Sensitivity of the isotope shift to the distribution of nuclear charge density

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It is usually assumed that the field isotope shift (FIS) is completely determined by the change of the averaged squared values of the nuclear charge radius $\langle r^2\rangle$. Relativistic corrections modify the expression for FIS, which is actually described by the change of $\langle r^{2\gamma}\rangle$, where $\gamma=\sqrt{1-Z^2\alpha^2}$. In the present paper we consider corrections to FIS which are due to the nuclear deformation and due to the predicted reduced charge density in the middle of the superheavy nuclei produced by a very strong proton repulsion (hole in the nuclear center). Specifically, we investigate effects which cannot be completely reduced to the change of $\langle r^2\rangle$ or $\langle r^{2\gamma}\rangle$.

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I. INTRODUCTION

Isotope shift (IS) phenomena in heavy atoms are an important way of probing various scenarios in nuclear physics and can aid the search for new physics beyond the standard model. Nuclear theory predicts the existence of long-lived isotopes for elements with $Z\geqslant 104$ (see, e.g., [1,2]), in particular isotopes with a magic neutron number $N=184$. However, producing these neutron-rich isotopes in laboratories by colliding lighter atoms is currently impossible. The Coulomb repulsion for nuclei grows as Z^2; in order to compensate for this with the attractive strong force, the neutron number N must grow faster than Z. Consequently, an isotope from the island of stability with $N=184$ cannot be produced from the collision of a pair of lighter isotopes with smaller N/Z ratios.

In contrast to laboratories, various astrophysical events such as supernovae explosions, neutron stars, and neutron star–black hole/neutron star mergers generate high neutron fluxes and may create environments favorable for the production of neutron-rich heavy elements. For example, a new mechanism of such kind due to the capture of the neutron star material by a primordial black hole has been suggested in [3]. Furthermore, neutron star–neutron star mergers are predicted to generate optimal environments for the production of heavy atoms [4,5].

As a consequence, astrophysical data may be the best place to observe superheavy metastable elements. It is possible that optical lines of elements up to $Z=99$ have already been identified in the spectra of Przybylski's star [6]. These elements include heavy, short-lived isotopes which may be products of the decay of long-lifetime nuclei near the island of stability [7].

IS calculations for superheavy elements (SHEs) can help trace the hypothetical island of stability in existing astrophysical data. It may be possible to predict a spectral line of a neutron-rich isotope ν' based on the experimental spectrum of a neutron-poor isotope ν and calculations of IS $\delta\nu$ as $\nu'=\nu+\delta\nu$. The results can then be used to search for the long-lifetime neutron-rich elements in complicated astrophysical spectra such as that of Przybylski's star.

Spectroscopic measurements of IS may also be relevant to the search for strange matter. Strange nuclei consist of up, down, and strange quarks (see [8], and references therein). A strange-matter nuclei of charge Z would have a very different radius in comparison to any regular isotope. Calculations of IS can be used to predict the effects of this change in radius on atomic spectra.

Calculations of IS allows one to estimate the King-plot nonlinearity of a given element. New long-range forces such as Yukawa-type interactions between electrons and nucleus can lead to nonlinearities in a King plot for a series of isotopes [9]. It is useful to understand other possible sources of nonlinearities in the IS in order to constrain new physics beyond the standard model.

It should be noted that relativistic corrections produce an important difference in the dependence of the field shift on the nuclear radius r. The traditional expression for field shift is known as $F_i\delta\langle r^2\rangle$ where F_i is an electronic structure factor and $\delta\langle r^2\rangle$ is a nuclear parameter. It is usually assumed that electron factor F_i is the same for all isotopes. In fact, it is only true when relativistic effects are sufficiently small to be neglected. An alternative formula allowing separation of nuclear and electron variables in the relativistic case should be written as $\tilde{F}_i\delta\langle r^{2\gamma}\rangle$, where $\gamma=\sqrt{1-Z^2\alpha^2}$, α is the fine-structure constant. The electronic factor $\tilde{F}_i$ is to be calculated. An analytical estimate of $\tilde{F}_i$ has been done in Ref. [10] (see also [11–14]), and relativistic many-body calculations for $Z=102$–109 have been done in Refs. [15–17]. The traditional formula for the field shift $F_i\delta\langle r^2\rangle$ still can be used for neighboring isotopes where change in F_i is small and can be neglected. The formula is useful for finding the change in nuclear root mean square (RMS) radius from the IS measurements.

Due to the relativistic effects in heavy atoms, the field shift of the $p_{1/2}$ orbital is comparable to that of the $s_{1/2}$: The ratio is $\sim(1-\gamma)/(1+\gamma)$ [10]. The $Z\alpha$ expansion gives the ratio $\sim Z^2\alpha^2/4$ but for $Z=137$, $\gamma\approx 0$ and for the superheavy elements the ratio tends to 1. For $j>1/2$ the direct mean-field single-particle field shift is small. However, the mean-field rearrangement effect (the correction to the atomic potential δV due to the perturbation of the s and $p_{1/2}$ orbitals by the

field-shift operator) produces the same dependence of field shift on nuclear radius for all orbitals: $\tilde{F}_i \delta\langle r^{2\gamma}\rangle$.

The difference between the nonrelativistic $\langle r^2\rangle$ and relativistic $\langle r^{2\gamma}\rangle$ expressions may be explained by the different dependence of the nonrelativistic and relativistic wave functions near the origin. Another relativistic effect is due to the variation of the electron density ρ_e inside the nucleus which for the s and $p_{1/2}$ orbitals is approximately presented by the following formula [10]:

$$\rho_e(r) \approx \rho_e(0)\left[1 - \frac{Z^2\alpha^2}{2}\left(\frac{r}{c}\right)^2\right] \tag{1}$$

where c is the nuclear radius. The r-dependent term gives us an additional sensitivity of IS to the nuclear charge distribution beyond the change of $\langle r^2\rangle$.

In this work we study the effect of the change in nuclear charge distribution on the field isotope shift. We consider four types of charge distribution variation: (a) A hole in the origin, where nuclear density is small in the origin and increases to the periphery; (b) nuclear quadrupole deformation; (c) change of the skin thickness; and (d) change in nuclear RMS radius. The questions we try to answer include the following: (a) Can isotope measurements be used to study nuclear structure beyond the change of nuclear RMS radius? (b) What is the best way of using isotope shift calculations to predict the spectra of neutron-rich SHEs with the aim to reach the hypothetical island of stability? (c) Can nuclear deformation lead to nonlinearity of the King plot?

We choose the $E120^+$ ion for numerical analysis. It is sufficiently heavy for the relativistic effects to be pronounced. On the other hand, the ion has relatively simple electron structure (one external electron above closed shells) so that all important points can be illustrated without getting into the trouble of complicated many-body calculations. We use the results of nuclear calculations [18] to get the parameters of nuclear deformation and nuclear RMS radius. We consider only even isotopes because nuclear calculations for them are more reliable. The work [18] considers a range of nuclear models which favor spherical nuclear shape at $Z = 120$ and $N = 172$. We use this spherical nucleus as the starting point in our study.

II. CALCULATIONS

We use an approach similar to the one in Refs. [19,20]. Electron potential V for valence orbitals is found by solving relativistic Hartree-Fock (RHF) equations for a closed-shell core

$$(\hat{H}^{\rm HF} - \epsilon_c)\psi_c = 0, \tag{2}$$

where c numerates states in the core from $1s$ to $7p_{1/2}$ and $7p_{3/2}$. States of valence electron (Brueckner orbitals) are obtained by solving the RHF-like equations for the valence orbitals

$$(\hat{H}^{\rm HF} + \lambda\Sigma^{(2)} - \epsilon_v)\psi_v^{\rm Br} = 0. \tag{3}$$

Here Σ is the correlation potential responsible for core-valence correlations [21], and index "2" indicates second order of the many-body perturbation theory. Σ is defined in such a way that the correlation correction to the energy ϵ_v is given by $\delta\epsilon_v = \langle\psi_v|\Sigma|\psi_v\rangle$ (see, e.g., [21] for details). We calculate Σ *ab initio*, limiting ourselves to the lowest order of the perturbation theory. λ is a scaling parameter introduced to simulate the effect of higher-order correlations. Its value ($\lambda = 0.75$) is chosen to fit the result of all-order calculations of Refs. [19,20].

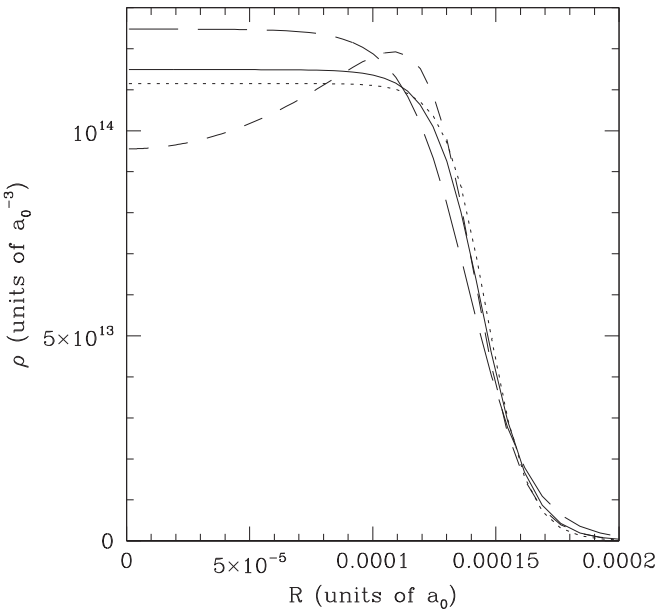

FIG. 1. Variations of nuclear density. Solid line: Fermi distribution (5); dashed line: Modified distribution with a hole in the origin, formula (6) with $k = 0.5$; dotted line: Fermi distribution with reduced skin thickness [parameter t in (5)] by 14.5% to simulate the effect of the hole; long dashed line: Fermi distribution with increased skin thickness by 30.5% to simulate the effect of quadrupole deformation, formula (7) with $\beta = -0.4$.

IS is calculated using the so-called random phase approximation (RPA; see, e.g., [21]), which can be described as linear response of self-consistent atomic field to a small perturbation. In our case the perturbation is the change in nuclear potential δV_N due to a change in nuclear charge distribution. The RPA equations are first solved for the core

$$(\hat{H}^{\rm HF} - \epsilon_c)\delta\psi_c = -(\delta V_N + \delta V_{\rm core}), \tag{4}$$

where $\delta\psi_c$ is the correction to the core orbitals due to the effect of δV_N, and $\delta V_{\rm core}$ is the correction to the electron potential of core electrons due to the changes in all core orbitals. IS for states of a valence electron is found as $\langle\psi_v^{\rm Br}|\delta V_N + \delta V_{\rm core}|\psi_v^{\rm Br}\rangle$.

We use Fermi nuclear charge distribution (solid line in Fig. 1)

$$\rho(r)_f = \frac{\rho_0}{1 + \exp 4\,\ln 3(r-c)/t}, \tag{5}$$

where c is nuclear radius, t is skin thickness, and ρ_0 is the normalization constant, $\int \rho(r)_f dV = Z$. Nuclear charge distribution with a hole in the origin is given by (dashed line in Fig. 1)

$$\rho(r)_h = \rho(r)_f\left[1 + k\left(\frac{r}{c}\right)^2\right]. \tag{6}$$

TABLE I. Nuclear parameters for the range of even isotopes from ^{292}E120 to ^{306}E120 isotopes taken from [18].

A	β	$\sqrt{\langle r^2\rangle}$ (fm)
292	0.0	6.220
294	−0.174	6.264
296	−0.205	6.294
298	−0.218	6.330
300	−0.221	6.358
302	−0.261	6.297
304	−0.290	6.484
306	−0.376	6.503

The normalization constant ρ_0 is adjusted to keep correct normalization. Nuclear quadrupole deformation is considered by replacing constant nuclear radius c in (5) by varying parameter $c(\theta)$,

$$c(\theta) = c[1 + \beta Y_{20}(\theta)], \tag{7}$$

and calculating the spherical average by integrating over θ. It is known that this is approximately equivalent to the increase in skin thickness [22,23]

$$t^2 \approx t_0^2 + (4\ \ln 3)^2(3/4\pi^3)c^2\beta^2. \tag{8}$$

We also consider the change of nuclear radius. We use the ^{292}E120 isotope as a reference one and we take nuclear parameters from nuclear calculations [18].

III. RESULTS

Table I lists the isotopes of SHE E120 used in this study with nuclear parameters taken from [18]. The results are presented in Tables II and III and Fig. 3. In all cases the IS for s and $p_{1/2}$ states is dominated by the $\langle\phi_a^{\rm Br}|\delta V_N|\phi_a^{\rm Br}\rangle$ term (see Table II); IS for states with $j > 1/2$ is dominated by the core polarization (CP) term $\langle\phi_a^{\rm Br}|\delta V_{\rm core}|\phi_a^{\rm Br}\rangle$. The largest contributions to the CP come from the core s states as shown in Fig. 2. Therefore, the effect of change in nuclear charge distribution is very similar for all states except the $p_{1/2}$ states.

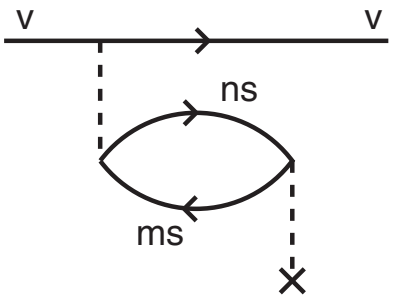

FIG. 2. Dominating contribution to the isotope shift of the single-electron states v with total angular momentum $j > 1/2$ ($p_{3/2}$, $d_{3/2}$, $d_{5/2}$, etc.). The cross stands for δV_N, the change of nuclear potential due to change in nuclear charge distribution.

TABLE II. Isotope shift for specific states of E120$^+$(in 10^{-3} cm^{-1}) due to a change in nuclear charge distribution. Reference IS is the IS between ^{292}E120 and ^{294}E120 calculated ($\langle\psi_a^{\rm Br}|\delta V_N + \delta V_{\rm core}|\psi_a^{\rm Br}\rangle$) with the nuclear parameters from Table I. "Hole" is the shift due to the difference between pure Fermi distribution (5) and the distribution with the hole in the origin, formula (6) with $k = 0.5$. The same IS is produced by reducing the skin thickness t in (5) by 14.5%. "Deformation" is the shift due to quadrupole deformation, formula (7) with $\beta = -0.4$. The same IS is produced by increasing the skin thickness t in (5) by 30.5%. Note that while changing the hole parameter k or the deformation parameter β we are also changing nuclear radius parameter c to keep the rms radius unchanged. "Change of $\sqrt{\langle r^2\rangle}$" is the IS due to change of nuclear RMS radius in pure Fermi distribution (5) from 6.220 to 6.211 fm. "Br" stands for IS given by $\langle\psi_a^{\rm Br}|\delta V_N|\psi_a^{\rm Br}\rangle$; "Br+CP" includes core polarization, $\langle\psi_a^{\rm Br}|\delta V_N + \delta V_{\rm core}|\psi_a^{\rm Br}\rangle$. Note that corresponding matrix elements may be interpreted as isotope shift corrections to the ionization potential for an electron on a given orbital.

	Reference	Hole		Deformation		Change of $\sqrt{\langle r^2\rangle}$	
State	IS	Br	Br+CP	Br	Br+CP	Br	Br+CP
$8s$	10134	743	813	−1986	−2172	−1988	−2172
$9s$	2377	182	191	−486	−510	−486	−510
$8p_{1/2}$	1705	130	131	−347	−351	−359	−365
$8p_{3/2}$	−485	$\sim 10^{-2}$	−38	$\sim 10^{-1}$	103	$\sim 10^{-2}$	104
$7d_{3/2}$	−1350	$\sim 10^{-3}$	−106	$\sim 10^{-3}$	284	$\sim 10^{-3}$	289
$7d_{5/2}$	−606	$\sim 10^{-8}$	−48	$\sim 10^{-7}$	128	$\sim 10^{-8}$	130

A. Hole in nuclear charge distribution and change of the nuclear skin thickness

A hole (or, more accurately, central depression) in nuclear density for E120 was considered in Refs. [24–26]. Its importance is related to the theoretical prediction of magic numbers for protons and neutrons. We study the effect of making a hole in nuclear charge distribution by comparing the energies of the ^{292}E120$^+$ ion in which nuclear charge distribution is pure Fermi distribution (5) to the energies of the ion in which nuclear density is modified according to (6) (see also Fig. 1). We use $k = 0.5$ while keeping the RMS radius fixed. The results are presented in Table II. We also present in this table reference IS which is the shift between ^{292}E120 and ^{294}E120 calculated with the nuclear parameters from Table I as a matrix element $\langle\psi_a^{\rm Br}|\delta V_N + \delta V_{\rm core}|\psi_a^{\rm Br}\rangle$. The ratio of the energy shifts due to a hole to the reference IS is about 8%.

TABLE III. Isotope shift between ^{292}E120 and ^{294}E120 (in cm^{-1}) for the frequencies of the $8s$-$8p$ and $9s$-$8p$ transitions in E120$^+$. Case A corresponds to nuclear parameters in Table I. Case B is a model case in which $\beta = 0$ for both isotopes and a change in RMS radius is chosen to fit the shift of s states.

Transition	A	B	A-B
$8s$-$8p_{1/2}$	8.42911	8.43187	−0.0028
$8s$-$8p_{3/2}$	10.6185	10.6183	0.0002
$9s$-$8p_{1/2}$	0.67423	0.67528	−0.0011
$9s$-$8p_{3/2}$	2.86118	2.86109	0.0001

This means that the effect is significant and deserves further study.

It turns out that a hole in the nuclear charge distribution is numerically equivalent to decreasing the value of the skin thickness [parameter t in ((5)]. The value $k = 0.5$ corresponds to the 14.5% decrease in the value of t. In both cases the effect is practically the same for all considered states.

B. Nuclear quadrupole deformation and change of nuclear radius

Next we study the effect of nuclear quadrupole deformation. We consider a model situation by comparing two nuclei with the same RMS radius but one has no deformation, and another has a deformation with $\beta = -0.4$ in (7). This value of β comes from nuclear calculations for the ^{316}E120 isotope [18]. The effect of quadrupole deformation is equivalent to increased skin thickness (see Fig. 1). Calculations show that for $\beta = -0.4$ equivalent increase in skin thickness t is 30.5% in good agreement with (8). The shift in energy is significant, ~ 2 cm^{-1} for s states (see Table II) or $\sim 20\%$ of the reference IS for all considered states. This leads to a question whether IS can be used to study nuclear deformation. Therefore, we check whether nuclear deformation can be distinguished from the change of nuclear RMS radius. The two last columns of Table II show the effect of the change in nuclear RMS radius in which the parameters were chosen to produce the same IS for the $8s$ state as in the case of quadrupole deformation. We see that the shift is the same for all states except the $8p_{1/2}$ state. The difference for the $8p_{1/2}$ state is 4% or 0.014 cm^{-1}. This is large enough to be detected in spectroscopic measurements. However, this is a model case. Let us now consider a more realistic case of isotope shift between two isotopes ^{292}E120 and ^{294}E120 in which nuclear parameters are taken from nuclear calculations [18]. We consider isotope shift for frequencies of electric dipole transitions in E120$^+$ for isotopes in Table I. IS for the $a \to b$ transition is given by $\delta\nu_{ab} = \langle\psi_b^{\rm Br}|\delta V_N + \delta_{\rm core}|\psi_b^{\rm Br}\rangle - \langle\psi_a^{\rm Br}|\delta V_N + \delta V_{\rm core}|\psi_a^{\rm Br}\rangle$. The results are presented as case A in Table III. In case B we perform model calculations to check whether IS can be reduced to the change in RMS radius. The answer is negative. We see that if we chose the change in RMS radius to fit the shift of s and $p_{3/2}$ states (they behave the same way; see above), then the shift for the $p_{1/2}$ state is slightly different leading to different IS in the ns-$mp_{1/2}$ transitions. The difference is ~ 0.003 cm^{-1} for the $8s$-$8p_{1/2}$ transition, which is probably large enough to be detected. This means that nuclear deformation can be studied by comparing IS in the s-$p_{1/2}$ and s-$p_{3/2}$ transitions. Both these ISs cannot be fitted by changing just one nuclear parameter, e.g., RMS radius. A change in nuclear deformation (β) is also needed. Note that this might be the only way to study nuclear deformation for even-even isotopes by means of atomic spectroscopy. In odd isotopes one can also measure electric quadrupole hyperfine structure. Note also that since three types of nuclear deformations (hole in the origin, quadrupole deformation, and change of thickness) are numerically equivalent in terms of producing similar IS, what is said above about nuclear deformation is also true about having a hole in nuclear charge distribution; i.e., it can be studied by comparing IS in the s-$p_{1/2}$ and s-$p_{3/2}$ transitions.

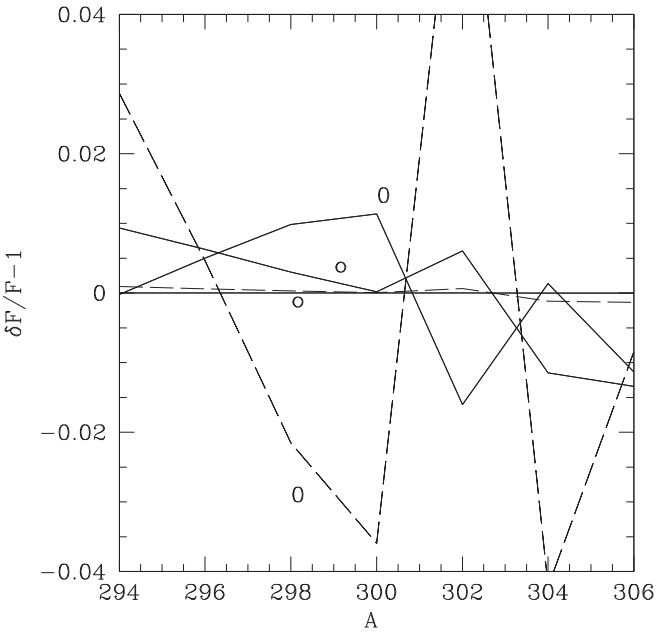

FIG. 3. Fractional deviation of the isotope shift constants from their average values in cases of spherical and deformed nuclei. Solid lines are for the field constant F; long dashed lines are for the modified field shift constant $\tilde{F}$. Lines corresponding to spherically symmetric nuclei marked with "o"; lines corresponding to deformed nuclei markded with "0."

C. Isotope shift for large change of neutron numbers

It was suggested in Ref. [7] to use isotope shift calculations to predict transition frequencies in SHEs from a hypothetical island of stability. These metastable SHEs differ from isotope-poor SHEs produced in laboratories by a large number of neutrons (large ΔN). This should be taken into account in the IS calculations. The calculations reported above use the RPA method, which assumes that the change in nuclear potential δV_N is a small perturbation and ignores nonlinear in δV_N contributions. In SHEs with large ΔN nonlinear in δV_N contributions are likely to be important and should not be thrown away. The most obvious way to do calculations properly is to calculate energy levels for each isotope and then take the difference. This does not work for light atoms because the IS is small and obtaining it as a difference of large almost equal numbers leads to numerical instabilities. Fortunately, IS in SHE is sufficiently large to ensure stable results. Even for neighboring isotopes taking the difference between two RHF calculations produces results which are very close to the RPA calculations. For large ΔN, the calculations based on the difference between two isotopes are preferable because they include nonlinear contributions.

It is customary to present FIS as a formula in which electron and nuclear variables are separated. The standard formula reads

$$\text{FIS} = F\delta\langle r^2\rangle. \tag{9}$$

It is assumed that the electron structure factor F does not depend on nuclear variables. This formula works very well in light atoms and is widely used even for atoms close to the

end of the known periodic table (e.g., for No, $Z = 102$, [15]). It was shown in Ref. [10] that relativistic corrections lead to a different formula

$$\mathrm{FIS} = \tilde{F}\delta\langle r^{2\gamma}\rangle, \tag{10}$$

where $\gamma = \sqrt{1-(\alpha Z)^2}$. New electron structure constant $\tilde{F}$ does not depend on nuclei. The formula was obtained by considering spherical nuclei with uniform change distribution. Below we study the performance of both formulas (9) and (10) for deformed nuclei. We calculate isotope shifts for the $8s$-$8p_{1/2}$ and $8s$-$8p_{3/2}$ transitions for all even isotopes of $E120^+$ from $A = 294$ to $A = 306$. We take nuclear parameters β and RMS radius from Ref. [18] (see Table I). We also consider a model case in which all considered nuclei are assumed to be spherically symmetric ($\beta = 0$). IS is calculated for pairs of neighboring isotopes using the RPA method as described above. The constants F and $\tilde{F}$ are found using (9) and (10). The calculations repeated for both transitions for seven pairs of isotopes. In the end we have 14 values of isotope shift and 14 values of F and $\tilde{F}$. We found that $\mathrm{FIS}(8s\text{-}8p_{1/2})/\mathrm{FIS}(8s\text{-}8p_{3/2}) \approx$ const for all considered isotopes. This means almost exact cancellation of a nuclear factor, as if F or $\tilde{F}$ in (9) or (10) does not depend on an isotope. However, the nuclear factor must be different from $\delta\langle r^2\rangle$ or $\delta\langle r^{2\gamma}\rangle$, since neither formula (9) nor (10) works well. Figure 3 shows variations of F and $\tilde{F}$ from isotope to isotope in terms of their deviation from the average values, e.g., $\delta(F/F)_i = (F_i - \langle F\rangle)/\langle F\rangle$, where $\langle F\rangle = \sum F_i/7$. The value of F in (9) tends to drift in one direction leading to large variations for large difference in neutron numbers. This is similar for both cases, symmetrical and deformed nuclei. In contrast, formula (10) works very well for spherical nuclei, showing only about 0.01% variation for $\tilde{F}$ in the considered interval. However, the formula does not work so well for deformed nuclei. The value for $\tilde{F}$ jumps up and down by several percent from one isotope to another. This is probably because the value of $\langle r^{2\gamma}\rangle$ depends on two nuclear parameters, nuclear deformation parameter β and nuclear RMS radius, making its behavior irregular.

Note that the difference in the value of F for neighboring isotopes usually does not exceed 1% for both spherical and deformed nuclei. With this accuracy formula (9) can be used for neighboring isotopes to extract the change of nuclear RMS radius from isotope shift measurements (see, e.g., [15]). Keeping in mind that the value of F depends on the isotope, the calculations should be performed for one of the isotopes of interest (or for both, taking then an average value).

Since neither of the formulas (9) or (10) works well for deformed nuclei there is a question whether there is any alternative to them. We suggest a two-dimensional fit of the calculated IS on the β, R plane, where $R = \sqrt{\langle r^2\rangle}$ is the nuclear RMS radius. High accuracy of fitting can be achieved by quadratic fit in both coordinates,

$$\delta\nu_i = ax_i + by_i + cx_iy_i + dx_i^2 + ey_i^2. \tag{11}$$

Here $x = \beta_i - \beta_0$, $y = R_i - R_0$, where index i numerates isotopes, 0 corresponds to a reference isotope, and isotope shift (11) is the change of frequency of an atomic transition between isotopes i and 0. Note that we do not need the results of nuclear calculations to do the fitting. We only need to know a reasonable range of change of x and y in (11) for isotopes of interest. We calculate $\delta\nu_i$ for nine points in the selected range, $x = 0, (\beta_{\max} - \beta_{\min})/2, (\beta_{\max} - \beta_{\min})$; $y = 0, (R_{\max} - R_{\min})/2, R_{\max} - R_{\min}$. Since it is always a possibility to have a spherically symmetric nucleus in the considered range of isotopes, it is reasonable to have $\beta = 0$ within the range. Having $\delta\nu_i$ for nine points we then find the values of the a, b, c, d, e parameters in (11) by least-square fitting. The results for four transitions in $E120^+$ are presented in Table IV. The range of change of β and R was $\beta_{\min} = -0.4$, $\beta_{\max} = 0$, $R_{\min} = 6.22$ fm, $R_{\max} = 6.62$ fm. The accuracy of the fitting IS by (11) in our case is ~0.1%. In the case of a large change of nuclear parameter from one isotope to another the accuracy for IS given by (11) can be lower.

TABLE IV. Parameters of formula (11) for isotope shifts in the $8s$-$8p$ and $9s$-$8p$ transitions in $E120^+$.

Transition	a	b	c	d	e
$8s$-$8p_{1/2}$	0.0752	−197.8919	−0.9775	11.6472	2.7319
$8s$-$8p_{3/2}$	0.0991	−253.3191	−1.2516	14.7876	3.1999
$9s$-$8p_{1/2}$	−0.0063	12.6461	0.0711	−0.8339	−0.3028
$9s$-$8p_{3/2}$	−0.0302	68.0734	0.3453	−3.9746	−0.7707

Having formula (11) is useful because different nuclear calculations produce significantly different nuclear parameters. Using formula (11) allows one to obtain the value of IS for any values of β and R within the considered range without performing new atomic calculations. Note that the parameters β and R can even go outside of the considered range, but the accuracy of the predicted IS in this case would be lower.

IV. CONCLUSIONS

We studied the effects of nuclear deformations on the field isotope shift in SHEs. We demonstrated that making a hole in the nuclear charge distribution and having quadrupole deformation can be reduced to changing nuclear skin thickness. On the other hand, the change in skin thickness is not totally equivalent to the change of nuclear RMS radius. There is a small difference in energy shift of the $p_{1/2}$ states compared to states of other symmetries. With sufficiently accurate measurements of the IS this difference can probably be used to study nuclear deformations in even nuclei.

The total effect of the nuclear hole on the isotope shift is up to ~8%; the effect of the deformation is up to ~20%.

We demonstrated that known formulas for the separation of nuclear and electron variables do not work for heavy deformed nuclei. However, in the considered examples the ratio of isotope shifts for two atomic transitions remained isotope independent. Therefore, the linearity of the King plot is not broken.

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