Decoherence and visibility enhancement in multipath interference

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We theoretically analyze the observations reported in a four-path quantum interference experiment via multiple-beam Ramsey interference [Phys. Rev. Lett. **86**, 559 (2001)]. In this experiment, a selective scattering of photons from just one interfering path causes decoherence. However, contrary to expectations, there is an increase in the contrast of the interference pattern, demonstrating that path selective decoherence can not only lead to a decrease but, under certain conditions, lead to an *increase* of the fringe contrast. Here we explain this seemingly counterintuitive effect based on a model for a multipath interference, with four to six slits, in the presence of decoherence. The effect of the environment is modeled via a coupling to a bath of harmonic oscillators. When decoherence is introduced in one of the multiple paths, an enhancement in fringe contrast is seen under certain conditions. A similar effect is shown to appear if instead of path-selective decoherence, a selective path detector is introduced. Our analysis points to the fact that while traditional fringe visibility captures the wave nature in the two-path case, it can fail in multipath situations. We explain the enhancement of fringe visibility and also show that *quantum coherence* based on the l_1 norm of coherence, in contrast to traditional visibility, remains a good quantifier of wave nature, even in such situations. The enhancement of fringe contrast in the presence of environmental decoherence underscores the limitations of traditional visibility as a good measure for wave nature in quantifying complementarity and also makes it an unlikely candidate for quantifying decoherence. Our analysis could lead to better insight in ways to quantify decoherence in multipath interference and in studies that seek to exploit quantum superpositions and quantum coherence for quantum information applications.

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I. INTRODUCTION

Quantum interference lies at the heart of quantum phenomena and is a subject of continual and intense interest, fascination, and study. Feynman *et al.* placed the double-slit interference experiment at the very core of quantum mechanics and famously declared that it contains "the only mystery" of quantum mechanics [1]. While the double-slit experiment for classical light is satisfactorily understood in terms of the wave nature of light, interference experiments in quantum systems continue to throw up conceptual challenges just as advances in technology push frontiers to implement hitherto impossible interference experiments with progressively larger quantum systems like cold atoms, large molecules, and Bose-Einstein condensates [2–8]. Larger interfering quantum objects are also vulnerable to decoherence, which is believed to wash out quantum behavior and drive the transition from quantum to classical systems [9–11]. Germane to the debate surrounding the double-slit experiment in quantum mechanics is Bohr's *Principle of Complementarity* [12] and discussions regarding the consequences of "which-way" detection on the interference pattern. The observation of an interference pattern is a signature of the wave nature of the quanton while the acquisition of which-way information marks its particle nature, both of which are considered complementary or mutually exclusive. In other words, the complete ignorance of which-way information gives a fringe pattern with perfect visibility while the presence of full which-way information washes out the fringes. In an intermediate regime where one obtains *incomplete* which-way information, interference fringes persist with *reduced* visibility. This was established in a pioneering result by Wootters and Zurek [13], following which Greenberger and Yasin [14] and Englert [15,16] established complementarity relationships between measures which quantitatively estimate the which-path information, and the visibility, which measures the fringe contrast. Subsequent studies involved the inclusion of which-way detectors in the interference experiment and the concept of the quantum eraser [17–20] and complementarity relations in two-path interference were also successfully validated by experiments [21–25]. Central to all investigations involving duality relations has been the traditional notion of visibility as a measure of the wave nature given by

$$\mathcal{V} = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}, \tag{1}$$

where $I_{\max}$ and $I_{\min}$ are the maximum and minimum of the interference pattern (intensity distribution) on the screen. Following the double-slit studies, it was natural to explore analogous forms of interferometric duality when there is a multipath interference by the quanton. Some significant studies in this direction were made [26–37], particularly by

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Dürr [26], where the inequality relations characterizing wave and particle properties in two-beam interferometers were generalized to multibeam interferometers. Many earlier studies of multipath interference were based on *predictability*, i.e., predicting which path the particle might have followed based on the asymmetry of the beam, rather than *distinguishability*, which involves using a path-detecting device to probe which path the particle followed. In an interesting study of multipath interference involving path detection, Jakob and Bergou [28] represented distinguishability in terms of predictability and a measure of the degree of entanglement between the particle paths and the path-detecting device (concurrence), introducing a new way of looking at duality, in terms of *three* measures: predictability, concurrence, and visibility [29]. This study involved a newly defined generalized visibility. For the particular case of three-slit interference, Siddiqui and Qureshi have recently derived a new duality relation which uses the conventional fringe visibility (1) [30]. However, in spite of these results, it is increasingly doubted whether the traditional notion of visibility is really compatible with the intuitive idea of complementarity, except in the case of two-slit interference [38–41]. Our analysis of Mei and Weitz's experiment [38,39] in this paper will also highlight the inadequacy of visibility as a satisfactory measure of wave nature, as understood conventionally.

The notion of coherence intimately captures the wave aspect of both classical light and quantum systems. Recently, a measure for quantum coherence was defined in the framework of quantum information theory by Baumgratz *et al.* [42]. This was used in defining a normalized quantum coherence measure by Bera *et al.* [31] to derive a wave-particle duality relation for arbitrary multipath quantum interference phenomena. Simply stated, the normalized coherence, also known as the l_1 norm of coherence, is defined as the sum of the absolute values of the off-diagonal elements of the density matrix describing the quantum system:

$$\mathcal{C} = \frac{1}{n-1}\sum_{i\neq j}|\langle i|\rho|j\rangle|, \tag{2}$$

where n is the dimensionality of the Hilbert space and the value of $\mathcal{C}$ always lies between 0 and 1. This measure is clearly basis dependent and $\rho_{ij} = \langle i|\rho|j\rangle$ are the matrix elements of the density operator of the system, in the basis formed by the set of n orthogonal states, which correspond to the quanton passing through the n different slits. Using the above measure, Bera *et al.* [31] stated the Bohr complementarity principle for multipath quantum interference as a duality relation between *quantum coherence* (in contrast to conventional visibility) and path distinguishability, whose measure was described in a particular fashion via unambiguous quantum state discrimination (UQSD) [43,44]. Bagan *et al.* [32] took an alternate approach to path distinguishability based on minimum error state discrimination and derived a different duality relation using the coherence given by (2) as a measure of wave nature. However, it was subsequently pointed out that this duality relation was not tight, and a similar looking, but tight, duality relation was derived by Qureshi and Siddiqui [33]. In another approach, the problem of wave-particle duality was formulated in terms of "discrimination games" by Bagan *et al.* [34], and a tight duality relation was obtained, again in terms of coherence. The results of Bera *et al.* [31], Qureshi and Siddiqui [33], and Bagan *et al.* [34] firmly established that quantum coherence and path distinguishability are truly complementary in nature, thus making quantum coherence superior to the notion of traditional visibility in a satisfactory understanding of wave particle duality. It is worthwhile to point out here that the two-slit interference remains an exceptional situation where the traditional visibility is the same as quantum coherence. Following the work of Bera *et al.*, Paul and Qureshi presented a method of experimentally measuring quantum coherence in a multislit interference experiment [45], putting it on a firmer footing as a true experimentally measurable signature of wave nature. Recently, Venugopalan *et al.* [46] analyzed the multislit interference experiment with which-way detectors in the presence of environmental coupling and obtained an expression for quantum coherence as a function of the parameters of the environment, thus providing a way to quantify decoherence through a measurement of quantum coherence. While all these studies seem to undermine the role of traditional visibility vis-à-vis quantum coherence, there is no denying the fact that traditional visibility remains an intuitive and directly observable and measurable sign of the underlying wave phenomenon in a wide variety of experiments. Indeed, for a two-slit interference, a decrease in the Michelson contrast (traditional visibility) is a certain sign of which-path information or the presence of decoherence. In view of this, it does seem difficult to discard traditional visibility completely in favor of coherence for situations involving more that two paths.

In the following, we theoretically analyze the observations reported in a four-path quantum interference experiment via multiple-beam Ramsey interference [38,39]. In this experiment, a selective scattering of photons from just one interfering path causes decoherence, revealing the which-path information, demonstrating that path-selective decoherence can not only lead to a decrease but, under certain conditions, lead to an *increase* of the fringe contrast. We analyze the experiment and explain this seemingly counterintuitive effect based on a model for a multipath interference in the presence of decoherence. We show that an enhancement of fringe visibility with path-selective decoherence and π phase in one path can be seen where the results are particularly interesting for three and four paths. However, we find that with a further increase in the number of interfering paths, the effect is not that significant. Our analysis reaffirms the fact that while traditional fringe visibility captures the intuitive idea of complementarity in the two-path case (where an increase in which-path information implies a decrease of fringe visibility for pure states), it can fail in multipath situations and under the particular conditions of Refs. [38,39], it can show unexpected and counterintuitive behavior. We correlate the effect to the recently introduced l_1 norm of quantum coherence which can be experimentally measured and show that coherence, as defined by the l_1 norm, in contrast to traditional visibility, captures complementarity and remains a good quantifier of wave nature. Further, we show that even in situations where visibility could increase with increasing decoherence (or which-path information), coherence always decreases and preserves its part in the complementarity relations. Moreover, in such situations, the

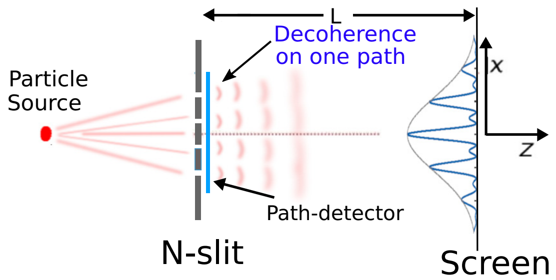

FIG. 1. Schematic diagram of an n-slit interference experiment, with a quantum path detector. The interfering quanton is affected by interaction with an environment only in one path which has an additional phase of π.

amount of decoherence can be estimated only by a measure of the residual coherence and cannot be inferred by changes (increase or decrease) in fringe contrast.

The rest of the paper is organized as follows: In Sec. II, we describe a simple model that captures a multipath interference set up with which-way detectors with the introduction of a π phase and the possibility of which-path information in one of the paths (see Fig. 1), to mimic the experiment of Mei and Weitz [38,39]. We look at the intensity distribution for n paths and analyze the cases for three and four paths. We explain the enhancement of visibility with which-path detection (and π phase) in one path and compare it with the l_1 norm of quantum coherence. In the three-path case, we show that there is initially a decrease in visibility as path information increases followed by an *increase*. We find that complete path information in one path does not lead to an overall enhancement of visibility compared to the case when all paths are indistinguishable. However, *partial* path information can lead to visibility enhancement as will be clear in the following. In the four-path case, we see that there is initially a decrease in visibility as path information increases, following which there is a significant *increase* in the visibility with increasing path information, in tune with the observations seen in the four-path experiment of Mei and Weitz. Similar trends are seen in the case of five and six slits. In Sec. III, we incorporate the effect of a real environment modeled by a bath of harmonic oscillators. This environmental coupling mimics the effect of photon scattering in a single path and we add a π phase to create the conditions of Mei and Weitz's experiment [38,39]. Once again, our analysis shows results predicted in Sec. I. We correlate our results for visibility with quantum coherence and decoherence and discuss its implications in Sec. IV before concluding in Sec. V.

II. MULTISLIT INTERFERENCE WITH WHICH-WAY PATH DETECTORS

A. Visibility

Let us consider a quanton passing through n paths (slits) such that the quantum state corresponding to the nth path is $|\psi_n\rangle$. Then the state of the quanton can be written as

$$|\Psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + c_3|\psi_3\rangle + \cdots + c_n|\psi_n\rangle, \tag{3}$$

where $\sum_i^n |c_i|^2 = 1$ and the states $|\psi_n\rangle$ are orthonormal to each other. Let this quanton be entangled with an ancilla system which can be considered as an effective environment or any other path-detecting device. The states of the ancilla $|\chi_n\rangle$ are assumed to be normalized but are not necessarily orthogonal to each other. The density operator corresponding to the combined state of the quanton and the ancilla after their interaction is the entangled state:

$$\rho = \sum_{j=1}^{n}\sum_{k=1}^{n} c_j c_k^* |\psi_j\rangle\langle\psi_k| e^{i(\theta_j-\theta_k)} \otimes |\chi_j\rangle\langle\chi_k|, \tag{4}$$

where θ_j represents the phase by which the beam coming out of the jth path is shifted. Since we are only interested in the dynamics of the quanton, we will trace over the ancilla states to obtain the density operator of the *reduced* state of the quanton as

$$\rho_{\text{red}} = \sum_{j=1}^{n}\sum_{k=1}^{n} c_j c_k^* |\psi_j\rangle\langle\psi_k| e^{i(\theta_j-\theta_k)} \otimes \langle\chi_k|\chi_j\rangle. \tag{5}$$

Let us assume that after emerging from the n slits, the beams are combined and split into new channels, whose states may be represented by $|\phi_i\rangle$. For simplicity, we assume that all the original beams have equal overlap with a particular output channel and that this overlap is given by α. The probability, I, of finding the quanton in the ith channel is then

$$\begin{aligned} I &= \langle\phi_i|\rho_{\text{red}}|\phi_i\rangle \\ &= |\alpha|^2\left[1 + \sum_{j\neq k} |c_j||c_j|\langle\chi_k|\chi_j\rangle\cos(\theta_j-\theta_k)\right]. \end{aligned} \tag{6}$$

Further, it is also assumed that the phases in the beams can be independently varied. We now look to incorporate the conditions of the experiment of Mei and Weitz [38,39], i.e., introduce the possibility of *which-path information* and a π phase in only one path. Let this be the nth path. This means that the rest of the $n-1$ detector states are identical with each other (i.e., their overlaps $\langle\chi_k|\chi_j\rangle = 1$) but these $n-1$ detector states all have an overlap with the nth detector states given by $\langle\chi_j|\chi_n\rangle = \beta$, where $j = 1, 2, 3, \ldots, n-1$, and the value of β lies between 0 and 1. We can represent the path information of the nth path by a quantity *one-path knowledge* which can be parametrized by $1-\beta$. If $\beta = 0$, it would imply complete one-path knowledge, with value 1. On the other hand, if $\beta = 1$, it implies zero one-path knowledge and all paths are indistinguishable. To mimic the conditions of the experiment, we add a π phase to the nth detector. Further, for simplicity we assume that the phases associated with the n paths are such that they can be written as $\theta_k = k\theta$ and $\theta_n = n\theta + \pi$. The density operator for this reduced state of the quanton can then be represented as the $n\times n$ matrix

$$\rho_{\text{red}} = \frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & 1 & . & . & -\beta \\ 1 & 1 & 1 & 1 & . & . & -\beta \\ 1 & 1 & 1 & 1 & . & . & -\beta \\ 1 & 1 & 1 & 1 & . & . & -\beta \\ . & . & . & . & . & . & -\beta \\ . & . & . & . & . & . & -\beta \\ -\beta & -\beta & -\beta & -\beta & -\beta & -\beta & 1 \end{pmatrix}, \tag{7}$$

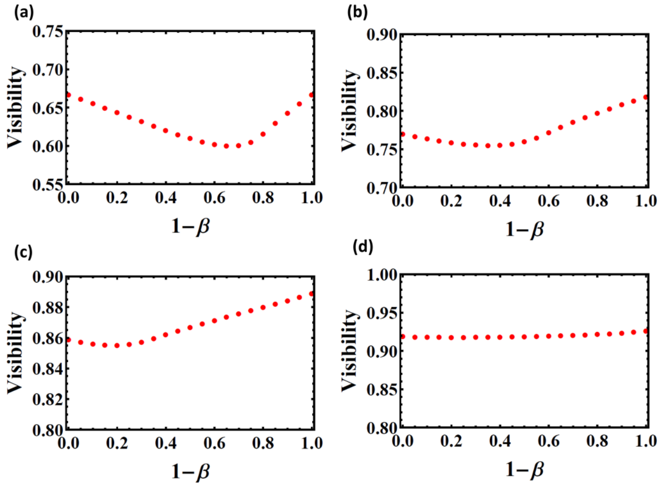

FIG. 2. Variation in visibility with respect to one-path knowledge $(1-\beta)$ for (a) three-path, (b) four-path, (c) five-path, and (d) six-path interference.

and the intensity (6) of the quanton in the ith channel can be shown to be

$$I = |\alpha|^2 \left\{ 1 + \sum_{j=3}^{n} (n+1-j)\cos[(j-2)\theta] - \frac{2\beta}{n}\cos[(n-1)\theta] - \frac{2\beta}{n} \frac{\cos\left[\frac{(n-1)\theta}{2}\right]\sin\left[\frac{(n-2)\theta}{2}\right]}{\sin\left(\frac{\theta}{2}\right)} \right\}. \tag{8}$$

One can see from (8) that the intensity for the two-path case ($n=2$) is $|\alpha|^2(1-\beta\cos\theta)$. As expected, the visibility, (1), $\mathcal{V}=\beta$ for the two-path case. Clearly, for full one-path knowledge ($\beta=0$), the visibility is 0 while for zero one-path knowledge ($\beta=1$) visibility is 1. As mentioned before, the two-path case is straightforward and in tune with all conventional notions of visibility and complementarity. Let us now look at the scenario where there are more than two paths ($n \geqslant 3$). For $n=3$, one can see that intensity (8) is given by

$$I = |\alpha|^2 \left[1 + \frac{2(1-\beta)}{3}\cos(\theta) - \frac{2\beta}{3}\cos(2\theta) \right]. \tag{9}$$

Note that (9) incorporates the possibility of path information in the third path (via β), and this path also has an additional π phase. From (9), the visibility as a function of the detector overlap β (which quantifies the one-path knowledge) is shown in Fig. 2(a). One can see that for both the case of complete path information ($\beta=0$) and no path information ($\beta=1$), the visibility, (1), for the $n=3$ case is $\frac{2}{3}$. Thus, in contrast to the observation of enhancement in visibility seen in the four-path experiment of Mei and Weitz [38,39], it appears as though the three-path case shows no such enhancement in visibility when one compares the case of complete path indistinguishability ($\beta=1$) and that of complete path information ($\beta=0$) with the π phase in the third path. However, note from Fig. 2(a) that if we look at an *intermediate* range of values for β (say, between 0.2 and 0.5), which corresponds to *partial path information*, the $n=3$ case shows a regime in which the visibility *enhances* with increasing path information [see Fig. 2(a)].

In an earlier work, Bimonte and Musto constructed a three-beam example to demonstrate that one could have a peculiar situation where increasing path information could lead to an increase in fringe visibility [27]. However, their analysis fell short of demonstrating that [47]. The preceding analysis completes the task left unfinished in the analysis of Bimonte and Musto. Clearly, this effect is counterintuitive to the notion of complementarity as understood in duality relations between conventional visibility and path information. Unlike in the two-path case, in the three-path case, it is clear that under conditions described above, the traditional visibility does not capture the intuitive idea of wave-particle duality. Since the preceding analysis shows that even in a three-beam interference experiment visibility can increase with increasing path information, it also implies that the duality relation for three-slit interference, formulated by Siddiqui and Qureshi [30], will not hold in such special scenarios.

Let us now analyze the observations reported in the four-path quantum interference experiment of Mei and Wietz [38,39]. In this experiment, a selective scattering of photons from just one interfering path can lead to an *increase* in the fringe contrast if an additional phase of π is included in the path. Although it is not possible to extract path information in this experiment, it is obvious that the scattered photons do carry path information about the particle. Our ancilla states can effectively mimic the role of photons. From (8), it can be shown that the intensity of the quanton for $n=4$ is

$$I = |\alpha|^2 \left[1 + \frac{2-\beta}{2}\cos(\theta) + \frac{1-\beta}{2}\cos(2\theta) - \frac{\beta}{2}\cos(3\theta) \right]. \tag{10}$$

From (10), the visibility as a function of the detector overlap β (which quantifies the path information) is shown in Fig. 2(b). Note that in the $n=4$ case, when there is complete path information ($\beta=0$) of the fourth path, the visibility *is more* than that of the case when all four paths are indistinguishable ($\beta=1$). This is in agreement with the experimental observations of Mei and Weitz. Also note that in the regime with partial path information, one can see regions where there is a definite enhancement of visibility with increasing path information. This trend is more dramatic in the $n=4$ case compared to the $n=3$ case. It is worth pointing out here that the inclusion of the π phase is very crucial to this effect since what would have been a perfect destructive interference of the fourth path with the other three is now being replaced by a more incoherent contribution of this path to the interference pattern. It is easy to verify from (6) that the inclusion of a π phase for $n=2$, i.e., the *two-path* case, will have no such effect. Figures 2(c) and 2(d) show similar plots for the cases $n=5$ and $n=6$, respectively, and one can see that an increase in the knowledge of path information is accompanied by increase in the visibility of the interference fringes. Upon further increasing the number of paths, we see that the variation in the fringe visibility with β (the degree of path information of the nth path) reduces and saturates to a value close to 1 as n becomes very large. This is akin to a diffraction grating where larger number of slits leads to a sharper fringe

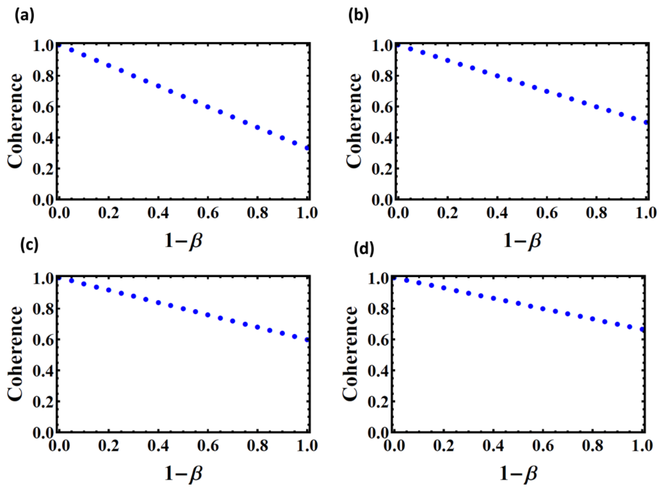

FIG. 3. Variation in coherence with respect to one-path knowledge $(1-\beta)$ for (a) three-path, (b) four-path, (c) five-path, and (d) six-path interference.

pattern or higher visibility. Clearly, as n becomes very large, the path information of just one path has a negligible effect on the visibility as more paths coherently contribute to it. Thus, one can conclude that the counterintuitive effect seen in the experiment of Mei and Weitz is significant only for a small number of interfering paths and is most appreciable in the $n = 4$ case (chosen for their experiment) and becomes negligible as we increase n beyond six paths. Even so, the enhancement in fringe contrast with increasing path information for $n = 3, 4, 5, 6$ is in conflict with the spirit of Bohr's complementary and points to the limitations of traditional visibility, (1).

B. Quantum coherence

Quantum entanglement and quantum coherence both intimately capture the intrinsic and uniquely quantum nature of systems. A measure for quantum coherence, called the l_1 *norm* recently defined in the framework of quantum information theory by Baumgratz *et al.* [42] has evoked a lot of interest and it has been argued that this l_1 *norm* is a better quantifier of wave nature than traditional visibility. The l_1 *norm* was generalized to a normalized quantum coherence measure by Bera *et al.* [31] to derive a wave-particle duality relation for arbitrary multipath quantum interference phenomena. It can be seen that the coherence, (2), for the state of the quanton, (7), is

$$C(\rho_{\rm red}) = \frac{2\beta}{n} + \frac{n-2}{n}. \tag{11}$$

We can see that for $n = 2$, $C(\rho_{\rm red}) = \beta$, and for $n = 3$ and $n = 4$, it is $\frac{2\beta}{3} + \frac{1}{3}$ and $\frac{\beta+1}{2}$, respectively. Note that the quantum coherence via l_1 norm is *the same as the traditional visibility* for the two-path case. In Fig. 3, we show the variation of coherence with the degree of one-path knowledge $(1-\beta)$ for the cases of three, four, five, and six paths. Clearly, unlike the visibility, the coherence measured by the l_1 norm *decreases* monotonically with increase in the path information, in the spirit of Bohr's complementarity. Note also that unlike visibility, the measure of coherence is unaffected by the presence or absence of the π phase in one path. Consequently, all the duality relations based on coherence [31,33,34] will be respected, irrespective of their definition of distinguishability. Equation (11) also shows that when the number of paths is large ($n \to \infty$), the degree of which path information, β, is of little significance and coherence approaches its maximum value of 1.

For illustration of the preceding sections, we used a very simple model to incorporate the effect of the n detectors by assuming that the overlaps of $n-1$ detectors is such that $\langle\chi_k|\chi_j\rangle = 1$ and that these $n-1$ detectors states all have an overlap with the nth detector state (which will provide the which-path information) given by $\langle\chi_j|\chi_n\rangle = \beta$, where $j = 1, 2, 3, \ldots, n-1$. In Mei and Weitz's experiment [38,39], done via multiple-beam Ramsey interference with cesium atoms, while there are no explicit detector states entangled with the quanton's state of four interfering paths, the degree of which-path information is presumably revealed by a scattering of photons from the fourth path. In the experiment, the four interfering paths are represented by the magnetic sublevels of the $F = 3$ hyperfine component of the cesium electronic ground state. Upon irradiating the atoms with resonant optical beams, a coherent superposition of four magnetic ground-state sublevels is created which simulates the four-path interference. The coherent superposition is probed via atom interferometry techniques to obtain the fringe pattern. Between the Ramsey pulses, one selected path is coupled to the environment by a sequence of microwave pulses resulting in path-selective decoherence due to scattering of photons which can reveal the which-path information. The initial motivation for Mei and Weitz's experiment was to study decoherence in the multipath interference scenario and they suggest from their counterintuitive observations that in the case of multiple-beam interference the Michelson fringe contrast (visibility) is not sufficient to quantify decoherence.

In the next section, we look at the multipath interference of Mei and Weitz with path selective decoherence and π phase by incorporating the effect of a realistic environment and analyze its effect on visibility and quantum coherence.

III. MULTISLIT INTERFERENCE WITH DECOHERENCE

Let us now assume that as the particle (quanton) comes out of the n slits and travels to the screen, it is affected by weak interaction with some kind of environment. We describe this environment as a reservoir of noninteracting quantum oscillators, each of which interacts with the particle. The Hamiltonian governing the particle can then be represented as

$$H = \frac{p^2}{2m} + \sum_j \frac{P_j^2}{2M_j} + \frac{1}{2}M_j\omega_j^2\left(X_j - \frac{g_j x}{M_j\omega_j^2}\right)^2, \tag{12}$$

where x, p are the position and momentum operators of the particle (quanton), m is its mass, X_j, P_j are position and momentum operators, M_j is the mass of the jth harmonic oscillator of frequency ω_j comprising the environment, and g_j are the respective coupling strengths. Such a system has been studied in great detail [48–53] and the solution for this

Hamiltonian is governed by the master equation given by

$$\frac{\partial\rho(x,x',t)}{\partial t}=\left\{\frac{-\hbar}{2im}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x'^2}\right)-\gamma(x-x')\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)-\frac{D}{4\hbar^2}(x-x')^2\right\}, \tag{13}$$

where $\rho(x,x',t)$ is the reduced density matrix of the quanton after tracing out the environmental degrees of freedom, γ is the Langevin friction coefficient, $D=2m\gamma k_BT$ can be interpreted as a diffusion coefficient, and T is the temperature of the harmonic oscillator heat-bath [48–50].

Let us assume that the state $|\psi_k\rangle$ which emerges from the kth slit is a Gaussian wave packet localized at the location of the kth slit, with a width equal to the width of the slit:

$$\langle x|\psi_k\rangle=\frac{1}{(\pi/2)^{1/4}\sqrt{\epsilon}}e^{-(x-k\ell)^2/\epsilon^2}, \tag{14}$$

where ℓ is the distance between the centers of two neighboring slits and ϵ is their approximate width. Assuming that which-path detectors present at each slit interact with the particle (quanton), the combined wave function of the particle and the detectors, as it emerges from the n slits, is given by

$$\langle x|\Psi\rangle=\frac{1}{(\pi/2)^{1/4}\sqrt{\epsilon}}\sum_{k=1}^{n}c_k e^{-(x-k\ell)^2/\epsilon^2}|\chi_k\rangle, \tag{15}$$

The density matrix corresponding to this can be written as

$$\rho_0(x,x')=\frac{1}{\sqrt{\pi/2}\epsilon}\sum_{j,k}c_jc_k^* e^{\frac{-(x-j\ell)^2}{\epsilon^2}}e^{\frac{-(x'-k\ell)^2}{\epsilon^2}}|\chi_j\rangle\langle\chi_k|. \tag{16}$$

If one were to look only at the particle, it amounts to tracing over the states of the path detectors, giving us the reduced density matrix for the quanton

$$\rho(x,x',0)=\frac{1}{\sqrt{\pi/2}\epsilon}\sum_{j,k}c_jc_k^* e^{\frac{-(x-j\ell)^2}{\epsilon^2}}e^{\frac{-(x'-k\ell)^2}{\epsilon^2}}\langle\chi_k|\chi_j\rangle. \tag{17}$$

This is the density operator of the particle at time $t=0$ as it emerges from the n slit. Its decoherent dynamics will be governed by Eq. (13). Under the assumption that the effect of the environment is so weak that dissipative time scales are much longer than the time the particle takes to reach the screen (t) after traveling a distance L, the diagonal component of the density operator $\rho(x,x',t)$ representing the probability density of the particle hitting the screen at a point x can be given by

$$\rho(x,x,t)=\frac{1}{\sqrt{\pi\alpha/2}}\Bigg[\sum_{j=1}^{n}|c_j|^2e^{-2\epsilon^2(x-jl)^2/(\lambda L/\pi)^2}+\sum_{j\neq k}|c_j||c_k||\langle\chi_k|\chi_j\rangle|e^{\frac{-\epsilon^2\{(x-j\ell)^2+(x-k\ell)^2\}}{(\lambda L/\pi)^2}}e^{\frac{-D(j-k)^2\ell^2t}{12\hbar^2}}\times\cos\left\{\frac{2\pi\ell(k-j)\left(x-\ell\frac{k+j}{2}\right)}{\lambda L}+\theta_k-\theta_j\right\}\Bigg], \tag{18}$$

where $\alpha=\epsilon^2+\frac{\hbar^2(1-e^{-2\gamma t})^2}{\epsilon^2m^2\gamma^2}+\frac{D[4\gamma t+4e^{-2\gamma t}-e^{-4\gamma t}-3]}{8m^2\gamma^3}$. Staying within the Fraunhoffer limits, the width of the slits (ϵ) is much much small in comparison to the fringe width ($\lambda L/l$), so the narrow width of the Gaussians in (17) will become very large in (18). So for all the practical purposes, the values of all the Gaussians, at any point x on the screen, is almost the same, and thus independent of j,k, and the expression for $\rho(x,x,t)$ can be approximated as

$$\rho(x,x,t)\approx\frac{e^{-2\epsilon^2x^2/(\lambda L/\pi)^2}}{\sqrt{\pi\alpha/2}}\left\{1+\sum_{j\neq k}|c_j||c_k||\langle\chi_k|\chi_j\rangle|e^{\frac{-D(j-k)^2\ell^2t}{12\hbar^2}}\cos\left[\frac{2\pi\ell(k-j)x}{\lambda L}+\theta_k-\theta_j\right]\right\}. \tag{19}$$

Equation (19) is a recently reported result by Venugopalan *et al.* [46] for an n-slit interference in the presence of environmentally induced decoherence. Note that in the limit of coupling with the environment going to zero, which is the regime when decoherence is completely absent, i.e., when $\gamma\to 0$, $D\to 0$, (19) reduces to Eq. (16) of Ref. [45], which is the known result for a multipath interference. Equation (19) shows that the decohering n-slit interference pattern is also built up from all possible *two-slit inteference terms* which is also the case when there is no coupling to the environment [45]. The environmental coupling has had the effect of modifying these pairwise contributions and this effect of decoherence is neatly condensed into the exponential factor $e^{-D(j-k)^2\ell^2t/12\hbar^2}$ in Eq. (19), which multiplies the cosine term giving rise to the interference. Notice that the exponential decay term in (19) whose argument is $-D(j-k)^2\ell^2t/12\hbar^2$ cannot be pulled out of the summation, as it depends on j,k.

It may also be noted that for each pair of slits, there will be a characteristic decoherence time, $\tau_d^{(jk)}=12\hbar^2/D(j-k)^2\ell^2$, and for n slits there will be a total of $n(n-1)/2$ timescales which will collectively contribute in the summation, leading to the degradation of the n-slit pattern. For the simplest case of two slits, there will be only one timescale, $\tau_d=12\hbar^2/D\ell^2$.

Now let us see how we can apply the general result (19) to the situation explored in the experiment of Mei and Weitz. As discussed in detail in the previous sections, in the n-path interference, there is an additional π phase added to the nth path and there is also the possibility of which-path information from decoherence via scattering of photons for this path. In the present model, decoherence will be instrumental in revealing the which-path information, and we assume that the effect of the environment applies only to the nth path. Note that this means we set all detector overlaps to 1, making them irrelevant for now. With these conditions taken into account, the probability density of the particle hitting the screen at a point x can be written as

$$\rho(x,x,t) \approx \frac{e^{-2\epsilon^2 x^2/(\lambda L/\pi)^2}}{\sqrt{\pi\alpha/2}} \Bigg\{ 1 + \sum_{j\neq k}^{n-1} |c_j||c_k| \times \cos\left[\frac{2\pi\ell(k-j)x}{\lambda L} + \theta_k - \theta_j\right] - 2\sum_{j}^{n-1} |c_j||c_n| \; e^{\frac{-D(n-j)^2\ell^2 t}{12\hbar^2}} \times \cos\left[\frac{2\pi\ell(n-j)x}{\lambda L} + \theta_n - \theta_j\right]\Bigg\}. \tag{20}$$

Equation (20) captures the multipath interference pattern as the position probability distribution on the screen when the nth path is impacted by environmental influence and has an additional π phase. Clearly, with the passage of time, one can see progressive decoherence happening from the contribution of the last term, which eventually kills the interference between the nth path and all the others, increasingly revealing which-path information. When $t \to \infty$, there is complete path information of the nth path. With $n = 4$, one can explore (20) to explain the results of Mei and Weitz, which we will do in the next section.

Next, we ask if it is possible to extract the amount of quantum coherence as defined in (2) from the interference (20) and have a way of measuring it in a real experiment. It has been shown by Paul and Qureshi [45] that it is possible to measure quantum coherence in a real experiment for a multislit interference with which-way detectors whose path distinguishability is tunable [45] between two modes: (a) one that makes all paths completely *indistinguishable* and (b) one that makes all paths fully *distinguishable*. If the two cases, (a) and (b), are denoted by $\parallel$ and $\perp$, respectively, Paul and Qureshi provide the following protocol: First, the intensity at a primary maximum $I^{\parallel}_{\max}$ is measured when the n paths are indistinguishable, i.e., $|\chi_i\rangle$s are all identical and parallel. Next, the path detector is switched to the mode (b) where all the n paths are fully distinguishable, and the intensity $I^{\perp}_{\max}$ is measured at the same location on the screen as before. Coherence of the incoming quanton can then be measured as [45]

$$\mathcal{C}_{\rm expt} = \frac{1}{n-1}\frac{I^{\parallel}_{\max} - I^{\perp}_{\max}}{I^{\perp}_{\max}}. \tag{21}$$

While this protocol works for most cases of n-path interference, in an experiment like that of Mei and Weitz, notice that there is a deliberate arrangement of the π phase in one particular path such that the knowledge of path information for this path, instead of degrading the interference pattern (as normally expected), leads to a sharper contrast, as has been elaborated in the previous sections. In such situations, the protocol of Paul and Qureshi fails to provide an experimental measure of coherence. However, we have seen from discussions in the previous sections that coherence (2) does indeed decrease with increasing path information, even while the interference pattern becomes sharper. How then, can we measure the coherence? The way around this is a recent result by Qureshi [54] where a modified and unconventional protocol is proposed which allows for the measurement of quantum coherence in situations such as the experiment of Mei and Weitz where there could be constraints on phases. Coherence can be measured in such cases by opening only one pair of paths at a time (while blocking the rest), measuring conventional visibility for each pair at a time, and finally taking an average over all $n(n-1)/2$ pairs of paths. Essentially, this casts quantum coherence as a sum of visibilities of interference from individual pairs of path. We can show that coherence (2) of the pattern (20) can then be measured in terms of pairwise visibility using this protocol [54] as

$$C(\rho(x,x,t)) = \frac{2}{n(n-1)}\sum_{j\neq k}^{n-1}\frac{2|c_j||c_k|}{|c_j|^2+|c_k|^2} + \frac{2}{n(n-1)}\sum_{j}^{n-1}\frac{2|c_j||c_n|}{|c_j|^2+|c_n|^2}e^{\frac{-D(n-j)^2\ell^2 t}{12\hbar^2}}. \tag{22}$$

We recall, once again, that the two-path scenario remains the exceptional case where quantum coherence is the same as traditional visibility. In the next section, we discuss and analyze the results (20) and (22).

IV. RESULTS AND DISCUSSION

In the previous sections, we have analyzed the multislit interference set up with which-way detectors for situations where there is a possibility of which-way path detection for one particular path either through the detector or by means of path-selective decoherence. Additionally, in tune with the experiment of Mei and Weitz, the selected path has an extra phase of π. In the preceding section, incorporating a model for the environment, we have obtained an expression for the intensity distribution of the quanton (20) and the quantum coherence (22) in terms of the parameters of the environment. Let us now try to analyze how the visibility and coherence of the system evolves with time for the case of an initial maximally coherent state, (7), i.e., when $c_i = \frac{1}{\sqrt{n}}$. The intensity distribution on the screen of the quanton is then

$$\rho(x,x,t) \approx \frac{e^{-2\epsilon^2 x^2/(\lambda L/\pi)^2}}{\sqrt{\pi\alpha/2}}\left\{1 + \frac{1}{n}\sum_{j\neq k}^{n-1}\cos\left[\frac{2\pi\ell(k-j)x}{\lambda L} + \theta_k - \theta_j\right] - \frac{2}{n}\sum_{j}^{n-1} e^{\frac{-D(n-j)^2\ell^2 t}{12\hbar^2}}\cos\left[\frac{2\pi\ell(n-j)x}{\lambda L} + \theta_n - \theta_j\right]\right\}, \tag{23}$$

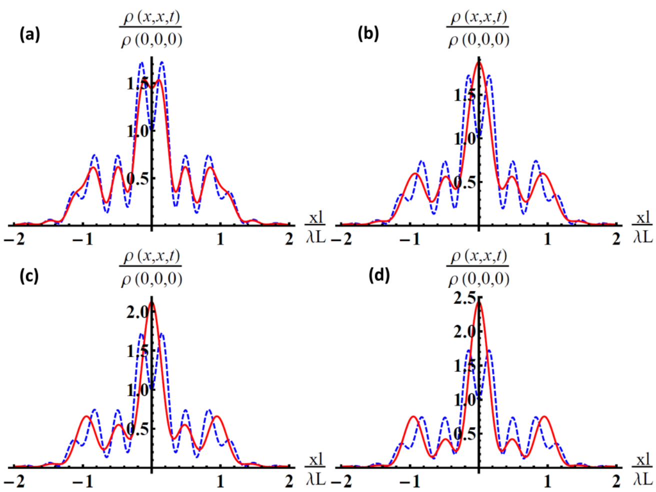

FIG. 4. Plots of $\rho(x,x,t)/\rho(0,0,0)$ at the screen, for four-path interference, showing how decoherence progressively affects interference. The following parameters used are adapted from the interference experiment on ultracold neon atoms [2]: $m = 3.349\times10^{-26}$ Kg, $T = 2.5$ mK, $\lambda = 0.018\ \mu$m, $\ell = 6\ \mu$m, $L = 37$ mm. The dashed blue plots are $t/\tau_d = 0$ while the red plots are for (a) $t/\tau_d = 1/12$, (b) $t/\tau_d = 1/4$, (c) $t/\tau_d = 1/2$, and (d) $t/\tau_d = 2$ where $\tau_d = 12\hbar^2/D\ell^2$. We can observe that with increase in time the there is an enhancement of the visibility.

and the quantum coherence is

$$C(\rho(x,x,t)) = \frac{n-2}{n} + \frac{2}{n(n-1)}\sum_{j}^{n-1} e^{\frac{-D(n-j)^2\ell^2 t}{12\hbar^2}}. \quad (24)$$

In Fig. 4, we plot the intensity pattern of the quanton with the passage of time, for the $n = 4$ case. Clearly, the pattern shows an enhancement of fringe contrast with increasing decoherence, as observed in the experiment of Mei and Weitz. The variation of visibility and coherence with time for $n = 3, 4, 5$, and 6 are plotted in Fig. 5.

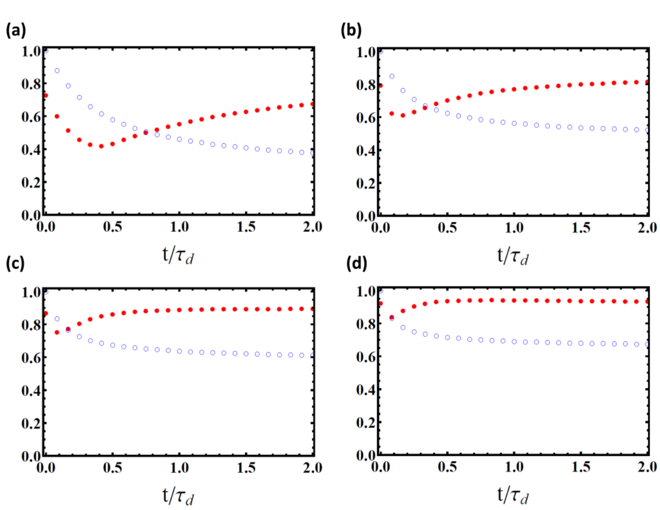

FIG. 5. Variation of visibility (in red filled circles) and coherence (in blue empty circles) with respect to scaled time t/τ_d for (a) three-path, (b) four-path, (c) five-path, and (d) six-path interference.

Let us now examine the initial visibility and quantum coherence of the quanton before decoherence starts in the nth path. Note that since all the which-way detectors are assumed to be identical, and the quanton is initially in a pure state, at $t = 0$ the initial quantum coherence of the quanton for all cases has the maximum value of 1. However, the visibility at $t = 0$ does not have its maximum value of 1. This is because the presence of an additional π phase in one of the paths affects the visibility and keeps it at a value that is less than 1 at $t = 0$. Thus, unlike coherence, even all identical which way detectors and a pure maximally coherent state does not ensure maximum visibility. Coherence is unaffected by the presence of the π phase. With the passage of time, there is path-selective decoherence in one of the paths and this affects the overall interference pattern and is also the mechanism for progressively revealing the path information. At $t = 0$ there is no decoherence and at $t \gg \frac{12\hbar^2}{Dl^2}$, i.e., $t \gg \tau_d$, we have a situation where there is full decoherence. For the n-path case, the coherence (24) of the quanton decays and saturates to the value of $\frac{n-2}{n}$ for large times when the decoherence has played its role to the fullest. Note that this was also seen in (11).

Let us now look at the specific case of the three-path interference [see Fig. 5(a)]. We can see that with the passage of time, while the coherence of the quanton decays from a maximum value of 1 to minimum value of $1/3$, the visibility shows a rather unusual variation. For short times, the plot shows a decrease in visibility but as time progresses there is an increase in the visibility. Thus, even though there is progressive decoherence over time, and hence increasing path information, there is a definite observable *increase* in the visibility which clearly goes against the spirit of what is understood conventionally as Bohr's complementary principle. Note that this was also seen in our analysis in Sec. II [see Fig. 2(a)]. Similarly, for four paths, one can see from Fig. 5(b) that with the passage of time while the coherence of the quanton decays from a maximum value of 1 to a minimum value of $1/2$, the visibility of the interference fringes *increases*, as seen in the experiment of Mei and Weitz. Also, for this case, it is clear from our analysis that the visibility for full path information (large t) is *greater* than in the case of no path information ($t = 0$). A similar trend for visibility and coherence can be seen for five- and six-path cases [see Figs. 5(c) and 5(d)], though they are less dramatic as compared to the $n = 4$ case. Note that with the increase in the number of paths, the effect of decoherence in one of the paths on the visibility and coherence decreases and both saturate to a value practically close to the maximum value 1. This can be understood from the fact that with large number of available paths the effect of decoherence in just one of the paths will have a negligible impact on the overall interference pattern, and a large number of interfering paths contribute to the enhancement of visibility and coherence.

V. CONCLUSIONS

To summarize, we have analyzed multipath quantum interference where which-path information for one particular path is revealed by which-way detectors with an additional π phase in the selected path. We have also analyzed multipath quantum interference where a path-selective decoherence plus an additional π phase in the selected path, is introduced, as in the experiment of Mei and Weitz. Through this analysis, we explain the observations of an experiment by Mei and Weitz where path-selective decoherence can lead to not only a decrease but an *increase* of the fringe contrast. The effect of the environment is modeled via a coupling to a bath of harmonic oscillators. When the effect of the environment (decoherence) is introduced in one of the four paths, an enhancement in fringe contrast is seen under the conditions of Refs. [38,39]. We show that a similar effect can be seen in the general case of multipath interference with path-selective decoherence and illustrate it additionally for three, five, and six slits. Our results suggest that traditional fringe visibility, while capturing well the intuitive idea of complementarity in the two-path case, fails in multipath situations such as those seen in the experiment of Mei and Weitz, raising serious doubts on its role as an indicator of coherence and wave nature. We also probe the effect seen in Refs. [38,39] using the recently introduced l_1 norm of *quantum coherence* which can be experimentally measured by a recently reported protocol [54]. We explain the enhancement of fringe visibility and show that even in situations where visibility could increase with increasing decoherence, or increasing which-path information, coherence always decreases and preserves its role as a quantifier of wave nature. The observation of increased fringe contrast in Mei and Weitz's experiments is for a four-path interference. A similar effect can be seen in certain regimes for three-path interference, and also for five and six paths. In Refs. [38,39], it is suggested that the four-path experiment could be extended toward an increased number of interfering paths or quantum systems of larger size. Our observation here is that the visibility enhancement effect is interesting only for the cases of a few paths and increasing the number of paths beyond these numbers tones down this dramatic effect. However, interference with few paths with larger quantum systems would certainly be interesting. In Refs. [38,39], it is suggested that traditional visibility fails as a signature for quantifying decoherence. Indeed, the enhancement of fringe contrast in the presence of environmental decoherence highlights the limitations of traditional visibility as a good measure for wave nature in complementarity and makes it an unlikely candidate for quantifying decoherence. Our results affirm the fact that the amount of decoherence can only be estimated by a measure of the residual coherence and cannot be inferred by changes in fringe contrast. This normalized l_1 norm of coherence can now be considered as a true and legitimate measure of the wave aspect of the quanton in a multipath experiment. Further, this quantum coherence can be measured experimentally in an n-path interference experiment by taking the average of *traditional visibility* for all available pairs of two-path interference by blocking the rest of the $n-2$ paths [54]. Our results and analysis could lead to better insight and understanding of wave-particle duality and complementarity as well as visibility and quantum coherence in multipath interference, and they will be of significance in studies that seek to exploit quantum superpositions and quantum coherence for quantum information applications.

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