Three-loop radiative corrections to the 1s Lamb shift in hydrogen

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The three-loop contributions to the Lamb shift of the 1s state in the hydrogen atom in order $\alpha^3(Z\alpha)^5m$, responsible for the second largest contribution to the uncertainty budget, have been known only partially. Here we estimate the remaining three-loop terms of that order. The total three-loop result in order $\alpha^3(Z\alpha)^5m$ for the 1s Lamb shift is found to be $-(3.3 \pm 10.5)(\alpha/\pi)^3(Z\alpha)^5m$, which contributes $-0.11(34)$ kHz to the energy of the ground state in the hydrogen atom and $-3(11)$ kHz in the helium ion. To verify our approach we have also estimated the known contributions in order $\alpha(Z\alpha)^5m$ and $\alpha^2(Z\alpha)^5m$. The obtained estimates are perfectly consistent with the well established results, found previously by other authors.

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I. INTRODUCTION

Energy levels in hydrogen atom are described by quantum electrodynamics (QED) with inclusion of nuclear-finite-size and nuclear-structure effects. The QED contributions are the largest ones and often the accuracy of their calculation determines the overall theoretical uncertainty. In particular, the computational QED accuracy of the complete theory of the Lamb shift in the ground state of the hydrogen atom is limited by our knowledge of the higher-order QED contributions within the external-field approximation as well as of some radiative-recoil corrections [1,2]. (The most recent CODATA's compilation is [1]; however, it does not contain any new theoretical input compared to the previous one [2], while only the earlier paper contains all the important references.) An accurate calculation of the 1s Lamb shift in hydrogen is crucial for a comparison with experiment in order both to test bound-state QED (see, e.g., Ref. [3]) and to extract the accurate values of the fundamental constants such as the Rydberg constant (see [1] for details).

The theoretical uncertainty is dominated by the two-loop and three-loop pure QED contributions in the external-field approximation (see [4] for more detail). While the two-loop contributions will be considered in detail elsewhere (see [4]), we study here the related external-field contributions of the order $\alpha^3(Z\alpha)^5m$ (the three-loop ones), which are known only partially [5] and we estimate the rest of them. The current estimation of the uncalculated terms in this order produces the second largest contribution to the uncertainty of the Lamb shift of the ns states in the hydrogen atom [1,2].

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The energy levels in the external-field approximation are parametrized up to the three-loop level as

$$\Delta E_L(ns) = \frac{\alpha(Z\alpha)^4m}{\pi n^3}\left[F^{(1)} + \frac{\alpha}{\pi}F^{(2)} + \left(\frac{\alpha}{\pi}\right)^2 F^{(3)}\right], \quad (1)$$

where

$$(Z\alpha)^4\,F^{(1)}(nl) = \sum_{kp} A_{kp}(Z\alpha)^k \ln^p\frac{1}{(Z\alpha)^2}, \quad (2)$$

$$(Z\alpha)^4\,F^{(2)}(nl) = \sum_{kp} B_{kp}(Z\alpha)^k \ln^p\frac{1}{(Z\alpha)^2}, \quad (3)$$

and

$$(Z\alpha)^4\,F^{(3)}(nl) = \sum_{kp} C_{kp}(Z\alpha)^k \ln^p\frac{1}{(Z\alpha)^2}. \quad (4)$$

Sometimes, an additional factor of $(m_r/m)^3$ is kept. For the sake of simplicity we ignore that reduced-mass factor here. This factor is important for low-order contributions, but not significant for higher-order terms, which we consider in this paper. Our concern is a calculation of C_{50} for an ns state. As a test calculation, we also obtain the results on A_{50} and B_{50}, the values of which are well known.

A total contribution of a certain order may involve several gauge invariant sets of Feynman diagrams. In this case we use the notation of (2), (3), and (4), but introduce subscripts or superscripts to identify which individual set of contributions is considered.

II. CONTRIBUTIONS IN ORDER $\alpha^k(Z\alpha)^5m$

In the scattering problem the diagrams with a different number of external photon "legs" are of different order in $Z\alpha$. In the case of the bound-state QED the diagrams with

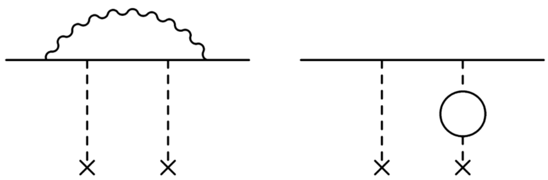

FIG. 1. Characteristic diagrams for the contact potential, responsible for $A_{50}^{\rm SE}(ns)$ (left) and $A_{50}^{\rm VP}(ns)$ (right).

a different number of the Coulomb photons may be of the same order. The contributions in order $\alpha^k(Z\alpha)^5m$ can be presented as a product of a contact term with the exchange by two Coulomb photons inside it [see, e.g., Fig. 1 for the $\alpha(Z\alpha)^5m$ contribution] and the squared value of the nonrelativistic Coulomb wave function at origin $|\Psi(r=0)|^2$ (see, e.g., Refs. [6,7]).

The "skeleton" two-photon-exchange "contribution" to the Lamb shift in the hydrogen atom (see Fig. 2) is of the form

$$\Delta E_{\rm sk}(ns) = \frac{(Z\alpha)^5m}{n^3}\left[-\frac{16m^3}{\pi}\int_0^\infty \frac{dq}{q^4}\right], \qquad (5)$$

where $q=|\mathbf{q}|$ and $\mathbf{q}$ is the three-dimensional momentum transfer through the left photon line in Fig. 2. The momentum transfer through the right one is $-\mathbf{q}$.

We put above the quotation marks on the "contribution", because the expression is divergent at low q and requires subtractions. However, in the case of k-loop radiative corrections, a dimensionless factor of

$$\left(\frac{\alpha}{\pi}\right)^k R^{(k)}(q^2)$$

is introduced and that leads to a substitution in the q integral

$$\int\frac{dq}{q^4}\to\int\frac{dq}{q^4}R(q^2). \qquad (6)$$

After the substitution the q integration is softened. The structure of the subtractions for radiative corrections is different from a correction to a correction, and we discuss below the subtractions in the context of the involved individual radiative corrections.

III. METHOD OF THE ESTIMATION OF TWO-PHOTON-EXCHANGE CONTRIBUTIONS

The purpose of this paper is to estimate three-loop two-photon-exchange contributions, for which we develop a method based on the evaluation of the asymptotics of the related q integral. While explaining the method, it is helpful to be more specific and give an example. Let us consider the contribution of the one-loop vacuum-polarization (VP) insertion into a Coulomb line (see the right graph in Fig. 1)

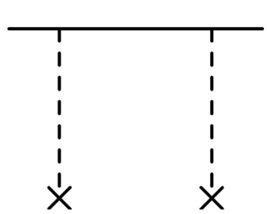

FIG. 2. Skeleton two-Coulomb-exchange diagram.

$$R_{\rm VP}(q^2) = 2\,q^2 I_{\rm VP}(q^2),$$

where (see, e.g., [6])

$$I_{\rm VP}(q^2) = \int_0^1 dv\,\frac{v^2(1-v^2/3)}{4m^2+(1-v^2)q^2}. \qquad (7)$$

To properly estimate the related q integral [cf. (6)], we have to consider two asymptotics: the low-q one at $q\ll 2m$ and the high-q one at $q\gg 2m$. Once we find them, we split the integral into two parts

$$\int_0^\infty dq\ldots = \int_0^{2m} dq\ldots + \int_{2m}^\infty dq\ldots$$

and estimate the low-q and high-q terms using the related asymptotics. (From the point of view of dispersion relations it is more natural to split the integral by $2m$ rather than by m [cf. (7)].) In the case of the VP correction, the asymptotics are

$$R_{\rm VP}(q^2)\simeq\begin{cases}\frac{2}{15}\frac{q^2}{m^2}-\frac{1}{70}\left(\frac{q^2}{m^2}\right)^2 & \text{at } q\ll 2m,\\ \frac{2}{3}\ln\frac{q^2}{m^2}-\frac{10}{9} & \text{at } q\gg 2m.\end{cases} \qquad (8)$$

Before starting any practical discussion, we have to return to the question of the convergence of the integral at low q and a need for related subtractions. The integral in (6) is divergent for $R_{\rm VP}(q^2)$ at low q. Such a divergence happens when the radiative correction contributes not only in order $\alpha^k(Z\alpha)^5m$, but also in order $\alpha^k(Z\alpha)^4m$. The total VP contribution does contain such a term. In the case of the presence of the $(Z\alpha)^4$ contribution, the expression for the $(Z\alpha)^5$ term should contain a subtraction. Usually the integral takes the form

$$\int\frac{dq}{q^4}R(q^2)\to\int\frac{dq}{q^4}[R(q^2)-q^2R'(0)]. \qquad (9)$$

The exception is the one-loop self-energy (SE) (see below in the Appendix). In the case of the one-loop SE the subtraction is more complicated because the leading term of the q^2 expansion of $R(q^2)$ contains $\ln(m^2/q^2)$.

In QED, power counting at $q\gg 2m$ for the skeleton Feynman diagrams and the related radiative correction to them is the same and the radiative corrections can have a logarithmic enhancement only [cf. (8)]. However, the subtraction changes the situation drastically. The asymptotics of the subtracted integrand are now

$$R_{\rm VP}(q^2)-q^2R'_{\rm VP}(0)\simeq\begin{cases}-\frac{1}{70}\left(\frac{q^2}{m^2}\right)^2 & \text{at } q\ll 2m,\\ -\frac{2}{15}\frac{q^2}{m^2} & \text{at } q\gg 2m.\end{cases} \qquad (10)$$

The subtracted radiative corrections at high q are enhanced by a factor q^2/m^2 compared both to the skeleton ($=1$) and the unsubtracted radiative correction [see (8)]. In other words, the high-q asymptotics [of the subtracted radiative correction $R(q^2)-q^2R'_{\rm VP}(0)$] is determined by the subtraction, which in its turn is determined by the q^2 term of the low-q asymptotics of the radiative correction $R(q^2)$.

Such an asymptotic behavior is rather standard for the radiative corrections. The asymptotics of the complete radiative corrections to the integrand are determined only by the low-q asymptotics of the (unsubtracted) radiative corrections. (In the case of the individual gauge-invariant sets the subtraction is not always present; however, the leading high-q terms for such individual contributions are subleading for the total one.)

As explained, the subtractions are due to the $(Z\alpha)^4$ terms. Let us consider the one-loop VP case more accurately. The leading VP contribution is of the order $\alpha(Z\alpha)^4 m$ with

$$A_{40}^{\rm VP} = -\frac{4}{15}.$$

The $A_{50}^{\rm VP}$ contribution according to our finding is

$$\Delta E_{\rm VP}(ns) = -\frac{16\alpha(Z\alpha)^5 m}{\pi^2 n^3}\, m^3 \int \frac{dq}{q^4} \times[R_{\rm VP}(q^2) - q^2 R'_{\rm VP}(0)]. \quad (11)$$

The high-q part of the contribution therefore reads

$$\Delta E^{>}_{\rm VP}(ns) = -\frac{16\alpha(Z\alpha)^5 m}{\pi^2 n^3}\, m^3 \int_{2m}^{\infty} \frac{dq}{q^4}\left[-\frac{2q^2}{15m^2}\right] \quad (12)$$

or

$$A_{50}^{\rm VP\,>}(ns) = -\frac{4}{\pi}\, A_{40}^{\rm VP}(ns). \quad (13)$$

The relation between the high-q part of $A_{50}(ns)$ and the coefficient $A_{40}(ns)$ is a generic one since both are expressed in terms of the value of $I_{\rm VP}(0)$. A similar relation takes place also for the two-loop and three-loop contributions:

$$B^{>}_{50}(ns) = -\frac{4}{\pi}\, B_{40}(ns),$$
$$C^{>}_{50}(ns) = -\frac{4}{\pi}\, C_{40}(ns). \quad (14)$$

The bottom equation is directly used for our evaluation of C_{50} below, while the upper one is suitable for the test of the method (see the Appendix).

We do not discuss here the low-q part of the VP contribution since its calculation somewhat differs from the one used below for $C_{50}(ns)$.

IV. THREE-LOOP CONTRIBUTIONS: ESTIMATION OF C_{50}

The state-of-the-art theory of the three-loop contributions to the Lamb shift of the ns state is the following. The leading term is of the order $\alpha^3(Z\alpha)^4 m$ and it is known [8–10]. The contribution in order $\alpha^3(Z\alpha)^5 m$ has been found only partially [5]: the pure VP contributions were found as well as the contributions of the two-photon-exchange diagrams with one radiative photon and all possible VP insertions. The result of [5] reads

$$C_{50}^{\rm known}(ns) = 8.331(2). \quad (15)$$

The leading higher-order three-loop logarithmic corrections, such as C_{63}, C_{62}, have been also considered [11].

Following the consideration above we first have to identify the diagrams which contribute to $C_{40}(ns)$. The three-loop self-energy (SE3) ones are plotted in Fig. 3, while those for the three-loop vacuum polarization (VP3) are depicted in Fig. 4. We are not interested in the asymptotics of the VP3 diagrams here since all the related three-loop VP contributions to C_{50}, reducible and irreducible, have been already found in [5]. We focus here on the SE3 ones.

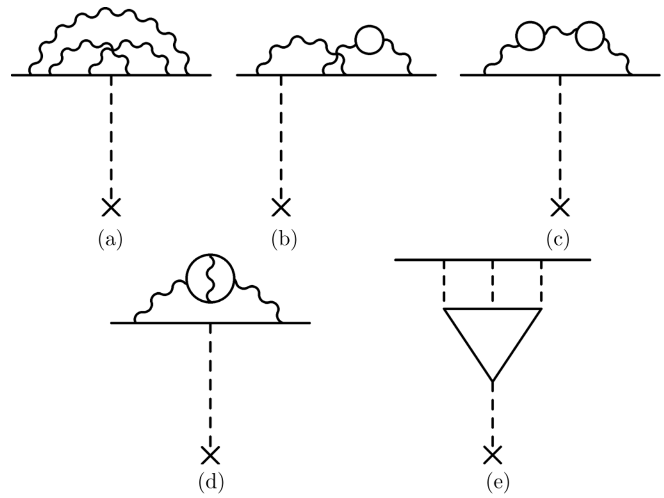

FIG. 3. Characteristic diagrams for the three-loop self-energy (SE3) contribution to $C_{40}(ns)$.

To find the high-q contribution we rely on (14). Meanwhile, we note that $C_{40}(ns)$ has been known [9,10]

$$C_{40}^{\rm SE3}(ns) \simeq 1.868\,12\ldots \quad (16)$$

for the total SE3 contribution, but the result for the slope of the Dirac form factor [10] is not given for the individual sets of the diagrams. The diagrams c and d in Fig. 3 relate to the sets already calculated in [5]. Therefore, we have to separate their contributions from the others.

The related $\alpha^3(Z\alpha)^5 m$ contributions are calculated in [5] by using the two-photon-exchange technique and the integrands are presented there with the explicit subtractions [see Eqs. (57) and (66) in [5]; cf. [12]]. We have calculated the involved subtraction terms and restored their contributions to $C_{40}(ns)$. Our result is

$$C_{40}^{\rm SE3:known}(ns) \simeq 0.9715\ldots. \quad (17)$$

Its sign is consistent with the sign of the related part of $C_{50}^{\rm known}(ns)$ [5].

Combining (16) and (17) we obtain

$$C_{40}^{\rm SE3:uknown}(ns) \simeq 0.896\,60\ldots \quad (18)$$

and therefore for the "unknown" contributions we arrive at the estimation

$$C_{50}^{\rm unknown,>}(ns) = -1.14, \quad (19)$$

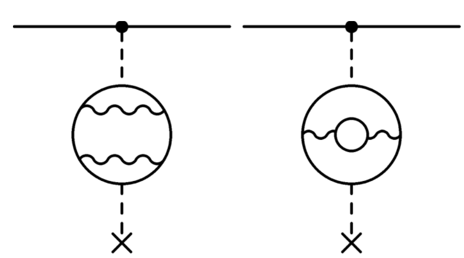

FIG. 4. Characteristic diagrams for $C_{40}^{\rm VP3}(ns)$: two types of irreducible diagrams. [Only irreducible pure VP graphs contribute to $C_{40}^{\rm VP3}(ns)$.]

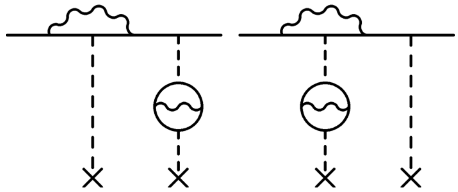

FIG. 5. Characteristic diagrams for $q^4\ln(m^2/q^2)$ asymptotic term for $C_{50}^{>}(ns)$ from SE × VP2. (Only irreducible VP2 graphs contribute.)

where we estimate the uncertainty of the calculation as 50%.

To estimate $C_{50}^{<}(ns)$ we note that, in contrast to the VP contributions, the SE ones often have logarithmic terms at low q, such as $\ln(m^2/q^2)$. It is not easy to find the complete q^4 asymptotics of the radiative corrections at low q, but it is possible to find its leading logarithmic term. To do that we note that the low-q logarithms appear in a manner similar to the leading logarithm at the one-loop self-energy (cf. [13]).

We find the leading logarithmic term in order q^4 for the three-loop radiative-correction factor $R(q^2)$. It is a single-logarithm one. A one-loop radiative-correction factor of order q^4 is linear in logarithm, while in the case of two loops there is a quadratic term (see below in the Appendix). One can expect that the three-loop radiative corrections to the skeleton diagrams have a cube of the logarithm. That is correct; however, a cubic term appears in order q^6, while the q^4 term is softer than that in the case of two loops (cf. [11]) and has only a logarithm.

The characteristic diagrams for the leading q^4 logarithmic contributions are presented in Fig. 5. For the sake of simplicity the plotted diagrams in Fig. 5 contain some VP insertions; however, that is not necessary. (The plotted diagrams are with a combination of the one-loop SE and the two-loop VP.) Pure SE diagrams are also important (see below). There are two types of the $q^4\ln(m^2/q^2)$ contributions. The left plot illustrates a combination of a contact potential (i.e., a potential which behaves at low q as $\propto$ const, which is $\propto q^2\times$ skeleton) and a logarithmic divergence in the electric form factor [$\propto q^2\ln(m^2/q^2)\times$ skeleton], which together produce the logarithmic $\propto q^4\ln(m^2/q^2)$ asymptotics. The right plot also involves the contact term and overall integral for the one-loop radiative correction. The contact term does not depend on momentum and therefore does not change the logarithmically divergent (at $q^2=0$) integration in the vertex. It also produces $q^4\ln(m^2/q^2)$.

The key detail of the calculation of the logarithmic contribution to $R(q^2)$ is the presence of a two-loop contact potential. The latter is the one responsible for $B_{40}(ns)$ [i.e., for the contribution in order $\alpha^2(Z\alpha)^4m$]. We present the characteristic diagrams for such a contact potential in Fig. 6. Note that of these three contributions only one (due to the last graph) corresponds to the corrections evaluated in [5]. The contributions to the asymptotics of the three-loop radiative corrections due to the two-loop contact potentials from the two first types of the diagrams correspond to the "unknown" part of $C_{50}(ns)$.

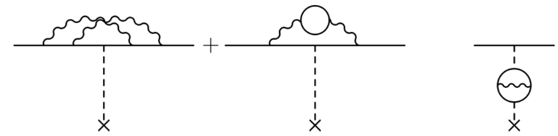

FIG. 6. Characteristic diagrams for the contact potential, responsible for $B_{40}^{\rm SE2}(ns)$ (the left and middle diagrams) and $B_{40}^{\rm VP2}(ns)$ (the right diagram).

The related low-q part of $C_{50}(ns)$ is

$$\begin{aligned} C_{50}^{<}(ns) &= -\frac{32}{\pi}\int_0^{2m}\frac{dq}{m}\left(\frac{2}{3}\ln\frac{q^2}{4m^2}\right)\left(-\frac{B_{40}}{4}\right)\\ &= -\frac{64}{3\pi}B_{40}(ns), \end{aligned} \tag{20}$$

where the set-by-set contributions to B_{40} are (see, e.g., [1,6])

$$\begin{aligned} B_{40}^{\rm pureSE}(ns) &= -\frac{9}{4}\zeta(3)+\frac{3\pi^2}{2}\ln 2-\frac{85\pi^2}{216}-\frac{163}{72}\\ &\simeq 1.409,\\ B_{40}^{\rm SE[VP]}(ns) &= \frac{5\pi^2}{216}-\frac{7}{81}\\ &\simeq 0.1420, \end{aligned} \tag{21}$$

and the notation is clear from Fig. 6.

As mentioned, we ignore $B_{40}^{\rm pureVP}$, since it is related to the known part of C_{50}, while here we are after the unknown one, for which we obtain

$$C_{50}^{\rm unknown,<}(ns) = -10.53, \tag{22}$$

with the uncertainty of 100% because we deal only with the logarithmic part of the asymptotics.

Combining the high-q [from (19)] and low-q parts of the unknown C_{50} contributions we find their estimation as

$$C_{50}^{\rm unknown}(ns) = -11.7(10.5) \tag{23}$$

or

$$C_{50}^{\rm total}(ns) = -3.3(10.5), \tag{24}$$

which is the eventual result of our calculations for the $\alpha^3(Z\alpha)^5m$ contributions to the Lamb shift of the ns state in hydrogen. That should be compared to an estimation of $C_{50}^{\rm total}(ns)=\pm30$ [1].

As a test of principles of our method, we estimate in the Appendix the values of the one-loop and two-loop two-photon-exchange contributions $A_{50}(ns)$ and $B_{50}(ns)$, which are well established. The achieved estimations are in good agreement with the known results [14] and [15] (see Table I).

TABLE I. Estimation of the total contributions to A_{50}, B_{50}, C_{50} for an ns state in hydrogenlike atoms and its comparison with known results.

Coefficient	Estimation	Result
A_{50}	9.4(5.3)	9.62 [14]
B_{50}	−19(18)	−21.6 [15]
C_{50}	−3.3(10.5)	

A success in the estimation of the well established coefficients $A_{50}(ns), B_{50}(ns)$ confirms that our estimation of the central value and the uncertainty of $C_{50}(ns)$ above is reasonable. To conclude our consideration of the two-photon-exchange contributions and the estimation of $A_{50}(ns), B_{50}(ns), C_{50}(ns)$, we note a few important properties of the deduced estimations.

(i) In all the cases we investigate in this section and in the Appendix the low-q and high-q asymptotics have the same sign. That is an important practical condition for the estimation of an integral through the use of its asymptotics.

(ii) The larger contributions come from the parts which contain logarithmic terms in the asymptotics. For the B_{50}, C_{50} coefficients that is the low-q part, while in the case of $A_{50}(ns)$ both asymptotics have logarithms and their contributions are comparable.

(iii) The largest value for the coefficients is that for $B_{50}(ns)$, only which has a *double* logarithmic asymptotics.

(iv) In the case of the one-loop and two-loop calculations the SE diagrams and, specifically, the "pure SE" ones (without any closed electron loops) dominate. That is consistent with the previous comments, because only they have logarithmic asymptotics in the case of one and two loops. In the case of three loops the diagrams with the VP loops (see Fig. 5) also produce the logarithmic q^4 terms and the pure SE contributions do not dominate.

All that tells us that the three-loop coefficients are organized somewhat differently than the one-loop and two-loop ones. The pure SE contributions cannot strongly dominate and the overall three-loop coefficient is smaller than the two-loop coefficients. The estimation ± 30 [1] is too conservative.

The SE and VP contributions often have different signs. As far as only one contribution dominates [as the pure SE one for $A_{50}(ns)$ and $B_{50}(ns)$], that does not really matter. In the case of three loops a massive cancellation between a "pure SE" contribution and various VP contributions is possible. That makes the partial result on the diagrams with the VP insertions [5] valuable for our estimation, eliminating an uncertainty which should be introduced otherwise.

V. CONCLUSIONS

Concluding, we performed above an estimation of the leading unknown three-loop contribution to the Lamb shift of the ns state in light hydrogenlike atoms. Our result for the total $\alpha^3(Z\alpha)^5m$ correction reads [4]

$$\Delta E_L^{(50)}(ns) = (-3.3 \pm 10.5)\left(\frac{\alpha}{\pi}\right)^3 \frac{(Z\alpha)^5 m}{n^3}. \tag{25}$$

That should be compared to the previous rough estimation [1], which has three times larger uncertainty. In contrast to that estimation, ours is based on an approximate calculation.

In the frequency units the contribution found is $-0.11(34)$ kHz for the 1s Lamb shift in hydrogen and deuterium and $-3(11)$ kHz for the 1s state in the He^+ ion.

The leading higher-order logarithmic correction is also known. In contrast to [1,2], it was established that $C_{63} = 0$ [11], while the leading logarithmic term has only a double logarithmic factor [11]

$$\Delta E_L^{(62)}(ns) \simeq -0.36\left(\frac{\alpha}{\pi}\right)^3 \frac{(Z\alpha)^6 m}{n^3} \ln^2 \frac{1}{(Z\alpha)^2}. \tag{26}$$

In the case of estimation of the uncertainty related to C_{63} in [1,2], the cubic logarithmic term would be comparable to the uncertainty in (25), being 0.2 kHz for hydrogen and 9 kHz for the helium ion. According to [11], where such a cubic term is eliminated, the leading double logarithmic contribution in (26) and the uncertainty due to subleading terms are both essentially below the current uncertainty of the $\alpha^3(Z\alpha)^5m$ in (25) and can be neglected. The double-logarithm term contributes -8 Hz for hydrogen and -0.4 kHz for the helium ion 1s Lamb shift.

The eventual three-loop contribution of order $\alpha^3(Z\alpha)^5m$ (and higher) to the 1s Lamb shift is therefore determined by (25). Together with the marginal higher-order correction, it is $-0.11(34)$ kHz for hydrogen and deuterium and $-4(11)$ kHz for the helium ion. The uncertainty is approximately three times smaller than in the previously published estimation [2].

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APPENDIX: ESTIMATION OF $A_{50}(ns)$ AND $B_{50}(ns)$

Let us start with the estimation of the one-loop contribution in order $\alpha(Z\alpha)^5m$ (cf. [16]). Using the substitution in (6) we find for the self-energy contribution

$$\Delta E(ns) = -\frac{16\alpha(Z\alpha)^5 m}{\pi^2 n^3} m^3 \int \frac{dq}{q^4}\left[R_{\rm SE}(q^2) - R_{\rm SE}^{(2)}(q^2)\right], \tag{A1}$$

and therefore

$$A_{50}^{\rm SE}(ns) = -\frac{16}{\pi} m^3 \int \frac{dq}{q^4}\left[R_{\rm SE}(q^2) - R_{\rm SE}^{(2)}(q^2)\right], \tag{A2}$$

where $R_{\rm SE}^{(2)}(q^2)$ is the q^2 part of the low-q asymptotics of $R_{\rm SE}(q^2)$. It contains a logarithmic term (see below) and cannot be present as $q^2 R'_{\rm SE}(0)$, similar to the VP case in Sec. III.

The radiative-correction factor $R_{\rm SE}(q^2)$ is considered in [17,18] (see also [19]). The related asymptotics are

$$R_{\rm SE}(q^2) - R_{\rm SE}^{(2)}(q^2) \simeq \begin{cases} -\frac{7}{30}\left(\frac{q^2}{m^2}\right)^2 \ln\frac{4m^2}{q^2} & \text{at } q \ll 2m, \\ \frac{q^2}{m^2}\left(-\frac{2}{3}\ln\frac{q^2}{m^2} + \frac{5}{9}\right) & \text{at } q \gg 2m, \end{cases} \tag{A3}$$

where for the q^2 term of the low-q asymptotics of $R_{\rm SE}(q^2)$ we find

$$R_{\rm SE}^{(2)} = -\frac{q^2}{m^2}\left(\frac{2}{3}\ln\frac{m^2}{q^2} + \frac{5}{9}\right).$$

In principle one can find a complete q^4 asymptotics from the expressions in [17,18]; however, to make the accuracy comparable with the calculation of C_{50} in Sec. IV we utilize here only its logarithmic part. We note that the q^2 asymptotics could be also found from the well known results for the electric form factor of the electron, the infrared logarithmic asymptotics of which are related to $A_{41}^{\rm SE}$.

The result for the coefficient of interest is found as

$$\begin{aligned} A_{50}^{\rm SE}(ns) &= A^{>}(ns) + A^{<}(ns) \\ &= \left[\frac{16m}{\pi}\int_{2m}^{\infty}\frac{dq}{q^2}\left(\frac{2}{3}\ln\frac{q^2}{4m^2} - \frac{5}{9} + \frac{4}{3}\ln 2\right)\right] \\ &\quad + \left[\frac{56m}{15\pi}\int_{0}^{2m}\frac{dq}{q^2}\frac{q^2}{m^2}\ln\frac{4m^2}{q^2}\right] \\ &= 4.3(2.1) + 4.8(4.8), \end{aligned} \tag{A4}$$

where we assign the 50% uncertainty for the calculation with the complete q^2 asymptotics and the 100% one for the calculation with the logarithmic q^4 asymptotics. Our final estimation reads

$$A_{50}^{\rm SE}(ns) = 9.1(5.2), \tag{A5}$$

which is in perfect agreement with the actual value of the coefficient [14]

$$A_{50}^{\rm SE}(ns) = \pi\left(\tfrac{139}{32} - 2\ln 2\right) \simeq 9.29. \tag{A6}$$

A similar consideration of the complete $A_{50}(ns)$ coefficients (for both the SE and VP contributions) also leads to agreement between our estimation

$$A_{50}^{\rm SE+VP}(ns) = 9.4(5.3) \tag{A7}$$

and the known analytic result [14]

$$A_{50}^{\rm SE+VP}(ns) = \pi\left(\tfrac{427}{96} - 2\ln 2\right) \simeq 9.62. \tag{A8}$$

Now we have to extend the estimation procedure to $B_{50}(ns)$. The high-momentum contribution is defined in (14). To find it we have to apply the results for the $\alpha^2(Z\alpha)^4m$ contribution, which are well known (see, e.g., [1,6]):

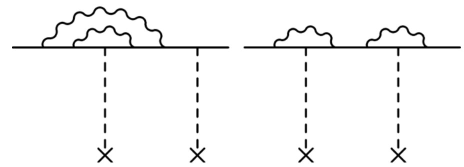

FIG. 7. Characteristic diagrams for $q^4\ln^2(m^2/q^2)$ asymptotics for $B_{50}^{<}(ns)$ from SE2.

$$\begin{aligned} B_{40}^{\rm SE2}(ns) &= \left(-\frac{9}{4}\zeta(3) + \frac{3}{2}\pi^2\ln 2 - \frac{10}{27}\pi^2 - \frac{1523}{648}\right), \\ B_{40}^{\rm VP2}(ns) &= -\frac{82}{81}. \end{aligned} \tag{A9}$$

The estimation for the high-q part of the SE2 contribution is

$$B_{50}^{\rm SE2,>}(ns) \simeq -2.0(1.0)\ . \tag{A10}$$

The leading q^4 term of the low-q asymptotics is the double logarithmic one (see Fig. 7). Having in mind the skeleton expression in (5), the well-known expression for the one-loop form factor, and the expression for the $q^4\ln^2(\lambda/m)$ term of the two-loop vertex (see [13]), we find

$$\begin{aligned} B_{50}^{<}(ns) &= -\frac{16}{\pi n^3}\int_0^{2m}\frac{dq}{m}\left(2\times\frac{1}{2}+1\right)\left(\frac{1}{3}\ln\frac{q^2}{(2m)^2}\right)^2 \\ &\simeq -18(18), \end{aligned} \tag{A11}$$

where for the application of the logarithmic term we assign the uncertainty of 100%.

The final result is

$$B_{50}^{\rm SE}(ns) = -20(18) \tag{A12}$$

to be compared with [15]

$$B_{50}^{\rm SE}(ns) = -24.269(4). \tag{A13}$$

One can also find a complete result

$$B_{50}^{\rm SE+VP}(ns) = -19(18), \tag{A14}$$

which is also in good agreement with the known value [15]

$$B_{50}(ns) = -21.558(4). \tag{A15}$$

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