Distributed sampling, quantum communication witnesses, and measurement incompatibility

Leonardo Guerini [1,*] Marco Túlio Quintino,[2] and Leandro Aolita[3]

[1] *International Centre for Theoretical Physics—South American Institute for Fundamental Research & Instituto de Física Teórica—UNESP, R. Dr. Bento Teobaldo Ferraz 271, São Paulo, Brazil*

[2] *Department of Physics, Graduate School of Science, The University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan*

[3] *Instituto de Física, Universidade Federal do Rio de Janeiro, P.O. Box 68528, Rio de Janeiro, RJ 21941-972, Brazil*

(Received 2 May 2019; published 14 October 2019)

We study prepare-and-measure experiments where the sender (Alice) receives trusted quantum inputs but has an untrusted state-preparation device and the receiver (Bob) has a fully untrusted measurement device. A distributed-sampling task naturally arises in this scenario, where the goal is for Alice and Bob to reproduce the statistics of his measurements on her quantum inputs using a fixed communication channel. Their performance of this task can certify quantum communication (QC), and this is formalized by measurement-device-independent QC witnesses. Furthermore, we prove that QC can provide an advantage (over classical communication) for distributed sampling if and only if Bob's measurements are incompatible. This gives an operational interpretation to measurement incompatibility and motivates a generalized notion of it related to a subset of quantum states. Our findings have both fundamental and applied implications.

DOI: 10.1103/PhysRevA.100.042308

I. INTRODUCTION

The prepare-and-measure scenario is a ubiquitous framework to investigate several foundational and communicational problems. There, one has two distant parties, Alice and Bob, and a referee, who sends them classical random inputs. Accordingly, Alice prepares a physical system, encoding a (classical or quantum) message that she sends to Bob. Bob then makes a measurement on the system and returns his outcome to the referee for final analysis. Depending on whether the message is classical or quantum, this framework provides a natural mindset for, e.g., classical and quantum dimension witnesses [1,2], quantum key distribution [3], classical and quantum random access codes [4,5], and self-testing [6,7]. All these tasks have been extensively studied in the so-called device-independent (DI) paradigm, where both Alice's state preparation and Bob's measurement stations are given by untrusted apparatuses effectively treated as black-box devices (the dimension of the communication channel is sometimes assumed, though). This implies that both devices admit only classical inputs and that Bob's device generates only classical outputs.

Alternatively, partially DI paradigms have also proven to yield extremely fruitful research lines. These consist of settings where the devices have both trusted (i.e., well-characterized and with full quantum control) and untrusted components. Notable instances thereof are the phenomena of Einstein-Podolsky-Rosen (EPR) steering [8], semiquantum instrumental causal networks [9], nonlocal correlations with quantum inputs [10] (which can be interpreted as measurement-device-independent entanglement certification [11]), and certification of quantum memories [12]. These studies have revealed interesting aspects of quantum theory that could not be properly addressed in the DI regime.

Here, we study the prepare-and-measure scenario with quantum inputs for Alice, which we call *semiquantum prepare and measure* (SQPM). More precisely, we consider a hybrid device for Alice, which admits trusted quantum-state preparations as inputs but is measurement DI, and a fully DI black-box device for Bob (see Fig. 1). In contrast to the usual prepare-and-measure scenario with classical inputs, in SQPM not all well-defined statistics admit a physical realization. Our first contribution is thus to characterize the set of SQPM statistics that arise from quantum experiments. Then we introduce a distributed-sampling (DS) problem where the goal is for Bob to simulate the outputs (i.e., sample from the outcome distribution) of measurements associated with his inputs on Alice's quantum states, using as little communication as possible. This is an information-theoretic task that can be used to certify quantum communication (QC) from Alice to Bob in the SQPM scenario. We formalize this through the notion of measurement-DI *quantum communication witnesses*, which can be efficiently obtained by means of semidefinite programs (SDPs). Furthermore, DS also turns out to be intimately connected to the fundamental problem of quantum measurement incompatibility [13]: We prove an equivalence between the quantum communication advantage for DS [over classical communication (CC)] and the incompatibility of the measurements implemented by Bob's black box. This provides a precise operational interpretation of measurement incompatibility and naturally leads to a generalized definition of compatibility relative to the input states in DS.

II. PRELIMINARIES

Let $\mathcal{H}$ be a d-dimensional complex Hilbert space and $\mathrm{Herm}(\mathcal{H})$ the set of linear operators acting on $\mathcal{H}$. Let $[n] =$

*guerini.leonardo@ictp-saifr.org

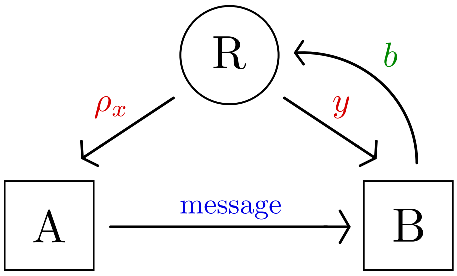

FIG. 1. Semiquantum prepare-and-measure scenario: A referee sends a random state ρ_x to Alice (unknown to her) and a random label y to Bob. Alice then sends a ρ_x-dependent message to Bob, who makes a y-dependent measurement on it and returns the outcome b to the referee.

$\{1,\ldots,n\}$. The states of a quantum system associated with $\mathcal{H}$ are given by linear operators $\rho\in\mathrm{Herm}(\mathcal{H})$ that are positive semidefinite and have unit trace, $\rho\geqslant 0$, $\mathrm{Tr}(\rho)=1$. The quantum measurements with o outcomes on this system are described by collections $\mathbf{M}_y=\{M_{b|y}\}_{b\in[o]}\subset\mathrm{Herm}(\mathcal{H})$ of positive semidefinite operators acting on $\mathcal{H}$ that sum up to the identity, $M_{b|y}\geqslant 0$, $\sum_b M_{b|y}=\mathbb{I}$. We denote by $\mathbb{S}(\mathcal{H})$ and $\mathbb{M}(\mathcal{H})$ the sets of all quantum states and measurements (with any number of outcomes) on $\mathcal{H}$, respectively.

A set of m quantum measurements $\mathcal{M}=\{\mathbf{M}_1,\ldots,\mathbf{M}_m\}$ with o outcomes is said to be *compatible*, or *jointly measurable* [13], if there exists a so-called mother measurement $\mathbf{N}=\{N_a\}_{a\in[n]}$ and response functions $f(\cdot|y,a):[o]\to[0,1]$, with $f(b|y,a)\geqslant 0$ and $\sum_b f(b|y,a)=1$ for all (b,y,a), such that

$$M_{b|y}=\sum_{a=1}^{n}N_a f(b|y,a) \tag{1}$$

for all $y\in[m]$ and $b\in[o]$. This expresses the fact that one can perform $\mathbf{N}$ and, depending on y and the mother measurement's outcome a obtained, sample b from $f(\cdot|y,a)$ to determine an outcome for $\mathbf{M}_y$. Denoting the (convex) set of compatible measurements **COMP**, we define the *generalized robustness of incompatibility* of $\mathcal{M}$ by

$$R_I(\mathcal{M})=\min\{\eta;\ \{(1-\eta)M_{b|y}+\eta Q_{b|y}\}_{b,y}\in\mathbf{COMP},\ \times\ \forall y\in[m]\ \{Q_{b|y}\}_b\in\mathbb{M}(\mathcal{H})\}, \tag{2}$$

i.e., the minimum amount of noise (represented by an arbitrary measurement $\mathbf{Q}$) needed to turn the combined measurement compatible.

A quantum channel is a completely positive trace-preserving linear map $\Lambda:\mathbb{L}(\mathcal{H})\to\mathbb{L}(\mathcal{H})$, forming a set denoted **CPTP**. We say that Λ is *non-steering-breaking* (NSB) if its adjoint $\Lambda^\dagger$ is *incompatibility-breaking*, i.e., $\{\Lambda^\dagger(M_{b|y})\}_{b,y}\in\mathbf{COMP}$ for all sets of measurements $\{M_{b|y}\}_{b,y}$ [14]. This follows from the fact that a set of measurements is incompatible if and only if it is useful for demonstrating EPR steering [15,16]. The *generalized robustness of the non-steering-breaking* of Λ is

$$R_{\mathrm{NSB}}(\Lambda)=\min\{\eta;\ \Gamma\in\mathbf{CPTP},\forall\{M_{b|y}\}_{b,y}\subset\mathbb{M}(\mathcal{H}),\ \times\ \{[(1-\eta)\Lambda+\eta\Gamma]^\dagger(M_{b|y})\}\in\mathbf{COMP}\}. \tag{3}$$

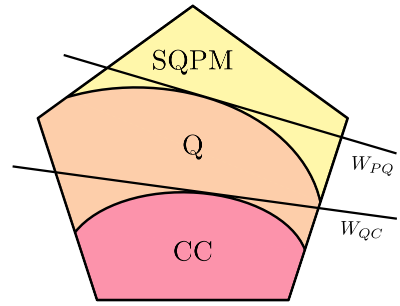

FIG. 2. General behaviors from a semiquantum prepare-and-measure (SQPM) scenario admit a polytope characterization and are illustrated by the external pentagon. The subset of quantum behaviors (Q) can be characterized by postquantum behavior witnesses (W_{PQ}) (see Appendix A) and the subset of behaviors that can be generated with classical communication (CC) can be characterized by quantum communication witnesses (W_{QC}).

III. THE SEMIQUANTUM PREPARE-AND-MEASURE SCENARIO

Consider the scenario where a sender, Alice, receives a quantum input ρ_x, and a receiver, Bob, is given a classical input y (see Fig. 1). We denote by $\mathcal{S}=\{\rho_x\}_x$ the set of quantum inputs for Alice. Alice's and Bob's inputs are randomly chosen from $\mathcal{S}$ and $[m]$, respectively, by a referee. Alice then prepares a (potentially quantum) message by implementing some (uncharacterized) operation on ρ_x and sends it to Bob, who extracts a classical output b from it through some (uncharacterized) measurement that may depend on y. This experiment is described by a state-conditioned behavior $\{(P(b|x,y),\rho_x)\}_{b,x,y}$, where $P(b|x,y)$ represents the conditional probability of b given x (the classical label of Alice's quantum input) and y. The state-conditioned behavior thus encapsulates the conditional probabilities of Bob's outcomes in explicit correspondence with the states ρ_x of Alice's inputs. Henceforth, we use the short-hand notation $\{P(b|\rho_x,y)\}_{b,x,y}$ for state-conditioned behaviors and refer to them simply as behaviors.

The standard prepare-and-measure scenario is recovered in the case where the states in $\mathcal{S}$ can be perfectly discriminated. Each choice $\mathcal{S}$ of trusted quantum states creates a different instance of the scenario, which is completely defined by the triple $(\mathcal{S},m,o)$, where m and o fix the range of values for the labels y and b, respectively. Thus, the standard probability constraints $P(b|\rho_x,y)\geqslant 0$, $\sum_b P(b|\rho_x,y)=1$, for all $b\in[o]$, $\rho_x\in\mathcal{S}$, and $y\in[m]$, define the polytope of behaviors from this scenario, whose extremal points are the $o^{m|\mathcal{S}|}$ deterministic behaviors (see Fig. 2).

The SQPM scenario is measurement device independent by definition. Given $P(b|\rho_x,y)$, the conditional on the quantum state ρ_x and on the classical label y can be completely arbitrary; in particular, we do not assume that the behavior can be obtained from quantum measurements performed on ρ_x. Whenever this is the case, we say that the behavior admits a quantum realization.

Definition 1. A behavior $\{P(b|\rho_x,y)\}_{b,x,y}$ *admits a quantum realization* if there exists a set of o-outcome measurements

$\{\mathbf{M}_y;\ y \in [m]\} \subset \mathbb{M}(\mathcal{H})$ such that

$$P(b|\rho_x, y) = \mathrm{Tr}(\rho_x M_{b|y}) \tag{4}$$

for any $b \in [o],\ \rho_x \in \mathcal{S},\ y \in [m]$. For simplicity, we refer to these as quantum behaviors and denote by $\mathbf{Q}$ the set formed by them.

In contrast with the prepare-in-measure scenario with classical inputs, in the SQPM not all behaviors are quantum; some statistics are incompatible with the given trusted quantum states (see Appendix A). However, deciding whether there exists a quantum realization for a given behavior can be done efficiently by means of SDP, as shown in Appendix A.

IV. DISTRIBUTED SAMPLING AND QUANTUM-COMMUNICATION WITNESSES

We now focus on quantum behaviors $\{P(b|\rho_x, y) = \mathrm{Tr}(\rho_x M_{b|y})\}_{b,x,y}$. We define a distributed sampling task by the following rules:

(1) A referee announces a set of states $\mathcal{S}$ and a set of m o-outcome measurements $\mathcal{M}$.

(2) The referee sends Alice a single copy of a randomly chosen $\rho_x \in \mathcal{S}$ and sends Bob a randomly chosen classical label $y \in [m]$.

(3) Alice applies an arbitrary quantum operation on ρ_x, producing a message that is sent to Bob.

(4) Conditioned on y and on the message, Bob generates an output $b \in [o]$ and sends it to the referee.

(5) Alice and Bob are successful if the conditional probability distributions observed by the referee after many rounds match the behavior $\{\mathrm{Tr}(\rho_x M_{b|y})\}_{b,x,y}$.

Note that $\mathcal{S}$ is broadcast, but Alice does not know the particular ρ_x (i.e., the value of x) sent to her in each run. The essence of this task appears in [17], where the inputs are not previously announced or restricted to limited sets, two-way communication is allowed, and the main interest lies in CC complexity. These differences allow us to focus on quantum properties of the involved objects.

This task is trivial if Alice and Bob have access to a perfect quantum communication channel, namely, the identity channel. In this case, Alice can simply send ρ_x to Bob, who will then hold both inputs and can implement $\mathbf{M}_y$, reproducing the statistics accurately. References [18,19] investigate the task of quantum compression, which can be interpreted as distributed sampling with perfect but lower-dimensional communication channels.

In contrast, consider now that Alice can only send classical messages to Bob. In this case, her most general strategy is to perform a quantum measurement $\mathbf{N} = \{N_a\}$ on the state ρ_x received by the referee and send the outcome a of her measurement to Bob. He then outputs a classical message b according to some response function, which may depend on the outcome a sent by Alice and the classical input y received from the referee.

Definition 2. A behavior $\{P(b|\rho_x, y)\}_{b,x,y}$ *admits a distributed sampling realization with classical communication* (CC realization) if there exists a quantum measurement $\mathbf{N} = \{N_a\}_{a=1}^n \in \mathbb{M}(\mathcal{H})$ and response functions $\{f(\cdot|y, a)\}_{y,a}$ such that

$$P(b|\rho_x, y) = \sum_{a=1}^{n} \mathrm{Tr}(\rho_x N_a) f(b|y, a) \tag{5}$$

for any $b \in [o],\ \ \rho_x \in \mathcal{S},\ \ y \in [m]$.

Characterizing the set $\mathbf{CC}$ of behaviors that admit a CC realization can be done by means of an SDP. Moreover, this SDP can quantify how far a given behavior $P = \{P(b|\rho_x, y)\}_{b,x,y}$ is from being CC realizable by calculating

$$\begin{aligned} R_{\mathrm{NCC}}(P) = \min\{\eta;\ (1-\eta)P + \eta q \in \mathbf{CC},\\ q = \{q(b|\rho_x, y)\}_{b,x,y} \in \mathbf{Q}\}. \end{aligned} \tag{6}$$

We call this quantity the *generalized robustness of non–CC realizability* of the behavior. Also, since $\mathbf{CC}$ is convex and compact, we can describe its border by means of witnesses.

Definition 3. A *quantum communication witness* is a pair $W_{\mathrm{QC}} = (\{\mu_{bxy}\}_{b,x,y}, \beta)$, with $\beta, \mu_{bxy} \in \mathbb{R}$, such that

$$\sum_{b,x,y} \mu_{bxy} P(b|\rho_x, y) \geqslant \beta \tag{7}$$

is satisfied by all CC-realizable behaviors, but violated by some behavior, in the scenario with trusted states $\mathcal{S} = \{\rho_x\}_x$.

Therefore, the violation of (7) is a measurement-DI way of certifying that Alice and Bob share a quantum communication channel.

Theorem 1. Let $\mathcal{S} \subset \mathbb{S}(\mathcal{H})$ be a finite set of states and o, m be positive integers. Then any behavior $P = \{P(b|\rho_x, y);\ \rho_x \in \mathcal{S},\ b \in [o],\ y \in [m]\} \notin \mathbf{CC}$ violates some quantum communication witness. Moreover, the maximal violation over all witnesses provides exactly the non-CC-realizability generalized robustness of this behavior,

$$R_{\mathrm{NCC}}(P) = \max_{W_{\mathrm{QC}}} \sum_{b,x,y} \mu_{bxy} P(b|\rho_x, y) - \beta. \tag{8}$$

In Appendix B we provide the proof of Theorem 1 and details on the SDP approach. As an application, we study the advantage for distributed sampling of a paradigmatic noisy quantum channel over classical ones. The results are graphically summarized in Fig. 3.

V. MEASUREMENT INCOMPATIBILITY

Next, we show that measurement compatibility is a "classical" property that matches precisely the classical communication case of distributed sampling. This implies that any distributedly sampled quantum behavior can be used to estimate both the degree of incompatibility of the implemented measurements and the degree of non-steering-breaking of the utilized channel.

Theorem 2. A set of measurements $\mathcal{M} \subset \mathbb{M}(\mathcal{H})$ is compatible if and only if the behavior $\{\mathrm{Tr}(\rho_x M_{b|y});\ \rho_x \in \mathcal{S},\ \mathbf{M}_y \in \mathcal{M}\}$ admits a distributed sampling realization with classical communication for any set of states $\mathcal{S} \subset \mathbb{S}(\mathcal{H})$ that spans $\mathrm{Herm}(\mathcal{H})$. Moreover, for any distributedly sampled quantum behavior $P = \{P(b|\rho_x, y) = \mathrm{Tr}(\widetilde{\Lambda}(\rho_x)\widetilde{M}_{b|y})\} \in \mathbf{Q}$, we have

$$R_{\mathrm{NCC}}(P) \leqslant R_I(\widetilde{\mathcal{M}}) \quad \text{and} \quad R_{\mathrm{NCC}}(P) \leqslant R_{\mathrm{NSB}}(\widetilde{\Lambda}), \tag{9}$$

where $\widetilde{\mathcal{M}}$ and $\widetilde{\Lambda}$ are the uncharacterized measurements and communication channel, respectively, used in the sampling of

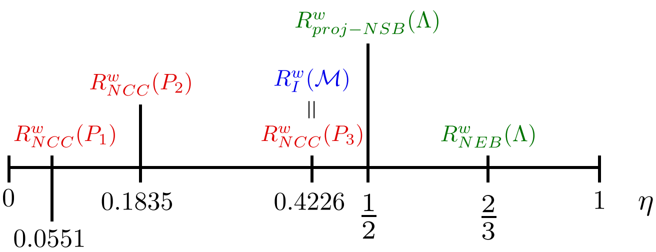

FIG. 3. Robustness of the properties of behaviors, set of measurements, and quantum channel involved in the distributed sampling with the depolarizing qubit channel $D_\eta : A \mapsto (1-\eta)A + \eta \mathrm{Tr}(A)\mathbb{I}/2$ of noise $0 \leqslant \eta \leqslant 1$. Consider the sets of states $\mathcal{S}_1 = \{|0\rangle\langle 0|, |+\rangle\langle +|\}$, $\mathcal{S}_2 = \{|0\rangle\langle 0|, |+\rangle\langle +|, |r\rangle\langle r|\}$, and $\mathcal{S}_3 = \{|0\rangle\langle 0|, |+\rangle\langle +|, |r\rangle\langle r|, \mathbb{I}/2\}$, where $|+\rangle$, $|r\rangle$, and $|0\rangle$ are, respectively, the positive-eigenvalue eigenstates of the Pauli matrices σ_x, σ_y, and σ_z, and the measurement set $\mathcal{M} \equiv \{\sigma_x, \sigma_y, \sigma_z\}$. The figure shows the robustness of the non-classical-communication realizability of the resulting behaviors P_1, P_2, and P_3, i.e., the critical noise at which the behavior admits a distributed sampling realization with classical communication. Since $D_\eta(\mathcal{M})$ is compatible for $\eta \geqslant 1 - 1/\sqrt{3} \approx 0.4226$, for this range D_η is replaceable by a classical channel regardless of the input states. In turn, for $\eta \geqslant 1/2$ the adjoint channel breaks the incompatibility of any set of projective measurements, and for $\eta \geqslant 2/3$ the channel is entanglement breaking. More details are given in Appendix E.

P. The equality holds in the first case if $\mathcal{S}$ spans Herm($\mathcal{H}$) and in the second case if, besides that, the measurements $\{\widetilde{\Lambda}^\dagger(\widetilde{M}_{b|y})\}_{b,y}$ present the greatest generalized robustness of incompatibility in its dimension.

The proof is presented in Appendix C.

Theorem 2 provides an operational interpretation for joint measurability in terms of a communicational task. The first inequality in (9) shows that any incompatible set of measurements generates some behavior that can certify quantum communication via distributed sampling. Similarly, the second inequality in (9) implies that a channel Λ is useless to certify quantum communication in this scenario if and only if $\Lambda^\dagger$ is incompatibility-breaking (i.e., Λ is steering-breaking).

A consequence of Theorem 2 is that a certification of quantum communication via distributed sampling also detects the incompatibility of the implemented measurements. Hence, quantum communication witnesses form a particular class of measurement incompatibility witnesses [21]. In general, the latter are defined by a set of Hermitian operators $\{F_{by}\}_{b,y}$ acting on the same space as the measurements and a scalar γ such that the condition

$$\sum_{b,y} \mathrm{Tr}(F_{by} M_{b|y}) \geqslant \gamma \tag{10}$$

is satisfied for any compatible set $\mathcal{M} = \{M_{b|y}\}_{b,y}$ but violated by some incompatible set. Our next result shows that, conversely, every measurement incompatibility witness also detects the exchange of quantum communication in the appropriate DS context.

Theorem 3. For any measurement incompatibility witness $W_{\mathrm{MI}} = (\{F_{by}\}, \gamma)$ there exists a set of states $\mathcal{S} = \{\rho_x\}_x$ and a quantum communication witness $W_{\mathrm{QC}} = (\{\mu_{bxy}\}, \beta)$ that detects the incompatibility of the same sets of measurements as W_{MI}.

The proof can be found in Appendix C.

VI. MEASUREMENT COMPATIBILITY ON A RESTRICTED SET OF STATES

Theorem 2 unveils a direct connection between distributed sampling and measurement compatibility, in which compatibility is equivalent to CC realizability with an informationally complete set of states. For more restricted sets of states, behaviors that can be CC realized are directly connected to a relaxed notion of compatibility which we now define.

Definition 4. A set of o-outcome measurements $\{M_{b|y}\}_{b,y}$ is *compatible on* $\mathcal{S} = \{\rho_x\}_x$ if there exists a mother measurement $\mathbf{N} = \{N_a\}_a$ and response functions $f(\cdot|y, a) : [o] \to [0, 1]$ such that

$$\mathrm{Tr}(\rho_x M_{b|y}) = \sum_{a=1}^{n} \mathrm{Tr}(\rho_x N_a) f(b|y, a) \tag{11}$$

for all $\rho_x \in \mathcal{S}$, $y \in [m]$, and $b \in [o]$.

Definition 4 can be applied to all situations where the experimenter is guaranteed that all states in the experiment lie in a restricted set $\mathcal{S}$, e.g., a qubit experiment where all states involved lie in the xz plane of the Bloch sphere. In such cases, the standard definition of joint measurability may represent an overkill, and the relaxed notion may be more suitable. See Fig. 3 and Appendix E for more on this topic.

VII. FINAL DISCUSSION

We have formalized the semiquantum prepare-and-measure scenario and presented a distributed sampling task potentially interesting on its own beyond the scope of this work. In turn, this leads to non-classical-communication certification via quantum communication witnesses. The underlying feature that allows this is the non-steering-breaking property of the communication channel, a notion strictly stronger than non-entanglement-breaking [20]. Hence, our framework cannot certify steering-breaking channels that are non-entanglement-breaking (and, therefore, still nonclassical). On the other hand, both properties usually require entanglement for their certification, while our framework uses single systems only (see Ref. [12] for non-entanglement-breaking channel certification in a scenario with two quantum inputs).

Measurement incompatibility is known to have operational interpretations in terms of EPR steering [15,16] and state discrimination games with postmeasurement information [21–23]. Our findings also provide a communication task that captures precisely the essence of this property, complementing the operational interpretations from previous results (although being essentially different from them; see Appendix D for a discussion of state discrimination).

Note that if two parties share an entangled quantum state and classical communication they can simulate a quantum channel via the teleportation protocol. Also, teleportation can be regarded as a protocol where Alice has a quantum input and Bob can do tomography on his final state. Since in our scenario Bob's measurements are untrusted, our results point towards a realization of quantum teleportation as DI as possible, where only Alice's input state is trusted.

Further open problems for future research are the characterization of the inner structure of the SQPM polytope and a quantitative study of the amount of classical or quantum communication required for approximate or probabilistic distributed sampling. For the classical case, it would be interesting to see how the results in Ref. [17] relate to measurement incompatibility. Finally, the quantum communication witnesses developed here are experimentally relevant and implementable with current technology.

All code and calculations can be found in the repository at Ref. [24].

ACKNOWLEDGMENTS

The authors thank Ranieri V. Nery and Gláucia Murta for fruitful discussions. L.A. thanks ICTP-SAIFR and the IFT-UNESP at São Paulo—where this work was started and partially developed—for the hospitality. L.G. was supported by São Paulo Research Foundation (FAPESP) under Grants No. 2016/01343-7 and No. 2018/04208-9. M.T.Q. acknowledges the Q-LEAP project of MEXT, Japan. L.A. was financially supported by the Brazilian agencies CNPq (PQ Grant No. 311416/2015-2 and INCT-IQ), FAPERJ (JCN E-26/202.701/2018), CAPES (PROCAD2013), FAPESP, and the Serrapilheira Institute (Grant No. Serra-1709-17173).

APPENDIX A: QUANTUM REALIZATIONS

Let $\mathcal{S} \in \mathbb{S}(\mathcal{H})$ be a set of quantum states and m, o be positive integers. Not all semiquantum prepare-and-measure behaviors $\{P(b|\rho_x, y); \rho_x \in \mathcal{S}, b \in [o], y \in [m]\}$ admit a quantum realization, i.e., can be written as $P(b|\rho_x, y) = \mathrm{Tr}(\rho_x M_{b|y})$, for some measurements $\{M_{b|y}\}_{b,y} \subset \mathbb{M}(\mathcal{H})$. Indeed, consider $\mathcal{H} = \mathbb{C}^2$, $\mathcal{S} = \{|0\rangle\langle 0|, |1\rangle\langle 1|, \mathbb{I}/2\}$, $o = 2$, $m = 1$, and the behavior specified by

$$P(1||0\rangle\langle 0|, 1) = P(1||1\rangle\langle 1|, 1) = 0, \tag{A1a}$$

$$P\left(1|\frac{\mathbb{I}}{2}, 1\right) = 1, \tag{A1b}$$

together with the normalization constraints. If this behavior is quantum samplable, then there exists a quantum measurement $\mathbf{M}_1 = \{M_{1|1}, M_{2|1}\}$ satisfying

$$0 = P(1||0\rangle\langle 0|, 1) + P(1||1\rangle\langle 1|, 1) \tag{A2a}$$

$$= \mathrm{Tr}(|0\rangle\langle 0|M_{1|1}) + \mathrm{Tr}(|1\rangle\langle 1|M_{1|1}) \tag{A2b}$$

$$= 2\mathrm{Tr}\left(\frac{\mathbb{I}}{2}M_{1|1}\right) \tag{A2c}$$

$$= 2P\left(1|\frac{\mathbb{I}}{2}, 1\right) = 2, \tag{A2d}$$

an obvious contradiction.

More generally, we can decide whether a behavior $\{P(b|\rho_x, y); \rho_x \in \mathcal{S}, b \in [o], y \in [m]\}$ admits a quantum realization with the semidefinite program

$$\begin{aligned}
&\text{given} \quad \{P(b|\rho_x, y)\}_{b,x,y} \\
&\min_{\{q(\cdot|\rho_x, y)\}, \{\mathbf{M}_y\}} \quad \eta \\
&\text{s.t.} \quad (1-\eta)P(b|\rho_x, y) + q(b|\rho_x, y) \\
&= \mathrm{Tr}(\rho_x M_{b|y}), \quad \forall \rho_x, b, y, \\
&\times q(b|\rho_x, y) \geqslant 0, \quad \forall \rho_x, b, y, \\
&\sum_b q(b|\rho_x, y) = \eta, \quad \forall \rho_x, y, \\
&M_{b|y} \geqslant 0, \quad \forall b, y, \\
&\sum_b M_{b|y} = \mathbb{I}, \quad \forall y.
\end{aligned} \tag{A3}$$

The probabilities $\{q(\cdot|\rho_x, y)/\eta\}_{x,y}$ represent an arbitrary noise, which is mixed with the given behavior until it accepts a quantum description $\{\mathrm{Tr}(\rho_x M_{b|y})\}_{b,x,y}$. Hence, the input behavior admits a quantum realization if and only if the optimal value obtained is $\eta^* \leqslant 0$.

For a fixed triple $(\mathcal{S}, o, m)$, it follows from the convexity of the set of quantum measurements that the set of quantum behaviors is convex, as well as compact. Hence, due to the separating hyperplane theorem [25] we have that such a set can be characterized by postquantum behavior witnesses.

Definition 5. A *postquantum behavior witness* is a pair $W_{\mathrm{PQ}} = (\{\lambda_{bxy}\}, \alpha)$ formed by real coefficients λ_{bxy} and a bound α such that

$$\sum_{b,x,y} \lambda_{bxy} P(b|\rho_x, y) \geqslant \alpha \tag{A4}$$

is satisfied for all quantum behaviors $\{P(b|\rho_x, y)\}$ but violated by some behavior of the scenario. Hence, even if a given behavior $\{P(b|x, y)\}_{b,x,y}$ may arise from some quantum experiment, a violation of (A4) ensures that this experiment does not involve states $\{\rho_x\}_x$, calculated from the dual formulation of SDP (A3).

We now show that postquantum behavior witnesses can be associated with the generalized robustness [26,27] of the property of the inputs under study (in our case, it is the 'postquantumness' of the behavior), since we optimize over all possible noises. Let us now fix an arbitrary noise, that is, consider $\{q(\cdot|\rho_x, y)/\eta\}_y$ not as variables but as inputs selected previously. This variation of SDP (A3) admits the Lagrangian

$$\mathcal{L}(\eta, \{M_{b|y}\}) = \eta\left(1 + \sum_{b,x,y} \lambda_{bxy}[P(b|\rho_x, y) - q(b|\rho_x, y)]\right) \tag{A5a}$$

$$- \sum_{b,y} \mathrm{Tr}\left(M_{b|y}\left[U_{by} + A_y - \sum_x \lambda_{bxy}\rho_x\right]\right) \tag{A5b}$$

$$- \sum_{b,x,y} \lambda_{bxy} q(b|\rho_x, y) + \sum_y \mathrm{Tr}(A_y) \tag{A5c}$$

for arbitrary scalars λ_{bxy} and operators $U_{by} \geqslant 0, A_y$. Each choice of these parameters yields an upper bound for the optimal value t^* of SDP (A3). Minimizing over them, considering feasibility constraints, and eliminating the slack variables U_{by}, we obtain the dual formulation[1] of SDP (A3),

$$\begin{aligned}
&\text{given} \quad \{P(b|\rho_x, y)\}_{b,x,y}, \{q(b|\rho_x, y)\}_{b,x,y} \\
&\max_{\{\lambda_{bxy}\},\{A_y\}} \quad \sum_y \mathrm{Tr}(A_y) - \sum_{bxy} \lambda_{bxy} P(b|\rho_x, y) \quad \text{(A6a)}\\
&\text{s.t.} \quad \sum_x \lambda_{bxy}\rho_x \geqslant A_y, \ \forall b, y \quad \text{(A6b)}\\
&\sum_{bxy} \lambda_{bxy}(q(b|\rho_x, y) - P(b|\rho_x, y)) = 1. \quad \text{(A6c)}
\end{aligned}$$

Consider the following result.

Theorem 4. Let $\mathcal{S} \subset \mathbb{S}(\mathcal{H})$ be a finite set of states and o, m be positive integers. Then every behavior $\{P(b|\rho_x, y);\ \rho_x \in \mathcal{S},\ b \in [o],\ y \in [m]\}$ that is not quantum violates some postquantum behavior witness.

The separating hyperplane theorem suffices to prove Theorem VII. Nonetheless, we now present a second demonstration, more constructive, that is based on SDPs (A3) and (A6). Duality theory says that the optimal solution for the dual problem is always an upper bound for the optimal solution of its primal. Sometimes, however, we may have strong duality between them, meaning that the problems are such that both optimal values coincide. A sufficient requirement for ensuring strong duality is called Slater's condition, which is satisfied whenever there is a feasible point satisfying all equality constraints and strictly satisfying the inequalities ones, for either one of the problems. Our second proof of Theorem 4 is based on proving that the two SDPs above satisfy Slater's condition and are therefore strongly dual.

Proof. The behavior $\{P(b|\rho_x, y)\}$ admits a quantum realization if and only if SDP (A3) yields an optimal solution $\eta^* \geqslant 1$. This SDP is strictly feasible: for $\eta = 1$ we obtain a solution by placing $q(b|\rho_x, y) = \mathrm{Tr}(\rho_x M_{b|y})$, for any arbitrary measurements $\{M_{b|y}\}_{b,y}$. In particular, choosing $M_{b|y} = \mathbb{I}/o$ for all b, y we have that $M_{b|y}$ are strictly positive operators, while $q(b|\rho_x, y)$ are strictly positive scalars. Hence, by Slater's condition [25] we have that (A3) and (A6) present strong duality. This implies that $\{P(b|\rho_x, y)\}$ admits a quantum realization if and only if the dual SDP (A6) yields an optimal solution less than or equal to 0 or, equivalently,

$$\sum_{bxy} \lambda^*_{bxy} P(b|\rho_x, y) \geqslant \sum_y \mathrm{Tr}(A^*_y), \quad \text{(A7)}$$

where λ^*_{bxy}, A^*_Y are provided by the optimal solution of (A6). Taking $\alpha = \sum_y \mathrm{Tr}(A^*_y)$ concludes the proof.

As an example, inputting the behavior defined in Eqs. (A1) in SDP (A6) we obtain the postquantum realization witness

$$-\tfrac{1}{2}[P(2|\rho_1, 1) + P(2|\rho_2, 1)] - P(1|\rho_3, 1) \geqslant -1. \quad \text{(A8)}$$

The left-hand side of the above witness equals -2 when evaluated on behavior (A1), confirming its "postquantumness."

[1] By fixing the noise we simplify SDP (A3); leaving the noise unspecified would lead to the same dual with an extra constraint, concerning the nonnegativity of variables $q(\cdot|\rho_x, y)$.

APPENDIX B: DISTRIBUTED SAMPLING WITH CLASSICAL COMMUNICATION

Recall that a behavior $\{P(b|\rho_x, y)\}_{b,x,y}$ admits a distributed sampling realization with classical communication (is CC realizable) if there exists a quantum measurement $\mathbf{N} \in \mathbb{M}(\mathcal{H})$ and response functions $\{f(\cdot|y, a)\}_{y,a}$ such that

$$P(b|\rho_x, y) = \sum_{a=1}^{n} \mathrm{Tr}(\rho_x N_a) f(b|y, a) \quad \text{(B1)}$$

for any $b \in [o],\ \rho_x \in \mathcal{S},\ y \in [m]$. Note that every such behavior is quantum realizable, associated with the measurements $M_{b|y} = \sum_a N_a f(b|y, a)$. Hence, admitting a quantum realization is a necessary condition for a behavior to be CC realizable.

The subset of behaviors that can be distributedly sampled with classical communication forms the convex and compact set **CC**. Consequently, Theorem 1 can also be seen as an application of the separating hyperplane theorem [25]. In what follows, we provide an SDP approach to the problem and an alternative proof of Theorem 1.

Note that we can always take Bob's response function to be deterministic by mapping its local randomness to the measurement **N**. In other words, if such a strategy is possible, then it can be done with **N** having at most $n = (o)^m$ outcomes, given by $\mathbf{a} = a_1 \dots a_m$, with $a_i \in [o]$. These outcomes already encode Bob's answer; he simply outputs the yth symbol of the classical message **a**, which corresponds to applying the map $p(b|y, \mathbf{a}) = \delta_{a_y, b}$.

Hence we can decide whether $\{P(b|\rho_x, y)\}_{b,x,y} \in \mathbf{CC}$ via the generalized robustness SDP:

$$\begin{aligned}
&\text{given} \quad \{P(b|\rho_x, y)\}_{b,x,y}\\
&\min_{\mathbf{N},\{\widetilde{M}_{b|y}\}} \quad \eta\\
&\text{s.t.} \quad (1-\eta)P(b|\rho_x, y) + \mathrm{Tr}(\rho_x \widetilde{M}_{b|y})\\
&= sum_{\mathbf{a}} \mathrm{Tr}(\rho_x N_{\mathbf{a}})\delta_{a_y,b}, \ \forall x, b, y\\
&\widetilde{M}_{b|y} \geqslant 0, \ \forall b, y\\
&= \sum_b \widetilde{M}_{b|y} = \eta\mathbb{I}, \ \forall y\\
&N_{\mathbf{a}} \geqslant 0, \ \forall \mathbf{a} \in \{1, \dots, o\}^{\times|\mathcal{M}|}\\
&\sum_{\mathbf{a}} N_{\mathbf{a}} = \mathbb{I}, \quad \text{(B2)}
\end{aligned}$$

where $\{\widetilde{M}_{b|y}/\eta\}_{b,y}$ is an arbitrary measurement that provides the quantum noise $\{\mathrm{Tr}(\rho_x \widetilde{M}_{b|y})/\eta\}_{b,x,y}$. Running SDP (B2), if the optimal value obtained is $\eta^* > 0$, then some amount of noise is needed and we know that $\{P(b|\rho_x, y)\}$ requires quantum communication to be distributedly sampled.

Let us now fix an arbitrary quantum noise $\{q(b|\rho_x, y) = \mathrm{Tr}(\rho_x \widetilde{M}_{b|y})\}_{b,x,y}$ (see footnote 1 in Appendix A). The obtained

SDP admits the Lagrangian

$$\mathcal{L}(\eta, \{N_a\}) = \eta\left(1 + \sum_{b,x,y} \mu_{bxy}[P(b|\rho_x, y) - q(b|\rho_x, y)]\right) \tag{B3a}$$

$$+ \sum_a \mathrm{Tr}\left(N_a\left[V_a + B - \sum_{b,x,y} \mu_{bxy}\rho_x\delta_{a_y,b}\right]\right) \tag{B3b}$$

$$\times\, \mathrm{Tr}(B) - \sum_{b,x,y} \mu_{bxy} q(b|x, y) \tag{B3c}$$

for arbitrary scalars λ_{bxy} and operators $V_a \geqslant 0, B$. Minimizing over these parameters, considering feasibility constraints, and eliminating the slack variables V_a, we obtain the dual formulation

$$\min_{\{\mu_{bxy}\},B} \quad \mathrm{Tr}(B) - \sum_{b,x,y} \mu_{bxy}\mathrm{Tr}(\rho_x M_{b|y}) \tag{B4a}$$

$$\text{s.t.} \quad \sum_{b,x,y} \mu_{bxy}\rho_x\delta_{a_y,b} \geqslant B, \ \forall \mathbf{a} \in [o]^{\times m} \tag{B4b}$$

$$\sum_{bxy} \mu_{bxy}[q(b|\rho_x, y) - P(b|\rho_x, y)] = 1. \tag{B4c}$$

We now show that these two problems indeed display strong duality. This implies that every behavior whose distributed sampling cannot be implemented only with classical communication violates a quantum communication witness and proves Theorem 1.

Proof. Since behaviors that are not quantum realizable cannot be CC realizable, we can restrict our proof to quantum realizable behaviors.

Let $\{P(b|\rho_x, y) = \mathrm{Tr}(\rho_x M_{b|y})\}_{b,x,y}$ be a quantum realizable behavior. Let $D_t : A \mapsto (1-\eta)A + \eta\mathrm{Tr}(A)\mathbb{I}/d$ be the depolarizing map, with robustness $\eta \in [0, 1]$. Given the above behavior, we obtain a strictly feasible solution for SDP (B2) by setting $q(b|x, y) = \mathrm{Tr}(M_{b|y})/d$ as white noise and $1 > \eta > 0$ close enough to 1 so that the depolarized measurements $\mathcal{M}_\eta = \{D_\eta(M_{b|y})\}$ are jointly measurable. From the connection between joint measurability and Einstein-Podolsky-Rosen steering [15,16], we know this happens for η strictly less than 1 [28]. If the corresponding mother measurement $\mathbf{N} = \{N_a\}_a$ possesses a zero eigenvalue, then the set $\mathcal{M}_{\eta'}$, with $1 > \eta' > \eta$, admits a mother $\mathbf{N}' = \{\Phi_{\eta'/\eta}(N_a)\}$, with strictly positive elements. Hence, by Slater's condition [25] the SDPs (B2) and (B4) present strong duality.

Therefore, if the behavior can be distributedly sampled with classical communication, the primal SDP yields $\eta^* \leqslant 0$. Strong duality ensures that the optimal solution for the dual matches η^*, being also $\leqslant 0$, and thus we have

$$\sum_{b,x,y} \mu_{bxy}\mathrm{Tr}(\rho_x M_{b|y}) \geqslant \mathrm{Tr}(B) \tag{B5}$$

for some real coefficients $\{\mu_{bxy}\}_{b,x,y}$ and some matrix B acting on $\mathcal{H}$ satisfying Eq. (B4b). Moreover, $R_{\mathrm{NCC}} = \eta^* = \max_{W_{QC}} \beta^* - \sum_{b,x,y} \mu^*_{bxy}\mathrm{Tr}(\rho_x M_{b|y})$, where this is the maximum violation of some witness for the given behavior.

As a concrete example, consider in dimension $d = 2$ the distributed sampling of the statistics generated by the set of states $\hat{\mathcal{S}} = \{|+\rangle\langle+|, |y\rangle\langle y|, |0\rangle\langle 0|, \mathbb{I}/2\}$, formed by one eigenstate of each Pauli matrix X, Y, and Z and the maximally mixed state, and the set of measurements $\hat{\mathcal{M}} = \{\mathbf{M}_x, \mathbf{M}_y\}$ associated with X and Y. Running SDP (B4) with q set as white noise, we obtain the quantum communication witness W_{QC} given by

$$\begin{aligned}
&\frac{-1}{2}[P(1|\rho_1, 1) + P(1|\rho_3, 2)] \\
&+ \frac{1}{2}[P(2|\rho_1, 1) + P(2|\rho_3, 2)] \\
&+ \frac{1}{2\sqrt{2}}[P(1|\rho_2, 1) + P(2|\rho_2, 1)] \\
&+ P(1|\rho_4, 2) - P(2|\rho_4, 1) \geqslant \frac{-1}{2\sqrt{2}}.
\end{aligned}$$

The behavior yielded by $\hat{\mathcal{S}}$ and $\hat{\mathcal{M}}$ violates it up to $1/(2\sqrt{2}) - 1$. Note that $\mathcal{S}$ spans the set of Hermitian operators acting in $\mathbb{C}^2$, therefore every incompatible set of measurements is detected by some W_{QC} constructed from $\mathcal{S}$.

APPENDIX C: PROOFS OF THEOREMS 2 AND 3

Here we restate and prove theorems from the text.

Theorem 2. A set of measurements $\mathcal{M} \subset \mathbb{M}(\mathcal{H})$ is compatible if and only if the behavior $\{\mathrm{Tr}(\rho_x M_{b|y});\ \rho_x \in \mathcal{S},\ \mathbf{M}_y \in \mathcal{M}\}$ admits a distributed sampling realization with classical communication for any set of states $\mathcal{S} \subset \mathbb{S}(\mathcal{H})$ that spans $\mathrm{Herm}(\mathcal{H})$. Moreover, for any distributedly sampled quantum behavior $P = \{P(b|\rho_x, y) = \mathrm{Tr}(\widetilde{\Lambda}(\rho_x)\widetilde{M}_{b|y})\} \in \mathbf{Q}$, we have

$$R_{\mathrm{NCC}}(P) \leqslant R_I(\widetilde{\mathcal{M}}) \quad \text{and} \quad R_{\mathrm{NCC}}(P) \leqslant R_{\mathrm{NSB}}(\widetilde{\Lambda}), \tag{C1}$$

where $\widetilde{\mathcal{M}}$ and $\widetilde{\Lambda}$ are the uncharacterized measurements and communication channel, respectively, used in the sampling of P. The equality holds in the first case if $\mathcal{S}$ spans $\mathrm{Herm}(\mathcal{H})$ and in the second case if, besides that, the measurements $\{\widetilde{\Lambda}^\dagger(\widetilde{M}_{b|y})\}_{b,y}$ present the greatest generalized robustness of incompatibility in its dimension.

Proof. Let $\mathcal{S} \in \mathcal{S}(\mathcal{H})$ be an arbitrary set of states and $\rho_x \in \mathcal{S}$. Assuming $\mathcal{M}$ to be jointly measurable, Alice can perform the mother measurement $\mathbf{N}$ on ρ_x and send the obtained outcome a to Bob, who applies $f(\cdot|y, a)$, one of the postprocessing distributions that accompanies $\mathbf{N}$. The statistics generated are described by

$$P(b|\rho_x, y) = \sum_{a=1}^{n} \mathrm{Tr}(\rho_x N_a) f(b|y, a) \tag{C2}$$

for any $b \in [o],\ \rho_x \in \mathcal{S},\ y \in [m]$, where the equality is guaranteed, independently of ρ_x, by the joint measurability hypothesis,

$$M_{b|y} = \sum_{a=1}^{n} N_a f(b|y, a), \tag{C3}$$

for all $y \in [m]$ and $b \in [o]$. It follows that $R_{\mathrm{NCC}}(P) \leqslant R_I(\mathcal{M})$.

On the other hand, suppose that there exists $\mathbf{N}$ and $\{p(\cdot|y, a)\}_{y,a}$ such that $\sum_a \mathrm{Tr}(N_a\rho)p(b|y, a) = \mathrm{Tr}(\rho_x M_{b|y})$ for all $(\rho_x, \mathbf{M}_y) \in \mathcal{S} \times \mathcal{M}$. If $\mathcal{S}$ spans Herm($\mathcal{H}$), this implies the relation between measurement operators given in Eq. (1). Hence $\mathcal{M}$ is jointly measurable, admitting $\mathbf{N}$ as a mother measurement and $\{f(\cdot|y,a)\}_{y,a}$ as postprocessing maps. Thus, $R_{\rm NCC}(P) \leqslant R_I(\mathcal{M})$ can be saturated for informationally complete sets $\mathcal{S}$.

Finally, we can write $P(b|\rho_x, y) = \mathrm{Tr}(\widetilde{\Lambda}(\rho_x)\widetilde{M}_{b|y}) = \mathrm{Tr}(\rho_x \widetilde{\Lambda}^\dagger(\widetilde{M}_{b|y}))$ to describe the effective communication channel $\widetilde{\Lambda}$ (which comprehends Alice's preparation on ρ_x and is a quantum-classical channel in the classical communication case) and the effective measurements $\{\widetilde{M}_{b|y}\}$ that generate the behavior. For large enough η, there exists some noise channel Γ such that $(1-\eta)\Lambda + \eta\Gamma$ is steering breaking. At this point, $\{[(1-\eta)\Lambda + \eta\Gamma]^\dagger(\widetilde{M}_{b|y})\}$ is compatible and its corresponding statistics are CC realizable, by the first part of the theorem. Therefore, $R_{\rm NSB}(\widetilde{\Lambda}) \geqslant R_{\rm NCC}(P)$, which is saturated if $\{\rho_x\}_x$ spans Herm($\mathcal{H}$)—thus detecting the standard incompatibility of the underlying measurements $\widetilde{\Lambda}(\widetilde{\mathcal{M}})$—and if those are the most incompatible acting in $\mathcal{H}$—thus implying that the noisy channel would break the incompatibility of any other set of measurements as well.

Theorem 3. For any measurement incompatibility witness $W_{\rm MI} = (\{F_{by}\}, \gamma)$ there exists a set of states $\{\rho_x\}_x$ and a quantum communication witness $W_{\rm QC} = (\{\mu_{bxy}\}, \beta)$ in the corresponding distributed sampling scenario that detects the incompatibility of the same sets of measurements as $W_{\rm MI}$.

Proof. Let $\{\rho_x\}_x$ be a set of states that spans $\{F_{by}\}$, i.e., satisfies

$$F_{by} = \sum_x \lambda_x^{by} \rho_x \tag{C4}$$

for some real coefficients $\{\lambda_x^{by}\}_{b,x,y}$. Such a set always exists, since any Hermitian operator can be written as the difference between two positive operators, which can be renormalized to be trace 1. Applying this procedure to a basis of the space of Hermitian operators acting in the underlying Hilbert space provides a set of quantum states that span the set of Hermitian operators.

Consider now the expression

$$\sum_{b,x,y} \lambda_x^{by} \mathrm{Tr}(\rho_x M_{b|y}). \tag{C5}$$

If the behavior $\{\mathrm{Tr}(\rho_x M_{b|y})\}_{b,x,y}$ admits a CC sampling, there exists a measurement $\mathbf{N}$ and response functions $f(\cdot|y,a)$ such that we can rewrite (C5) as

$$\sum_{b,x,y} \lambda_x^{by} \mathrm{Tr}(\rho_x M_{b|y}) = \sum_{b,x,y} \lambda_x^{by} \sum_a \mathrm{Tr}(\rho_x N_a) f(b|y,a) \tag{C6a}$$

$$= \sum_{b,y} \mathrm{Tr}\left(\sum_x \lambda_x^{by} \rho_x \left[\sum_a N_a f(b|y,a)\right]\right) \tag{C6b}$$

$$= \sum_{b,y} \mathrm{Tr}(F_{by}[\widetilde{M}_{b|y}]) \tag{C6c}$$

for some set of measurements $\{\widetilde{M}_{b|y} := \sum_a N_a f(b|y,a)\}_{b,y}$, which is compatible by definition. Since $(\{F_{by}\}, \gamma)$ is an incompatibility witness, the expressions above are lower-bounded by γ.

By definition, $(\{F_{by}\}, \gamma)$ detects some incompatible set of measurements $\{\hat{M}_{b|y}\}_{b,y}$ and, hence, $(\{\lambda_{bxy}\}, \gamma)$ detects the quantum communication in the distributed sampling of the behavior $\{\mathrm{Tr}(\rho_x \hat{M}_{b|y})\}$. Therefore, $(\{\lambda_b^x y\}, \gamma)$ defines a quantum communication witness, which detects the same incompatible measurements (with the trusted states ρ_x) as $(\{F_{by}\}, \gamma)$ by construction.

APPENDIX D: RELATION WITH STATE DISCRIMINATION

We start by pointing out that if a set of states $\mathcal{S} = \{\rho_x\}_x$ can be perfectly discriminated, then Alice can identify the label x and send it to Bob, who will again hold both inputs. Therefore, perfect discrimination implies a CC realization in our DS task. Consequently, a violation of a quantum communication witness detects not only the quantum communication needed for realizing the behavior and the incompatibility of the measurements, but also that $\mathcal{S}$ is not perfectly discriminable.

It is shown in Theorem 3 that every incompatibility witness corresponds to a quantum communication witness. As proven in Refs. [21–23], the former can also be phrased in terms of a discrimination game of suitable ensembles. Hence, we detail here how to translate from one formulation to the other. Note that the distributed sampling related to one set of states $\{\rho_x\}$ ensures as much incompatibility as the discrimination of a *different* set of states $\{\sigma_b^y\}$, reflecting the fact that the translation is nontrivial and the tasks are intrinsically different.

Recall that our witnesses are given by

$$\sum_{b,x,y} \mu_{bxy} \mathrm{Tr}(\rho_x M_{b|y}) \geqslant \mathrm{Tr}(B), \tag{D1}$$

where the sum runs over $b = 1, \dots, o$, $x = 1, \dots, |\mathcal{S}|$, and $y = 1, \dots, m$. Applying the reasoning depicted in Theorem 1 of [21], we find that a violation of (D1) provides the advantage the set $\mathcal{M} = \{M_{b|y}\}_{b,y}$ presents over compatible sets when discriminating states from the subensembles $\mathcal{E}_1 = (\{\sigma_b^1\}, p^1(b))_b, \dots, \mathcal{E}_m = (\{\sigma_b^m\}, p^m(b))_b$ given by

$$p^y(b)\sigma_b^y = \alpha\left(\sum_x \lambda_{bxy}\rho_x + \nu\mathbb{I}\right), \tag{D2}$$

where $\nu = \sum_{b,y} \max_x |\lambda_{bxy}|$ and $\alpha = (\sum_{bxy} \lambda_{bxy} + om\nu/d)^{-1}$.

Returning to the example of the witness $W_{\rm QC}$ in Appendix B, measurements that provide its maximum violation can optimally discriminate the ensembles given by

$$\sigma_1^1 = \frac{\mathbb{I} - 0.0551(X+Y)}{2}, \quad p^1(1) = 0.2711,$$

$$\sigma_2^1 = \frac{\mathbb{I} + 0.0551(X-Y)}{2}, \quad p^1(2) = 0.2289,$$

$$\sigma_1^2 = \frac{\mathbb{I}}{2}, \quad p^2(1) = 0.2711,$$

$$\sigma_2^2 = \frac{\mathbb{I} + 0.1306Y}{2}, \quad p^2(2) = 0.2289.$$

APPENDIX E: DEPOLARIZING CHANNELS AND RESTRICTED MEASUREMENT COMPATIBILITY

In the case where Alice and Bob share a perfect quantum channel $\Lambda : \rho \mapsto \rho$, distributed sampling is a trivial task, since Alice can send Bob her input state ρ_x. In Fig. 3, we consider the case where both players exchange imperfect quantum communication, represented by a depolarizing quantum channel given by $D_t : \mathrm{Herm}(\mathbb{C}^2) \to \mathrm{Herm}(\mathbb{C}^2)$, $\rho \mapsto t\rho + (1-t)\mathbb{I}/2$, for a fixed transmittance rate $t = 1 - \eta \in [0, 1]$. Thus, the parameter η marks the amount of white noise acquired in the communication. Also, the depolarizing channel is self-adjoint, meaning that $\mathrm{Tr}[D_t(\rho)M_b] = \mathrm{Tr}[\rho D_t(M_b)]$, for any state ρ and measurement element M_b.

We now calculate the critical parameter t that makes D_t replaceable by a classical channel, in the distributed sampling context. This will be the threshold for our quantum communication detection, and the white-noise robustness of non-steering-breaking of the identity channel. Our main tool to answer this question is a variation of SDP (B2) in which we fix the noise to be completely random (white noise), $\widetilde{M}_{b_y} = \mathrm{Tr}(M_{b|y})\mathbb{I}/d$, where $\{M_{b|y}\}$ is the underlying measurement that yielded the behavior. At the obtained critical transmittance rate $t^* = 1 - \eta^*$, the transmitted information is classical enough to camouflage any quantumness from the channel. The quantity η^* is then defined to be the behavior's *white-noise robustness of non–CC realizability*, denoted $R^w_{\mathrm{NCC}}n(P)$.

Let $\mathcal{H} = \mathbb{C}^2$ and $\mathcal{M}$ be the set of qubit measurements associated with the Pauli observables σ_x, σ_y, σ_z. For any $\mathcal{S} = \{\rho_x\}_x$, we denote $D_t(\mathcal{S}) \equiv \{D_t(\rho_x)\}_x$, and similarly for $D_t(\mathcal{M})$.

If $\mathcal{S}_1 = \{|0\rangle\langle 0|, |+\rangle\langle +|\}$, where

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \tag{E1}$$

then by SDP (B2) we see that the behavior generated by depolarized states $(D_{t_1^*}(\mathcal{S}_1), \mathcal{M})$ is CC realizable at the critical parameter $t_1^* = 0.9449$. Similarly, for $\mathcal{S}_2 = \{|0\rangle\langle 0|, |+\rangle\langle +|, |r\rangle\langle r|\}$, where

$$|r\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} \tag{E2}$$

is an eigenstate of σ_y, we obtain $t_2^* = 0.8165 \approx \sqrt{2/3}$. For $\mathcal{S}_3 = \{|0\rangle\langle 0|, |+\rangle\langle +|, |r\rangle\langle r|, \mathbb{I}/2\}$ we achieve the critical parameter $t_3^* = 0.5774 \approx 1/\sqrt{3}$. We note that the set $\mathcal{S}_3$ spans the qubit Hermitian operators space, and the critical value $t_3^* = 1/\sqrt{3}$ can also be obtained analytically [29].

From the point of view of measurement incompatibility of a restricted set of states, defined in the text, we can interpret these t^* values as the critical parameters for which set $\mathcal{M}$ becomes compatible with each of the sets $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. Since $\mathcal{S}_3$ spans $\mathrm{Herm}(\mathbb{C}^2)$, we have that $D_{t_3^*}(\mathcal{M})$ is compatible in the standard sense. By Theorem 2, this in turn implies that the behavior generated by $(D_{t_3^*}(\mathcal{S}), \mathcal{M})$ is CC realizable for any set of states $\mathcal{S}$, hence $D_{t_3^*}$ does not offer any advantage over classical channels.

The general qudit depolarizing channel D_t is known to be incompatibility-breaking for qudit projective measurements if and only if $t \leqslant t_d^{\mathrm{proj}} := \frac{1}{d-1}(-1 + \sum_{k=1}^d \frac{1}{k})$ [8,14–16,30]. Thus the qubit critical transmittance is 1/2. This implies that if $t \leqslant t_d^{\mathrm{proj}}$, the set of states ρ_x received by Alice lies in $\mathrm{Herm}(\mathbb{C}^d)$, and Bob performs projective measurements, then no quantum communication can be certified. On the other hand, if $t > t_d^{\mathrm{proj}}$ and the set of states ρ_x received by Alice spans $\mathrm{Herm}(\mathbb{C}^d)$, then any incompatible set of projective measurements in $\mathrm{Herm}(\mathbb{C}^d)$ implemented by Bob certifies quantum communication.

For general POVMs, by noting that the local hidden variable model in Ref. [30] can be transformed into a local hidden state model (“steering model”) and using the connection between joint measurability established in Refs. [15] and [16], we can show that the qudit depolarizing channel D_t is incompatibility-breaking for all measurements if $t \leqslant t_d^{\mathrm{all}} := \frac{(3d-1)(d-1)^{d-1}}{(d-1)d^d}$. Hence when $t \leqslant t_d^{\mathrm{all}}$ and the set of states $\{\rho_x\}_x$ lies in $\mathrm{Herm}(\mathbb{C}^{\mathrm{d}})$, all behaviors admit a CC realization, regardless of the measurements performed by Bob.

For the case where the number of measurements performed by Bob is a finite $n \in \mathbb{N}$, it follows from Theorem 2 that a channel can be used to certify QC if and only if it is n-incompatibility-breaking [14], i.e., it breaks the incompatibility of any n measurements. For this situation, there exists a systematic numerical method with a sequence of algorithms that converge to the exact critical value for the depolarizing channel to be n-incompatibility-breaking [31]. This numerical approach provides upper and lower bounds for the n-incompatibility-breaking critical value of D_t in finitely many steps.

We also remark that D_t is entanglement-breaking if and only if $t \leqslant \frac{1}{d+1}$ [20]. Also, a channel is entanglement-breaking if and only if it admits a measure-and-prepare realization, that is, it can be described as $\rho \mapsto \sum_b \mathrm{Tr}(\rho N_b)\sigma_b$, for some measurement $\mathbf{N}$ and states $\{\sigma_b\}_b$. This description provides a clear recipe for a CC realization of distributed sampling that simulates these channels.

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