = 0. \tag{4}$$

Equation (4), together with Eq. (1), implies a generalized time-dependent Schrödinger-like equation [13,22]

$$i\frac{d}{dt}\,|\Psi(t)\rangle = \Lambda(t)\,|\Psi(t)\rangle, \tag{5}$$

where $\Lambda(t)$ is the generator of time displacement given by

$$\Lambda(t) = H(t) + iK(t),$$

with $K(t) = -\eta^{-1}(t)\dot{\eta}(t)/2$. Replacing the state vectors in Eq. (4) by their time evolution $U(t, t_0) = \hat{\mathbf{T}}\exp[-i\int_{t_0}^{t}\Lambda(\tau)d\tau]$ ($\hat{\mathbf{T}}$ denotes time ordering) and using Eq. (5) one obtains

$$i\dot{\eta} = \Lambda^\dagger\eta - \eta\Lambda, \tag{6}$$

where the dot denotes the time derivative. Equation (5) can be rewritten conveniently by introducing a covariant derivative $D_t = d/dt - K(t)$. We thus find

$$iD_t\,|\Psi(t)\rangle = H(t)\,|\Psi(t)\rangle. \tag{7}$$

We conclude this section by highlighting the physical consequences of the dynamical model presented here. Note that we imposed the Hermiticity condition (under the metric η) for all times t [see Eq. (1)]. For this to be true, the Schrödinger equation of conventional QM has to be replaced by the Schrödinger-like equation (5) to satisfy the unitarity condition, Eq. (4) [22]. If one wants to retain the original Schrödinger equation $i\,|\dot{\Psi}\rangle = H\,|\Psi\rangle$, then Eq. (1) is violated whenever the metric becomes time dependent. This can be seen by replacing Λ by H in Eq. (6). In this case, H would no longer be an observable for times $t > t_0$ [13]. Up until now, this seems to be not fully understood, and a number of different approaches to handle this problem were proposed. However, the model presented here does not produce any contradiction with conventional quantum mechanics and, as it was shown in Ref. [22], a proper mapping to conventional QM is possible. We expect that, whatever the final formulation of pseudo-Hermitian QM might look like, it will embody the physical demands made in this model up to a matter of notation.

III. DERIVATION OF THE HOLONOMY

The occurrence of non-Abelian geometric phases (holonomies) is, in terms of differential geometry, associated with a connection, i.e., a unique (adiabatic) separation of the (tangent) Hilbert space $\mathscr{H} = \mathscr{H}_{\text{exc}} \oplus \mathscr{H}_0$ into an n_0-fold degenerate ground-state subspace $\mathscr{H}_0$ and the space $\mathscr{H}_{\text{exc}}$ containing all excited states. Such a separation can be technically realized by a gauge field (a local connection one-form). Because a dynamical system leads in general to a time-dependent Hilbert space, we demand that this separation holds while the quantum states undergo a time evolution during the period T. Thus, any initial preparation $|\Psi(0)\rangle \in \mathscr{H}_0$ is mapped onto a final state $|\Psi(T)\rangle = U(T)\,|\Psi(0)\rangle$ lying also in $\mathscr{H}_0$. Such an isodegenerate mapping is nothing but the adiabatic condition [7]. Under which circumstances such a separation is valid needs to be checked for each physical realization individually. Notwithstanding, it should be noted that a generalization of the adiabatic limit to nonunitary evolutions, in the context of an open system approach, can be found in Ref. [25].

Returning to the question of time evolution, we now seek an explicit representation of the final state $|\Psi(T)\rangle$. By applying the adiabatic condition, Eq. (7) takes the form

$$iD_t\,|\Psi(t)\rangle = E_0(t)\Pi_0(t)\,|\Psi(t)\rangle, \tag{8}$$

where $\Pi_0(t)=\sum_{a=1}^{n_0}|\phi_0^a(t)\rangle\,\langle\tilde{\phi}_0^a(t)|$ for times $t\in[0,T]$ is the (pseudo-Hermitian) ground-state projector and E_0 denotes the lowest eigenvalue of H. As the state is initially prepared in $\mathscr{H}_0$ and will stay there while the evolution takes place, we can expand it in terms of the basis $\{|\phi_0^a(t)\rangle\}_{a=1}^{n_0}$, i.e.,

$$|\Psi(t)\rangle = \sum_{a=1}^{n_0} c_a(t)\,\big|\phi_0^a(t)\big\rangle, \tag{9}$$

with complex expansion coefficients $c_a(t)$. Inserting the expansion (9) into Eq. (8) it is readily shown that

$$i\sum_{a=1}^{n_0}\big(\dot{c}_a\,|\phi_0^a\rangle + c_a\,|\dot{\phi}_0^a\rangle\big) = \sum_{a=1}^{n_0} c_a\big(E_0\,|\phi_0^a\rangle + iK\,|\phi_0^a\rangle\big), \tag{10}$$

where we used the definition of the covariant derivative D_t.

Contracting both sides of Eq. (10) with $\langle\tilde{\phi}_0^b|$ and noting that $\langle\tilde{\phi}_0^b|\phi_0^a\rangle=\delta_{ba}$, one obtains

$$i\dot{c}_b + i\sum_{a=1}^{n_0} c_a\,\langle\tilde{\phi}_0^b|\dot{\phi}_0^a\rangle = E_0 c_b + i\sum_{a=1}^{n_0} c_a\,\langle\tilde{\phi}_0^b|\,K\,|\phi_0^a\rangle,$$

which can be rearranged as

$$\dot{c}_b + iE_0c_b + \sum_{a=1}^{n_0} c_a\,\langle\tilde{\phi}_0^b|\,D_t\,|\phi_0^a\rangle = 0. \tag{11}$$

A formal solution to Eq. (11) can be given in terms of a time-ordered integral [12]. By introducing $(A_t)^{ba}=\langle\tilde{\phi}_0^b|\,iD_t\,|\phi_0^a\rangle$, a solution to Eq. (11) is

$$c_b(T) = \sum_{a=1}^{n_0}\left[\hat{\mathbf{T}}\exp\int_0^T[-iE_0(t)\mathbb{1}+iA_t(t)]dt\right]^{ba} c_a(0). \tag{12}$$

An evolution in time is associated with a path $\gamma:[0,T]\to\mathscr{M}$ in a control manifold of the underlying quantum system. The d-dimensional manifold $\mathscr{M}$ is (locally) parametrized by a set of coordinates $\boldsymbol{\lambda}=\{\lambda^\mu\}_{\mu=1}^d$. These are the so-called control fields which drive the evolution of the Hamiltonian, i.e., $H(\boldsymbol{\lambda})=H_{\gamma(t)}$. In this framework, the time ordering for the integral over A_t can be replaced by a path ordering $\hat{\mathbf{P}}$ with respect to the parametrization by the coordinate chart $\{\lambda^\mu\}_{\mu=1}^d$.

Inserting the solution for the coefficients (12) into the expansion (9), we find an explicit form for the quantum state after its evolution:

$$|\Psi(T)\rangle = \sum_{a,b=1}^{n_0} c_a(0)\exp\left[-i\int_0^T E_0(t)dt\right] \times\left[\hat{\mathbf{P}}\exp\left(i\int_{\boldsymbol{\lambda}(0)}^{\boldsymbol{\lambda}(T)}A\right)\right]^{ba}\big|\phi_0^b(0)\big\rangle, \tag{13}$$

where we introduced the gauge field (local connection one-form) $A=\sum_{\mu=1}^d A_\mu d\lambda^\mu$. Its matrix-valued components A_μ are given by

$$(A_\mu)^{ba} = i\big\langle\tilde{\phi}_0^b(\boldsymbol{\lambda})\big|\,[\partial/\partial\lambda^\mu - K_\mu(\boldsymbol{\lambda})]\,\big|\phi_0^a(\boldsymbol{\lambda})\big\rangle, \tag{14}$$

with $K_\mu(\boldsymbol{\lambda})=-\eta^{-1}(\boldsymbol{\lambda})\partial_\mu\eta(\boldsymbol{\lambda})/2$ ($\partial_\mu=\partial/\partial\lambda^\mu$). Note that the components in Eq. (14) contain a part that can be found in conventional QM and a metric-dependent term K_μ. This has already been observed in Ref. [20] for the Abelian case. One recovers the Abelian result by setting $a=b$ and simplifying Eq. (14) using $\langle\tilde{\phi}_0^a|\,K_\mu\,|\phi_0^a\rangle=[\langle\tilde{\phi}_0^a|\,\partial_\mu\,|\phi_0^a\rangle+\partial_\mu(\langle\phi_0^a|)\,|\tilde{\phi}_0^a\rangle]/2$. In this notation $(A_\mu)^{aa}=-\Im\,\langle\tilde{\phi}_0^a|\,\partial_\mu\,|\phi_0^a\rangle$. Additionally, our gauge field (14) differs from the one derived in Ref. [13], not only by a Lie-algebra factor i but also by the term $\langle\tilde{\phi}_0^b|\,\partial_\mu\,|\phi_0^a\rangle$.

It can be straightforwardly shown that under a pseudounitary transformation $|\psi_0^a\rangle=\sum_{c=1}^{n_0}U_{ca}\,|\phi_0^c\rangle$ the components of A_μ transform like a proper gauge field (see Appendix A). Furthermore, the term iA_μ obeys a generalized anti-Hermiticity condition, that is, $[i(A_\mu)^{ba}]^*=-i(A_\mu)^{ab}_{\phi\leftrightarrow\tilde{\phi}}$, where $\phi\leftrightarrow\tilde{\phi}$ means an interchange of $|\phi_0^a\rangle$ and $|\phi_0^b\rangle$ by $|\tilde{\phi}_0^a\rangle$ and $|\tilde{\phi}_0^b\rangle$, respectively. The condition was derived by noting that $\langle\tilde{\phi}_0^a|\,\partial_\mu\,|\phi_0^b\rangle=-\partial_\mu(\langle\tilde{\phi}_0^a|)\,|\tilde{\phi}_0^b\rangle$.

The appearance of a gauge field in non-Hermitian QM was of course expected, as we started from an adiabatic separation (i.e., a connection) $\mathscr{H}=\mathscr{H}_{\mathrm{exc}}\oplus\mathscr{H}_0$. In non-Hermitian systems designed by an open system approach, the adiabatic theorem has to be suitably modified. In particular, due to the leakage to the environment, the adiabatic approximation might only be applicable during a limited time interval [25]. An extension to nonadiabatic holonomic gates as in conventional HQC [26,27] should be possible with a similarly strong analogy as for the adiabatic holonomy developed here.

Returning to Eq. (13) and assuming that the state $|\Psi\rangle$ returns after a full period into its initial state up to a pseudounitary rotation, $|\Psi(0)\rangle\to|\Psi(T)\rangle$, where the initial state is assumed to be one of the eigenstates $|\phi_0^k(0)\rangle$ rather than a superposition of them, we find $[c_a(0)=\delta_{ak}]$

$$|\Psi(T)\rangle = \exp\left[-i\int_0^T E_0(t)dt\right]\sum_{b=1}^{n_0}[\mathcal{U}_A(\gamma)]^{bk}\,|\phi_0^b(0)\rangle, \quad k\in\{1,\dots,n_0\} \tag{15}$$

where the cyclic time evolution corresponds to a loop $\gamma(0)=\gamma(T)$ in the parameter space $\mathscr{M}$. The mapping of the initial state $|\phi_0^k(0)\rangle$ described by Eq. (15) is nothing but a unitary transformation with respect to the modified inner product $\langle\cdot,\cdot\rangle_\eta$. The exponential factor in Eq. (15) is a dynamical phase factor, while the second term

$$\mathcal{U}_A(\gamma) = \hat{\mathbf{P}}\exp\left(i\oint_\gamma A\right) \tag{16}$$

has purely geometric origin and is indeed a holonomy.

Because the transformation (16) is geometric in nature, $\mathcal{U}_A$ is robust against parametric noise and invariant under general reparametrization. Indeed, the path-ordered integral in Eq. (16) can be turned into a surface integral using the non-Abelian Stokes theorem [28]. Consequently, two loops in $\mathscr{M}$ generate the same pseudounitary gate, if they enclose the same surface in parameter space. This means that, analogously to conventional HQC, pseudounitary holonomic gates show a certain robustness against stochastic fluctuations in the control fields $\lambda^\mu(t)$. In addition, $\mathcal{U}_A(\gamma)$ does not depend on the way in which the path is traversed, i.e., $\mathcal{U}_A\{\gamma[f(t)]\} = \mathcal{U}_A[\gamma(t)]$, where f is any function (diffeomorphism) of t.

IV. CONSTRUCTION OF PSEUDO-HERMITIAN SYSTEMS

For the purpose of illustration we shall consider a benchmark Hamiltonian on which the previously developed theory can be studied. There are mainly two approaches to construct artificial pseudo-Hermitian systems. The first route is to implement pseudo-Hermiticity via a *top-down* approach in a gain-loss system. For that one usually starts with an effective non-Hermitian Hamiltonian H describing an open system phenomenologically. The eigenvectors of this non-Hermitian Hamiltonian result directly in a biorthogonal basis as used in the previous sections. This approach has the advantage that it is directly connected to a physical system. For example, typical experimental realizations exist in the realm of optics, where the similarity of the paraxial Helmholtz equation with the Schrödinger equation allows one to design non-Hermitian characteristics with lossy waveguide systems [29,30]. An approach using parity-time-symmetric lasing in an optical fiber network has been pursued in Ref. [31], and in parity-time synthetic photonic lattices in Ref. [32].

The second approach to non-Hermitian quantum theory is provided by biorthogonal quantum mechanics [3]. Given any biorthogonal basis, one can construct different pseudo-Hermitian systems from a *bottom up* approach [21]. Let us investigate the relation between these two approaches in more detail by considering a benchmark system. In the following, $H = L - i\Gamma$ is a complex Hamiltonian, with L and Γ being Hermitian operators given by

$$L = \frac{1}{2\Omega}\begin{pmatrix} 0 & (\Omega-\Delta)\kappa_0^* & (\Delta+\Omega)\kappa_- + (\Delta-\Omega)\kappa_-^* & (\Delta-\Omega)\kappa_+^* \\ (\Omega-\Delta)\kappa_0 & 0 & (\Delta+\Omega)\kappa_+ & 0 \\ (\Delta+\Omega)\kappa_-^* + (\Delta-\Omega)\kappa_- & (\Delta+\Omega)\kappa_+^* & 0 & \frac{\alpha^2\kappa_0^*}{\Delta-\Omega} \\ (\Delta-\Omega)\kappa_+ & 0 & \frac{\alpha^2\kappa_0}{\Delta-\Omega} & 0 \end{pmatrix},$$

$$\Gamma = \frac{\alpha}{2\Omega}\begin{pmatrix} |\kappa_-|^2 & \kappa_+^* & 0 & \kappa_0^* \\ \kappa_+ & 0 & \kappa_0 & 0 \\ 0 & \kappa_0^* & -|\kappa_-|^2 & -\kappa_+^* \\ \kappa_0 & 0 & -\kappa_+ & 0 \end{pmatrix},$$

where $\Omega(\alpha) = \sqrt{\Delta^2 - \alpha^2}$, with Δ being a real constant, and time-dependent parameters $\alpha(t) \in \mathbb{R}$, $\kappa_c(t) \in \mathbb{C}$. We assume that $0 < \alpha^2 < \Delta^2$ so that Ω stays real valued. We can decompose H as

$$H = \sum_{c=0,\pm} (\kappa_c\, |G^c(\alpha)\rangle \langle \tilde{E}(\alpha)| + \kappa_c^*\, |E(\alpha)\rangle \langle \tilde{G}^c(\alpha)|), \tag{17}$$

where

$$|E\rangle = \mathcal{N}_1 \begin{pmatrix} i(\Omega-\Delta) \\ 0 \\ \alpha \\ 0 \end{pmatrix}, \quad |G^0\rangle = \mathcal{N}_1 \begin{pmatrix} 0 \\ i(\Omega-\Delta) \\ 0 \\ \alpha \end{pmatrix},$$

and

$$|G^-\rangle = \mathcal{N}_2 \begin{pmatrix} -i(\Omega+\Delta) \\ 0 \\ \alpha \\ 0 \end{pmatrix}, \quad |G^+\rangle = \mathcal{N}_2 \begin{pmatrix} 0 \\ -i(\Omega+\Delta) \\ 0 \\ \alpha \end{pmatrix},$$

with normalization factors $\mathcal{N}_1 = 1/\sqrt{2\Omega(\Delta-\Omega)}$ and $\mathcal{N}_2 = i/\sqrt{2\Omega(\Delta+\Omega)}$. Together with the associated states $|\tilde{E}\rangle = |E\rangle^*$ and $|\tilde{G}^c\rangle = |G^c\rangle^*$ for $c = 0, \pm$, they form a biorthogonal basis. Note that, in our example, the relation between the states and their associated counterparts is merely complex conjugation, which is solely due to the chosen biorthogonal basis.

The Hamiltonian H in Eq. (17) possesses a twofold degenerate dark subspace (zero-eigenvalue eigenspace) and is therefore suitable for generating a pseudounitary holonomic gate. The Hamiltonian H is the pseudo-Hermitian analog of a typical light-matter coupling Hamiltonian that can be found in a variety of physical applications, for instance, in semiconductor quantum dots [15], trapped ions [33], and neutral atoms [34]. They all fall into the class of tripod systems. By considering a controlled driving of the coupling parameters $\kappa_c = \kappa_c(t)$ a generalized stimulated Raman adiabatic passage process is induced [35]. The system described by H can

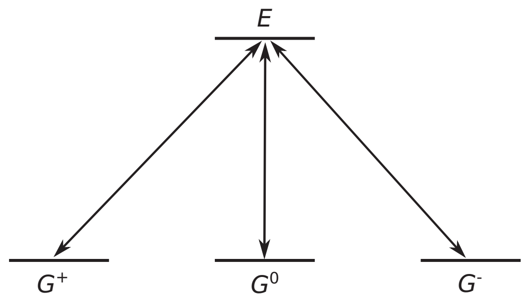

FIG. 1. Representation of the level scheme of the pseudo-Hermitian Hamiltonian from Eq. (17) in the time-varying Hilbert space $\mathscr{H}_{\eta(t)}$. In $\mathscr{H}_{\eta(t)}$ the Hamiltonian describes a tripod system.

therefore be seen as such a process taking place in a Hilbert space with a varying inner product structure $\langle\cdot,\cdot\rangle_{\eta(t)}$ (see Fig. 1).

In the following, we show that the pseudo-Hermitian system as defined by the Hamiltonian (17) can indeed be associated with physically equivalent Hermitian systems [2]. A first example is obtained by noting that in general a Hermitian Hamiltonian h can be expanded in an orthonormal basis, i.e., $\{|g^c\rangle, |e\rangle\}$ in the present case with $\langle g^c|g^d\rangle = \delta_{cd}$, $\langle g^c|e\rangle = 0$ and $\langle e|e\rangle = 1$. This basis is related to the nonorthogonal states $\{|E\rangle, |G^c\rangle\}$ by a generally nonunitary matrix u, i.e., $|g^c\rangle = u\,|G^c\rangle$ and $|e\rangle = u\,|E\rangle$. Similarly, for the associated states we have $|g^c\rangle = v\,|\tilde{G}^c\rangle$ and $|e\rangle = v\,|\tilde{E}\rangle$, where v is some nonsingular matrix. By construction, we have

$$\begin{aligned}\delta_{cd} &= \langle g^c|g^d\rangle = \langle\tilde{G}^c|\, v^\dagger u\,|G^d\rangle,\\ 0 &= \langle g^c|e\rangle = \langle\tilde{G}^c|\, v^\dagger u\,|E\rangle,\\ 1 &= \langle e|e\rangle = \langle\tilde{E}|\, v^\dagger u\,|E\rangle,\end{aligned}$$

only if $v^\dagger u = \mathbb{1}$.

We shall assume that $v \neq u$ to ensure that the problem is nontrivial. A relation of u and v to the metric operator η is readily obtained. For example, starting from the state $|E\rangle$ we find that

$$1 = \langle\tilde{E}|E\rangle = \langle E|\,\eta\,|E\rangle = \langle e|\,v\eta u^{-1}\,|e\rangle\,,$$

hence $\eta = v^{-1}u = u^\dagger u$. Finally, we observe that the Hermitian counterpart h to H is given by $h = uHv^\dagger$ [3]. In the particular case of the Hamiltonian (17) we find

$$h = \sum_{c=0,\pm}(\kappa_c\,|g^c\rangle\,\langle e| + \kappa_c^*\,|e\rangle\,\langle g^c|).$$

Another Hermitian counterpart of the pseudo-Hermitian Hamiltonian (17) can be defined by $\tilde{h} = \eta H$ as long as the spectrum of H is real valued, which can be seen from Eq. (1). However, $\tilde{h}$ is then represented in a nonorthogonal basis. A transformation to h, which is expanded in an orthonormal basis, is given by $\tilde{h} = (u^\dagger u)(v^\dagger h u) = u^\dagger h u$.

One could also utilize the positive definiteness of η to establish a transformation to yet another Hermitian system $\hat{h} = \rho H\rho^{-1}$, where $\rho = \sqrt{\eta}$ defines a unitary equivalence of the original inner product $\langle\cdot|\cdot\rangle$. Any of the associated Hermitian systems are different implementations of the same physical system. This means that, although they all share the same spectral properties, their actual experimental implementations may differ considerably.

V. EVOLUTION IN DARK SUBSPACES

We now turn to the Hamiltonian H from Eq. (17) to investigate its dynamics under an adiabatic evolution. At this point, one should recall that the metric operator of a pseudo-Hermitian system is in general not unique. It is well possible that a whole class of pseudo-Hermitian Hamiltonians is Hermitian under a certain metric operator. There might be even a time-independent metric under which H is Hermitian. In order to resolve this ambiguity, we demand that the metric under which the observable H is Hermitian is the *proper* metric η given by the dyadic products of the left-handed eigenstates of H [24].

We now investigate the dynamics induced by the Hamiltonian in Eq. (17) with the aim to compute a holonomy. To do so, we have to consider a cyclic time evolution or, equivalently, a closed loop γ in the parameter space $\mathscr{M}$. The evolution is assumed to be driven adiabatically by the time dependence of the parameters $\kappa_c = \kappa_c(t)$. The holonomy will be generated in the degenerate dark subspace $\mathscr{H}_{\mathcal{D}} = \mathrm{Span}(\{|\mathcal{D}^1\rangle, |\mathcal{D}^2\rangle\})$. This is suitable for our computational purposes, as it neglects the uncontrollable dynamical phase [$E_{\mathcal{D}}(t) = 0$ for all t]. Throughout the dynamical process the parameter α will be assumed to be constant. As we will see, this will substantially reduce the computational effort.

We seek a complete set of single-qubit gates, thus ensuring that any pseudounitary gate with respect to the metric η can be implemented over the dark subspace. For that, it is sufficient to design a pair of noncommuting single-qubit gates U_1 and U_2 [36]. For the gate U_1 we choose the parametrization $\kappa_- = 0$, $\kappa_+ = -\kappa\sin(\vartheta/2)e^{i\varphi}$, and $\kappa_0 = \kappa\cos(\vartheta/2)$. In this case, the dark states are

$$\begin{aligned}|\mathcal{D}^1\rangle &= |G^-\rangle,\\ |\mathcal{D}^2\rangle &= \cos(\vartheta/2)\,|G^+\rangle + \sin(\vartheta/2)e^{i\varphi}\,|G^0\rangle.\end{aligned}\tag{18}$$

The remaining bright states (with eigenvalues $\pm\kappa$) read

$$\begin{aligned}|\mathcal{B}^+\rangle &= \frac{1}{\sqrt{2}}[\sin(\vartheta/2)\,|G^+\rangle - e^{i\varphi}\cos(\vartheta/2)\,|G^0\rangle + e^{i\varphi}\,|E\rangle],\\ |\mathcal{B}^-\rangle &= \frac{1}{\sqrt{2}}[\sin(\vartheta/2)\,|G^+\rangle - e^{i\varphi}\cos(\vartheta/2)\,|G^0\rangle - e^{i\varphi}\,|E\rangle].\end{aligned}\tag{19}$$

Using the left-sided eigenstates associated with Eqs. (18) and (19), we compute the full metric operator

$$\begin{aligned}\eta &= \sum_{a=1,2}|\tilde{\mathcal{D}}^a\rangle\,\langle\tilde{\mathcal{D}}^a| + |\tilde{\mathcal{B}}^+\rangle\,\langle\tilde{\mathcal{B}}^+| + |\tilde{\mathcal{B}}^-\rangle\,\langle\tilde{\mathcal{B}}^-|\\ &= |\tilde{E}\rangle\,\langle\tilde{E}| + \sum_{c=0,\pm}|\tilde{G}^c\rangle\,\langle\tilde{G}^c|.\end{aligned}\tag{20}$$

We recognize that as long as α stays constant the metric operator η does not depend on the parametrization of $\mathscr{M}$. In terms of the geometry of the underlying Hilbert space, a change of the parameter α leads to a contribution of the connection K_μ. Hence, for $\alpha = \text{const}$, we have $K_\mu = 0$. Thus, the gauge field (14) reduces to

$$(A_\mu)^{ab} = i\,\langle\tilde{\mathcal{D}}^a|\,\partial_\mu\,|\mathcal{D}^b\rangle.$$

Evaluating the gauge field with respect to the coordinates $\lambda^\mu \in \{\vartheta, \varphi\}$ of $\mathscr{M}$, we get $(A_\varphi)^{22} = -\sin^2(\vartheta/2)$ as the only nonvanishing component of A. With this, we can compute the associated holonomy, and express the gate $U_1(\gamma)$ in terms of the Pauli matrices $\{\sigma^x, \sigma^y, \sigma^z\}$ with respect to the basis of dark states $\{|\mathcal{D}^1\rangle, |\mathcal{D}^2\rangle\}$, viz.,

$$U_1(\gamma) = e^{i\beta_1(\gamma)|1\rangle\langle\tilde{1}|}, \tag{21}$$

where $\beta_1(\gamma) = -\oint_\gamma \sin^2(\vartheta/2)d\vartheta\,\mathrm{d}\varphi$. Note that our computational basis is $|0\rangle = |\mathcal{D}^1(\mathbf{0})\rangle = |G^-\rangle$ and $|1\rangle = |\mathcal{D}^2(\mathbf{0})\rangle = |G^+\rangle$. In Eq. (21), path ordering can be neglected, as the chosen parametrization effectively generates an Abelian geometric phase, i.e., the matrix-valued components A_ϑ and A_φ commute.

For the second gate U_2, we choose the parametrization $\kappa_0 = \kappa\cos(\vartheta)$, $\kappa_- = \kappa\sin(\vartheta)\cos(\varphi)$, and $\kappa_+ = \kappa\sin(\vartheta)\sin(\varphi)$, and repeat the previous calculation in a similar fashion, starting with the new dark states

$$|\mathcal{D}^1\rangle = \cos(\vartheta)[\cos(\varphi)\,|G^-\rangle + \sin(\varphi)\,|G^+\rangle] - \sin(\vartheta)\,|G^0\rangle,$$
$$|\mathcal{D}^2\rangle = \cos(\varphi)\,|G^+\rangle - \sin(\varphi)\,|G^-\rangle.$$

Together with the associated bright states, we obtain in this case the same metric operator as in Eq. (20). Hence, $K_\mu = 0$ as long as $\alpha \neq \alpha(t)$. We find the components of the gauge field in $\mathscr{H}_\mathcal{D}$ to be

$$A_\vartheta = 0, \quad A_\varphi = \cos(\vartheta)\sigma^y, \tag{22}$$

so that path ordering can be neglected again.

The associated holonomy U_2 to A is thus given by inserting Eq. (22) into Eq. (16). Explicitly, we have

$$U_2(\gamma') = e^{i\beta_2(\gamma')\sigma^y},$$

where $\beta_2(\gamma') = \oint_{\gamma'} \cos(\vartheta)d\vartheta d\varphi$ for a path γ' in $\mathscr{M}$. From here, one is able to compute the commutator of U_1 and U_2, that is,

$$[U_1, U_2] = \sin(\beta_2)(1 - e^{i\beta_1})\sigma^x. \tag{23}$$

In general, Eq. (23) does not vanish for generic loops γ and γ'. Hence, by constructing the two noncommuting holonomies U_1 and U_2, we have found a universal set of pseudounitary single-qubit gates on which HQC could be based [36]. This is the key result of this paper. Arbitrary pseudo-$U(n_0)$ transformations for universal pseudounitary HQC can be implemented in a similar fashion by applying the developed theory to n_0-fold degenerate dark subspaces.

The presented procedure shows how a lossy system, which generates the Hamiltonian for a generic holonomic computation, can be described effectively in the pseudo-Hermitian picture. Given that the efficient implementation of an HQC protocol is rather demanding in terms of the accessible parameters [4,5], the benefit of our scheme is to add new control parameters such as gain and loss to the experimentalists' tool box, thereby providing a richer structure of the control space $\mathscr{M}$. The range of new applications that could stem from this extension of the theory needs further investigations and is beyond the scope of this paper.

VI. DISCUSSION AND CONCLUDING REMARKS

In this paper, we have shown how the holonomic approach to quantum computation can be extended to pseudo-Hermitian systems. We derived a non-Abelian geometric phase generating a pseudounitary holonomy over the degenerate eigenspace. The gauge field associated with the non-Abelian phase contains an additional term due to the modified inner product structure induced by a pseudo-Hermitian quantum system, which is absent in conventional quantum mechanics.

This general framework was applied to a benchmark Hamiltonian that can be implemented in terms of a gain-loss system. By choosing a suitable biorthogonal basis, the system has the form of a tripod Hamiltonian. An explicit calculation showed that the considered system allows for the implementation of arbitrary pseudounitary transformations over the two-dimensional dark subspace.

Furthermore, we investigated the underlying geometry of this Hamiltonian. In particular, we have shown that the inner product structure could be held constant throughout an adiabatic evolution. This can be done by choosing a suitable loop in the parameter space such that the additional term, appearing in the geometric phase, vanishes. Therefore, this loop only changes the coupling between certain tripod levels but does not involve the biorthogonal basis, i.e., the inner product structure in which the Hamiltonian is represented. Generalized to arbitrary pseudo-Hermitian systems, this enables clear analysis of pseudounitary holonomies and their dependence on the changing inner product structure.

Our paper paves the way to further investigate known concepts of conventional HQC in pseudo-Hermitian systems such as error-avoiding and error-correcting techniques, and whether or not these approaches are equally applicable to the pseudo-Hermitian case studied here.

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APPENDIX A: TRANSFORMATION LAW FOR THE GAUGE FIELD

Here we show that A indeed transforms like a proper gauge field [37] under a change of basis $|\psi^a\rangle = \sum_{i=1}^n U_{ia}\,|\phi^i\rangle$, where $U_{ia} \in \mathbb{C}$. The transformation is mediated by a pseudounitary matrix

$$\mathcal{U}(\boldsymbol{\lambda}) = \sum_{i,j=1}^{n} U_{ij}(\boldsymbol{\lambda})\,|\phi^i(\boldsymbol{\lambda})\rangle\,\langle\tilde{\phi}^j(\boldsymbol{\lambda})| \in U_\eta(n_0).$$

Here, $U_\eta(n)$ is the group of n-dimensional η-pseudo-unitary matrices [38]. We find the usual transformation law

$$(A'_\mu)^{ab} = i\,\langle\tilde{\psi}^a|\,(\partial_\mu - K_\mu)\,|\psi^b\rangle$$
$$= i\sum_{i,j=1}^{n}\langle\phi^i|\,U^*_{ia}\eta(\partial_\mu - K_\mu)U_{jb}\,|\phi^j\rangle$$

$$= i \sum_{i,j=1}^{n} [\langle\phi^i| U_{ia}^* \eta(\partial_\mu U_{bj}) |\phi^j\rangle + \langle\phi^i| U_{ia}^* \eta U_{bj} \partial_\mu |\phi^j\rangle - \langle\phi^i| U_{ia}^* \eta K_\mu U_{bj} |\phi^j\rangle]$$

$$= \sum_{i,j=1}^{n} U_{ia}^* i\partial_\mu U_{bj}\delta_{ij} + U_{ia}^* U_{bj}(A_\mu)^{ij}$$

$$= \sum_{i=1}^{n} U_{ia}^* i\partial_\mu U_{bi} + \sum_{i,j=1}^{n} U_{ia}^* U_{bj}(A_\mu)^{ij},$$

or, in matrix notation,

$$A_\mu \mapsto \mathcal{U}^{-1} A_\mu \mathcal{U} + \mathcal{U}^{-1} i\partial_\mu \mathcal{U}.$$

APPENDIX B: NATURAL GEOMETRIC PICTURE OF PSEUDO-HERMITIAN HAMILTONIANS

So far, our treatment of pseudo-Hermitian Hamiltonians did not involve the language of fiber bundles. In conventional QM it is well known that the projector formalism used in HQC involves more advanced concepts such as Grassmann and Stiefel manifolds [39]. To the best of our knowledge, these notions have not been established for pseudo-Hermitian systems yet.

Let us consider a pseudo-Hermitian Hamiltonian $H \in \mathrm{End}(\mathscr{H})$, with $R+1$ different eigenvalues, defined over the N-dimensional Hilbert space $\mathscr{H}$. Suppose H has a real spectrum so that its spectral decomposition reads

$$H = \sum_{l=0}^{R} E_l \Pi_l,$$

where $\{E_l\}_{l=0}^R$ are the eigenvalues corresponding to the pseudo-Hermitian projector $\Pi_l = \sum_{k=1}^{n_l} |\phi_l^k\rangle\langle\tilde{\phi}_l^k|$ with n_l being the degeneracy of the lth level. The states $\{|\phi_l^k\rangle\}_{k=1}^{n_l}$ of the lth eigenspace of H form a biorthogonal frame

$$V_l = \sum_{k=1}^{n_l} |\phi_l^k\rangle\langle\tilde{k}| \cong \big(|\phi_l^1\rangle, \dots, |\phi_l^{n_l}\rangle\big)_{|\tilde{k}\rangle}, \tag{B1}$$

where $\{|\tilde{k}\rangle\}_{k=1}^{n_l} \subset \mathbb{C}^{n_l}$ constitutes a complete, biorthogonal basis with $\{|k\rangle\}_{k=1}^{n_l}$, where $|\tilde{k}\rangle = \eta_{\mathrm{a}}|k\rangle$. Note that this basis is of no physical relevance and acts merely as a tool to represent the frame V_l. One can indeed choose $\eta_{\mathrm{a}} = \mathbb{1}_{n_l}$ so that $\{|k\rangle\}_{k=1}^{n_l}$ forms an orthonormal basis.

The notion of biorthogonal frames gives rise to a more subtle issue. Usually, in the study of pseudo-Hermitian and pseudounitary operators, one is confronted with square matrices. Because V_l is not an observable, we have to modify the pseudo-Hermiticity condition (1) for nonsquare matrices. In the case of a biorthogonal frame, we can define the pseudoadjoint matrix of V_l as

$$V_l^\ddagger = \eta_{\mathrm{a}}^{-1} V_l^\dagger \eta,$$

where $\eta \in \mathbb{C}^{N\times N}$ is the metric operator formed from the left-handed eigenstates $|\tilde{\phi}_l^k\rangle$, that is,

$$\eta = \sum_{l=0}^{R}\sum_{k=1}^{n_l} |\tilde{\phi}_l^k\rangle\langle\tilde{\phi}_l^k|.$$

Note that η serves as a metric for the projector Π_l, i.e.,

$$\Pi_l^\dagger \eta = \Pi_l^\dagger \eta_l = \eta_l \Pi = \eta\Pi,$$

where $\eta_l = \sum_{k=1}^{n_l} |\tilde{\phi}_l^k\rangle\langle\tilde{\phi}_l^k|$.

Representing the biorthogonal frame (B1) by a complex $(N\times n_l)$ matrix we find its pseudoadjoint to be

$$V_l^\ddagger = \sum_{k=1}^{n_l} |k\rangle\langle\tilde{\phi}_l^k| \cong \begin{pmatrix} \langle\tilde{\phi}_l^1| \\ \vdots \\ \langle\tilde{\phi}_l^{n_l}| \end{pmatrix} \in \mathbb{C}^{n_l\times N}.$$

By construction, we have $V_l^\ddagger V_l = \mathbb{1}_{n_l}$, which verifies that the set $\{|\phi_l^k\rangle\}_{k=1}^{n_l}$ constitutes a biorthogonal basis for the ground-state eigenspace. The set of all biorthogonal frames is called the Stiefel manifold defined by

$$S_{N,n_l,\eta} = \{V_l \in \mathbb{C}^{N\times n_l} \,|\, V_l^\ddagger V_l = \mathbb{1}_{n_l}\}.$$

It is noteworthy that the projector Π_l can be expressed in terms of a biorthogonal frame in S_{N,n_l} (we have dropped η for ease of notation), i.e., $\Pi_l = V_l V_l^\ddagger$. It is easily checked that the so-defined projector belongs to the Grassmann manifold

$$G_{N,n_l} = \{\Pi_l \in \mathbb{C}^{N\times N} \,|\, \Pi_l^2 = \Pi_l, \; \Pi_l^\ddagger = \Pi_l, \; \mathrm{tr}(\Pi_l) = n_l\}.$$

Because the projector is a square matrix, its pseudoadjoint is defined in the usual sense [2] as $\Pi_l^\ddagger = \eta^{-1}\Pi_l^\dagger\eta$.

We are now in a position to illuminate the gauge freedom within the projector Π_l. More precisely, we can define a projection π from the Stiefel manifold to the Grassmann manifold by $V_l \mapsto V_l V_l^\ddagger$. It is not hard to show that the image of this map stays invariant under a group action by a pseudounitary matrix $\mathcal{U} \in U_{\eta_{\mathrm{a}}}(n_l)$,

$$\pi(V_l\mathcal{U}) = (V_l\mathcal{U})(V_l\mathcal{U})^\ddagger = V_l\mathcal{U}\mathcal{U}^\ddagger V_l^\ddagger = V_l V_l^\ddagger,$$

where we applied the useful relation

$$(V_l\mathcal{U})^\ddagger = \eta_{\mathrm{a}}^{-1}(V_l\mathcal{U})^\dagger\eta = \eta_{\mathrm{a}}^{-1}\mathcal{U}^\dagger\big(\eta_{\mathrm{a}}\eta_{\mathrm{a}}^{-1}\big)V_l^\dagger\eta = \mathcal{U}^\ddagger V_l^\ddagger.$$

In conclusion, we have constructed a $U_{\eta_{\mathrm{a}}}(n_l)$-principal bundle, that is,

$$S_{N,n_l} \xrightarrow{\pi} G_{N,n_l}. \tag{B2}$$

The bundle structure, Eq. (B2), is a direct generalization of the one found in conventional QM (for a review, see, e.g., Ref. [39]). The standard theory is recovered for $\eta = \mathbb{1}_N$. It is therefore not surprising that the Stiefel manifold can be written, in analogy to their counterparts in conventional QM, as a coset space, i.e.,

$$G_{N,n_l} \cong S_{N,n_l}/U_{\eta_{\mathrm{a}}}(n_l).$$

Note how, for $n_l = 1$ (i.e., a nondegenerate situation), the Grassmann manifold reduces to the projective Hilbert space containing the pseudo-Hermitian density operators for a pure state, i.e., $|\phi_l\rangle\langle\tilde{\phi}_l| \in G_{N,1} \cong \mathbb{C}P^{N-1}$. The structure group of this principal bundle is $U_{\eta_{\mathrm{a}}}(1)$, which is identical to the conventional unitary group $U(1)$.

We conclude this section by recalling that it is rather demanding for a parameter space $\mathscr{M}$ to be mapped one to one (bijectively) onto G_{N,n_l}. In other words, a realistic quantum system, given by a family $\{H(\lambda)\}_{\lambda\in\mathscr{M}}$ of isospectral pseudo-Hermitian Hamiltonians, may have a smaller control manifold than the whole Grassmann manifold. Nevertheless, there is a map Φ from $\mathscr{M}$ onto G_{N,n_l} defined by $\Phi(\lambda) = \Pi_l$. A natural way to study the geometry of such systems is given in terms of the pullback bundle of the Stiefel manifold:

$$\Phi^* S_{N,n_l} = \{(\lambda, V_l) \in \mathscr{M} \times S_{N,n_l} \,\big|\, \pi(V_l) = \Phi(\lambda)\}. \quad \text{(B3)}$$

In order to construct the rest of the bundle structure of Eq. (B3), we can establish the fiber $\mathscr{F}_\lambda$ of $\Phi^* S_{N,n_l}$ over the point λ in $\mathscr{M}$. This fiber is just a copy of the fiber $\mathscr{F}_{\Phi(\lambda)}$ defined over the projector (point in G_{N,n_l}) Π_l. The latter is formally defined as the preimage of the projection $\pi(V_l) = \Pi_l$, that is, $\mathscr{F}_{\Phi(\lambda)} \cong \pi^{-1}(\Pi_l)$. Then

$$\Phi^* S_{N,n_l} \xrightarrow{\pi_\Phi} \mathscr{M},$$

where $\pi_\Phi : (\lambda, V_l) \mapsto \lambda \in \mathscr{M}$, constitutes a $U_{\eta_a(\lambda)}(n_l)$-principal fiber bundle. By construction, the sections of this bundle are just $\lambda \mapsto (\lambda, V_l)$.

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