Quantum metrology with generalized cat states

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We show a general relationship between a superposition of macroscopically distinct states and sensitivity in quantum metrology. Generalized cat states are defined by using an index which extracts the coherence between macroscopically distinct states, and a wide variety of states, including a classical mixture of an exponentially large number of states, has been identified as the generalized cat state with this criterion. We find that if we use the generalized cat states for magnetic field sensing without noise, the sensitivity achieves the Heisenberg scaling. More importantly, we even show that sensitivity of generalized cat states achieves the ultimate scaling sensitivity beyond the standard quantum limit under the effect of dephasing. As an example, we investigate the sensitivity of a generalized cat state that is attainable through a single global manipulation on a thermal equilibrium state and find an improvement of a few orders of magnitude from the previous sensors. Clarifying a wide class that includes such a peculiar state as metrologically useful, our results significantly broaden the potential of quantum metrology.

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I. INTRODUCTION

High-precision metrology is important in both fundamental and applicational senses [1–4]. In particular, magnetic field sensing has been attracting much attention [5–7] due to the potential applications in various fields from the determination of the structure of chemical compounds to imaging of living cells [8]. Numerous efforts have been made to increase the sensitivity of the magnetic field sensors [9–26], and various types of magnetometers have been studied [27–29]. A qubit-based sensing [30–38] is an attractive approach where quantum properties are exploited to enhance the sensitivity. By using superpositions of states, the standard Ramsey-type measurement without feedback can be implemented to measure the magnetic field, where the magnetic field information is encoded in the relative phase between the states in accordance with the magnetic field strength. If we use N qubits in separable states, it is known that the uncertainty (that is, the inverse of the sensitivity) scales as $\Theta(N^{-1/2})$, which is called the standard quantum limit (SQL) [39]. On the other hand, quantum physics allows one to beat the SQL. The ultimate scalings are known to be $\Theta(N^{-1})$, i.e., the Heisenberg scaling, in the absence of noise and $\Theta(N^{-3/4})$ in the presence of realistic decoherence [10,12,20,23,24,26,40–42].

In the standard Ramsey-type measurement protocol, the ultimate scalings seem to be attainable by using the quantum superposition. However, a general relationship between a quantum superposition and sensitivity is not yet known. Therefore, it is essential to clarify what type of superposition gives higher sensitivity in metrology than classical sensors.

Superpositions of macroscopically distinct states, i.e., "cat" states, have attracted many researchers due to the fundamental interest since its introduction by Schrödinger [43]. Although a cat state contains a superposition, not all types of superpositions can be considered as the cat state. The Greenberger-Horne-Zeilinger (GHZ) [44–46] state is one of the typical cat states. Since this cat state is useful in quantum metrology, we may expect other cat states to be useful as well. However, there was no unified criteria to judge if a given state contains such macroscopically distinct states [47], preventing the further understanding of the relation between cat states and sensors. Among many possible measures, we especially focus on the index q [48]. Importantly, q is defined for both pure and mixed states, and is measurable in experiments by measuring a certain set of local observables.

In this paper, we prove that generalized cat states, i.e., the superposition of macroscopically distinct states characterized by the index q, are all capable of achieving the ultimate scalings. We give the upper bound of the uncertainty when $q = 2$ states are used as a sensor state. First, we show the Heisenberg scaling in the absence of noise. Second, we analyze the case with a realistic decoherence. We prove that the SQL is still beaten; the generalized cat states achieve the ultimate scaling uncertainty $\Theta(N^{-3/4})$. Third, we present a nontrivial example and numerically show its advantage. Since there are states with low purity among the generalized cat states (Fig. 1), wide varieties of states have the potential to achieve the ultimate scalings.

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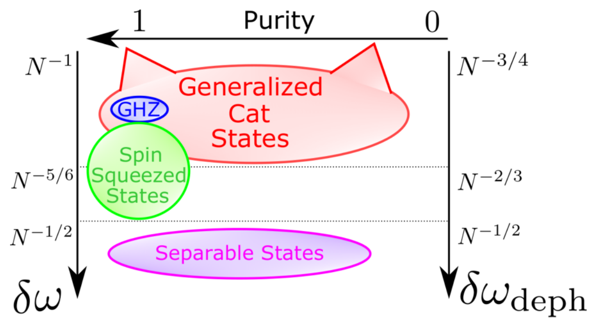

FIG. 1. The relationship between the purity and the scaling of the uncertainty for given quantum states when we use the quantum states for the Ramsey-type quantum sensing. The ultimate scaling of the uncertainty without [with] dephasing is $\delta\omega = \Theta(N^{-1})$ [$\delta\omega_{\mathrm{deph}} = \Theta(N^{-3/4})$]. Only special pure entangled states such as GHZ states are known to achieve such a scaling. The GHZ state is a pure state, and the uncertainty scales as $\Theta(N^{-1})$ in the absence of dephasing and $\Theta(N^{-3/4})$ in the presence of dephasing. One-axis and two-axis spin squeezed states [49] are pure states beating the SQL. Separable states, whether pure or mixed, do not beat the SQL. In this paper, we show that all the generalized cat states achieve the ultimate scalings, even if it is a classical mixture of exponentially large number of states.

II. GENERALIZED CAT STATES

To begin with, we introduce a concept of a *generalized cat state*, which is discussed in detail in the Appendix of [50]. We refer to the index q [7,48,51–55], which is a real number satisfying $1 \leqslant q \leqslant 2$. It is defined as

$$\max\left\{N, \max_{\hat{A},\hat{\eta}} \mathrm{Tr}(\hat{\rho}[\hat{A},[\hat{A},\hat{\eta}]])\right\} = \Theta(N^q), \tag{1}$$

where $\hat{A} = \sum_{l=1}^{N} \hat{a}(l)$ is an additive observable and $\hat{\eta}$ is a projection operator. Since the states with $q = 2$ have the interesting features that we would like to focus on in this paper, we simplify the definition for this case as follows. A quantum state $\hat{\rho}$ has $q = 2$ if there exist an additive observable $\hat{A}$ and a projection operator $\hat{\eta}$ such that

$$\mathrm{Tr}(\hat{\rho}[\hat{A},[\hat{A},\hat{\eta}]]) = \Theta(N^2). \tag{2}$$

We call a state with $q = 2$ a generalized cat state. By contrast, e.g., separable states have $q = 1$.

We can understand the physical meaning of q by expressing the left-hand side of Eq. (2) as follows: $\mathrm{Tr}(\hat{\rho}[\hat{A},[\hat{A},\hat{\eta}]]) = \sum_{A,\nu,A',\nu'}(A-A')^2\langle A,\nu|\hat{\rho}|A',\nu'\rangle\langle A',\nu'|\hat{\eta}|A,\nu\rangle$, where $|A,\nu\rangle$ denotes an eigenvector of $\hat{A}$ with eigenvalue A, and ν denotes the degeneracy. This shows that if $\hat{\rho}$ has $q = 2$, there exist terms such that $\langle A,\nu|\hat{\rho}|A',\nu'\rangle\langle A',\nu'|\hat{\eta}|A,\nu\rangle \neq 0$ for $|A - A'| = \Theta(N)$. For $N \gg 1$, the term $\langle A',\nu'|\hat{\rho}|A,\nu\rangle$ with $|A - A'| = \Theta(N)$ corresponds to a quantum coherence between states that are distinguishable even on a macroscopic scale. Therefore, the state with $q = 2$ can be considered to contain a superposition of macroscopically distinct states.

For pure states, $q = 2$ guarantees the existence of an additive observable such that $\mathrm{Tr}[\hat{\rho}(\Delta\hat{A})^2] = \Theta(N^2)$. As suggested from other measures of macroscopic quantum states [51,56], such a large fluctuation is available only when $\hat{\rho}$ has a superposition of macroscopically distinct states (for details, see the Appendix of [57]).

As an example, let us consider a state $|\psi\rangle := (|\downarrow\rangle^{\otimes N} + |\uparrow\rangle|\downarrow\rangle^{\otimes N-1} + |\uparrow\rangle^{\otimes 2}|\downarrow\rangle^{\otimes N-2} + \cdots + |\uparrow\rangle^{\otimes N})/\sqrt{N+1}$. Since this state is much more complicated than the well-known GHZ state, it may be difficult to intuitively judge whether this is a cat state, but we can actually show that this state has $q = 2$ by taking $\hat{A} = \hat{M}_z$ and $\hat{\eta} = |\psi\rangle\langle\psi|$. Pure states with $q = 2$ are known to have several "catlike" properties, such as fragility against decoherence and instability against local measurements [58].

For mixed states, q correctly identifies states that contain pure cat states with a significant ratio in the following sense (see, e.g., the Appendix of [50]). Without losing generality, we can perform a pure-state decomposition of a mixed state with $q = 2$ as $\hat{\rho} = \sum_{j=1}^{N}\lambda_j|\psi_j\rangle\langle\psi_j|$, where $|\psi_j\rangle\langle\psi_j|$ has $q = 2$ ($q < 2$) for $j = 1, 2, \ldots, m$ ($j = m+1, m+2, \ldots, N$) for $0 < m < N$. In this case, we can show $\sum_{j=1}^{m}\lambda_j = \Theta(N^0)$, and this intuitively means that a mixed state with $q = 2$ contains a significant (or nonvanishing) amount of pure states with $q = 2$. For example, $\hat{\rho}_{\mathrm{ex}} = w|\psi\rangle\langle\psi| + (1-w)\hat{\rho}_{\mathrm{sep}}$ has $q = 2$ for N-independent $w > 0$, where $\hat{\rho}_{\mathrm{sep}}$ is an arbitrary separable state.

III. DEFINITION OF SENSITIVITY

Since we will later discuss the relationship between the generalized cat states and quantum sensing, we review the concept of quantum metrology. Here we discuss the case of a spin system to exemplify in the context of magnetometry, although our results are, in principle, applicable to any physical systems, e.g., interferometry in optical systems [2].

Suppose that a sensor consists of N free spins that interact with a magnetic field with a Hamiltonian $\hat{H}_0(\omega) = \omega\hat{A}$, where ω denotes the Zeeman frequency shift of the spins and $\hat{A}$ is the sum of local spin operators [hence, $\|\hat{A}\| = \Theta(N)$]. We assume that the frequency has a linear scaling with respect to the magnetic field B (such as $\omega \propto B$). Also, we decompose magnetic field B into the "applied field" B_0 (corresponding Zeeman shift ω_0) and the "target field" B' (corresponding Zeeman shift ω'); $\omega = \omega_0 + \omega'$. Here, we assume that we know the amplitude of the applied magnetic field B_0 while the target small magnetic field B' is unknown. For metrological interest, we consider $\omega' \to 0$ throughout this paper. Also, to include the effect of the dephasing, we add the noise effect to the total Hamiltonian as $\hat{H} = \hat{H}_0(\omega) + \hat{H}_{\mathrm{int}}$, where $\hat{H}_{\mathrm{int}}$ denotes the interaction with the environment.

The following is the standard Ramsey-type protocol to detect the magnetic field by using spins. First, prepare the spins in the state $\hat{\rho}$. Second, let $\hat{\rho}$ evolve under the Hamiltonian $\hat{H}$ for an interaction time t_{int} to become $\hat{\rho}(t_{\mathrm{int}})$. Third, read out the state via a measurement described by a projection operator $\hat{\mathcal{P}}$. Fourth, repeat these three steps within a given total measurement time T. We assume that state preparation and projection can be performed in a short time interval much smaller than t_{int}. In this case, the number of the repetition is approximated to be T/t_{int}, and therefore the uncertainty $\delta\omega$ of the estimation of our protocol is described as

$$\delta\omega = \frac{\sqrt{P(1-P)}}{\left|\frac{dP}{d\omega}\right|}\frac{1}{\sqrt{T/t_{\rm int}}}, \tag{3}$$

where $P = \mathrm{Tr}[\hat{\rho}(t_{\rm int})\hat{\mathcal{P}}]$ denotes the probability that the projection described by $\hat{\mathcal{P}}$ occurs at the readout process.

IV. HEISENBERG SCALING IN THE IDEAL ENVIRONMENT

Here, we show that we can achieve the Heisenberg scaling, i.e., $\Theta(N^{-1})$ uncertainty, by using a state with $q = 2$ as a sensor of the target field if decoherence is negligible.

Suppose that we have a generalized cat state $\hat{\rho}$ satisfying Eq. (2) for an additive observable $\hat{A}$ and a projection operator $\hat{\eta}$. If the target field couples with the spins via $\hat{A}$ as $\hat{H}_0(\omega) = \omega\hat{A}$, which induces an energy change, we can use the state with $q = 2$ to sensitively estimate the value of ω. By setting the projection operator for the readout as $\hat{\mathcal{P}} = \hat{\eta}$, we can use the standard sensing protocol described in the previous paragraph. We find that for a certain positive constant p_1, there exist $\Omega_1 = \Theta(N^0)$ and $N_1 > 0$ such that

$$\delta\omega \leqslant \left(p_1 p_2^2 N t_{\rm int}\right)^{-1}\left(\sqrt{T/t_{\rm int}}\right)^{-1} \tag{4}$$

is satisfied for $p_2 := \omega t_{\rm int} N = \Theta(N^0) \leqslant \Omega_1$ and $N \geqslant N_1$. This is because the numerator of Eq. (3) satisfies $\sqrt{P(1-P)} = \Theta(N^0)$ for $\omega t_{\rm int} N = \Theta(N^0)$, whereas $|dP/d\omega|$ in the denominator has a lower bound,

$$\left|\frac{dP}{d\omega}\right| \geqslant \left|\left|\omega t_{\rm int}^2 \mathrm{Tr}(\hat{\rho}[\hat{A},[\hat{A},\hat{\eta}]])\right| - \left|it_{\rm int}\mathrm{Tr}(\hat{\rho}[\hat{A},\hat{\eta}])\right|\right| - 2t_{\rm int}\|\hat{A}\|\left(e^{2\omega t_{\rm int}\|\hat{A}\|} - 1 - 2\omega t_{\rm int}\|\hat{A}\|\right). \tag{5}$$

Since we assume Eq. (2), the term $u := |\omega t_{\rm int}^2 \mathrm{Tr}(\hat{\rho}[\hat{A},[\hat{A},\hat{\eta}]])| = p_2\Theta(t_{\rm int}N)$, whereas the term $v := |it_{\rm int}\mathrm{Tr}(\hat{\rho}[\hat{A},\hat{\eta}])| \leqslant \Theta(t_{\rm int}N)$. Therefore, we obtain $|u - v| = p_2\Theta(t_{\rm int}N)$ by tuning $p_2 = \Theta(N^0) < 1$ correctly. The remaining term in Eq. (5) is $-\Theta(t_{\rm int}N)p_2^2$, which can be made much smaller than $|u - v|$ by taking $p_2 \ll 1$. More precisely, we find that there exists a positive constant $\Omega_1 \ll 1$ such that $\forall p_2 = \Theta(N^0) \leqslant \Omega_1$ satisfies $|dP/d\omega| \geqslant p_1 p_2^2 t_{\rm int} N$ for a certain positive constant p_1. If we tune ω_0 in such a way that $\omega = \omega_0 + \omega'$ scales as $\omega = \Theta(N^{-1})$, and choose the interaction time $t_{\rm int} = \Theta(N^0)$ as to realize the condition of $\omega t_{\rm int} N = \Theta(N^0)$, then we have $\delta\omega \leqslant 1/\Theta(N)$, achieving the Heisenberg scaling.

V. ULTIMATE SCALING IN THE PRESENCE OF DECOHERENCE

In reality, dephasing is one of the major challenges to be overcome for beating the SQL. For example, the GHZ state acquires the information of the target field as a relative phase $\exp(i\omega' tN)$ on the off-diagonal terms of the density matrix. However, the dephasing induces a rapid decay of the amplitude of such off-diagonal terms, making it nontrivial whether or not the quantum sensor really has an advantage.

Upon discussing the dephasing, we must take into account the correlation time τ_c of the environment. Historically, the Markovian dephasing was considered for evaluating the performance of the quantum sensor [13,21,59,60]. This implies that τ_c was assumed to be much smaller than any other timescales such as the coherence time T_2^* and $t_{\rm int}$. Then, if we reasonably assume the independent dephasing, the decay of the off-diagonal terms behaves as $\exp(-tN/T_2^*)$, which is not slower than the phase accumulation $\exp(i\omega' tN)$. In this case, it was concluded that beating the SQL is impossible even with the optimal interaction time [which is $t_{\rm int} = \Theta(1/N)$].

However, in most of the solid-state qubits, $\tau_c \gg T_2^*$ in contradiction to the Markovian dephasing. By taking this point into account, Refs. [10,12,20,23,26,40] recently found that $t_{\rm int}$ should be taken in the so-called Zeno regime, i.e., $t_{\rm int} \ll \tau_c$, where the non-Markovian effect plays a crucial role. The decay of the off-diagonal terms in this regime behaves as $\exp[-(t/T_2^*)^2 N]$, which is much slower than the decay in Markovian dephasing. With the optimal interaction time $t_{\rm int} \sim T_2^*/\sqrt{N}$, it was proven that the GHZ state and spin squeezed states can beat the SQL, achieving the ultimate scaling $\delta\omega \propto N^{-3/4}$ [10,12,20,23,24,26,40]. However, these investigations were limited to some specific states, leaving an open question of whether or not there are any other metrologically useful superpositions. Moreover, although most of the previous research assumed that pure states can be prepared, quantum states for sensing may be mixed in experiments. So, for understanding the full potential of quantum metrology, it is crucial to explore the sensitivities of sensing using other, nontrivial and nonideal, states.

Here, we discuss the performance of the generalized cat states satisfying Eq. (2) as a magnetic field sensor under the effect of dephasing with τ_c longer than $t_{\rm int}$. We model the dephasing by adding Hamiltonian $\hat{H}_0(\omega)$ to the following interaction with the environment [40,61]: $\hat{H}_{\rm int} = \sum_{l=1}^{N}\lambda f_l(t)\hat{a}(l)$, where λ denotes the amplitude of the noise and $f_l(t) (l = 1, 2, \ldots, N)$ denotes a random classical variable at the site l. We assume $f_l(t)$ satisfies $\overline{f_l(t)} = 0$ and $\overline{f_l(t)f_{l'}(t')} = \exp(-|t-t'|/\tau_c)\delta_{l,l'}$, where the overline denotes the ensemble average. Taking $t_{\rm int} \ll \tau_c$, we can approximate $\exp(-|t-t'|/\tau_c) \simeq 1$ because $|t-t'| \leqslant t_{\rm int}$. When there is such a dephasing, the state after the time evolution is a classical mixture of $\exp(-i\omega\hat{A}t_{\rm int})\hat{\rho}\exp(i\omega\hat{A}t_{\rm int})$ [with a weight of $(\frac{1+\exp(-2\lambda^2 t_{\rm int}^2)}{2})^N$] and other states. The former state corresponds to the generalized cat state that has evolved in the magnetic field without dephasing. Although we have shown that the former state can achieve the Heisenberg scaling, the latter state has a complicated form, and so the calculation of the sensitivity of the latter state is not straightforward. Fortunately, by tuning $p_2[= \omega t_{\rm int} N = \Theta(N^0) \ll 1]$ and $t_{\rm int}$, the former contribution can be set to be larger than the latter contribution, and the uncertainty can be bounded as follows:

$$\delta\omega_{\rm deph}\sqrt{T} \leqslant (N\sqrt{t_{\rm int}})^{-1}\left\{p_1 p_2^2\left(\frac{1+e^{-2\lambda^2 t_{\rm int}^2}}{2}\right)^N - 2e^{2\omega t_{\rm int}\|\hat{A}\|}\frac{\|\hat{A}\|}{N}\left[1 - \left(\frac{1+e^{-2\lambda^2 t_{\rm int}^2}}{2}\right)^N\right]\right\}^{-1}. \tag{6}$$

By taking $t_{\rm int} \propto p_2^2/\sqrt{N}$, we obtain $\delta\omega_{\rm deph}\sqrt{T} \leqslant \Theta(N^{-3/4})$, and this achieves the ultimate scaling beyond the SQL. We can see the optimality of this scaling as follows. As we increase $t_{\rm int}$, the term $(N\sqrt{t_{\rm int}})^{-1}$ on the right-hand side of (6) becomes smaller, which contributes to achieve a better sensitivity. However, since we need to have a finite weight of $\exp(-i\omega\hat{A}t_{\rm int})\hat{\rho}\exp(i\omega\hat{A}t_{\rm int})$, its weight $(\frac{1+e^{-2\lambda^2 t_{\rm int}^2}}{2})^N$ should be nonvanishing in the limit of large N, hence scaling of $t_{\rm int}$ should be $\Theta(1/\sqrt{N})$ at most. Also, we should tune $t_{\rm int} \propto p_2^2$ so the right-hand side of Eq. (6) is positive. Thus we find $t_{\rm int} \propto p_2^2/\sqrt{N}$ is optimal. Then the scaling of the sensitivity is enhanced $N^{1/4}$ times more than that of the SQL, agreeing with Refs. [10,12,20,23], in which the GHZ beats the SQL by a factor of $N^{1/4}$ with $t_{\rm int} \propto 1/\sqrt{N}$. Other works also showed that this scaling is the best in the presence of dephasing [41,42]. Therefore, we have proven that the generalized cat states can achieve the sensitivity with $\delta\omega_{\rm deph} = \Theta(N^{-3/4})$ that is considered as the ultimate scaling under the effect of dephasing.

VI. EXAMPLE

We now discuss a possible application of our results to realize a sensitive magnetic field sensor by using a current technology. Recently, it was found that a single measurement of the total magnetization $\hat{M}_z$ converts a certain thermal equilibrium state into a generalized cat state [50]. The conversion procedure consists of two steps: (1) apply a magnetic field along a specific direction (that we call the x axis) and let the system equilibrate, (2) perform a projective measurement $\hat{\eta}_z$ on $\hat{M}_z = M$ subspace, where the z axis is defined as an orthogonal direction to the applied magnetic field. Then the postmeasurement state has $q = 2$ for $M \neq \pm N + o(N)$. Obviously, for finite temperature, the premeasurement state is a mixture of $\exp[\Theta(N)]$ states because it is a Gibbs state, and the projection measurement is a projection onto a subspace with dimension of $\exp[\Theta(N)]$. This means that the postmeasurement state is a mixture of an exponentially large number of states. Since this state can be prepared from a thermal equilibrium state, this protocol has a potential of generating metrologically useful states easily at moderate temperature. Below we discuss the sensitivity when we use this state for the sensing $\hat{M}_x$ with the readout projection $\hat{\eta}_z$.

Let us consider phosphorus donor electron spins with the density of $\sim 10^{15}$ cm^{-3} in a ^{28}Si substrate with a size of $32 \times 32 \times 1$ μm. Then there are approximately $N = 10^6$ electron spins in the substrate. We assume the applied magnetic field is 10 mT and the temperature is 10 mK, where the thermal energy ($k_B T/2\pi \simeq 208$ MHz) is comparable with the Zeeman splitting ($g\mu_b B/2\pi \simeq 280$ MHz) so that the spin is not fully polarized. Via a projective measurement of the total magnetization (that can be implemented by a superconducting circuit, for example), we can prepare the generalized cat state with $q = 2$. With the coherence time of one electron in this system being around 10 s [62], we numerically optimize the interaction time and find that the uncertainty takes its minimum $\delta\omega_{\rm deph}\sqrt{T} = 5.2 \times 10^{-5}/\sqrt{\rm Hz}$ at $t_{\rm int} = 5.4$ ms, which corresponds to $\delta B\sqrt{T} = 0.30\,{\rm fT}/\sqrt{\rm Hz}$. The optimal interaction time $t_{\rm int} = 5.4$ ms is consistent with our theoretical prediction that $t_{\rm int}$ should be comparable with the coherence time divided by $\sqrt{N}$. As a comparison, we consider using a thermal equilibrium state in the same conditions as above without converting it into the generalized cat state, and we obtain $\delta\omega_{\rm deph}\sqrt{T} = 9.8 \times 10^{-4}/\sqrt{\rm Hz}$. This shows that the use of the generalized cat states provides us with 20 times better sensitivity than the classical states with this system, which demonstrates the practical advantage of the metrology using the generalized cat states.

Let us compare our results with known theoretical results. If a fully polarized separable state with the same electron spins is used, $\delta\omega_{\rm deph}\sqrt{T} = 8.1 \times 10^{-4}/\sqrt{\rm Hz}$ is estimated [23]. Also, by squeezing the fully polarized spin state via nonlinear interactions, it is, in principle, possible to achieve a sensitivity of $\delta\omega_{\rm deph}\sqrt{T} = 7.1 \times 10^{-5}/\sqrt{\rm Hz}$ [23], and this sensitivity is comparable to our results. However, these proposals can be implemented only if a perfect initialization of the electron spins is available, which could be difficult due to the small Zeeman energy of the electron spins. On the other hand, the sensor state we discuss in this section, i.e., a generalized cat state in the Si substrate at finite temperature, is initially a thermal equilibrium spin state with the polarization ratio around 0.6, which is more feasible to prepare. This clearly shows the advantage to use our generalized cat states. According to the size of the substrate, the spatial resolution of the sensor is $\sim 10^{-5}$ m. Experimentally achieved sensitivities with similar spatial resolution are as follows. A superconducting flux qubit, a superconducting quantum interference device (SQUID), and an ensemble of NV centers showed sensitivities of 3.3 pT/$\sqrt{\rm Hz}$ with 5 μm resolution [33], 1.4 pT/$\sqrt{\rm Hz}$ with 100 μm resolution [63], and 150 fT/$\sqrt{\rm Hz}$ with 100 μm resolution [8,35], respectively.

Therefore, we can conclude that our proposed sensor has a sensitivity of at least a few orders of magnitude better than those of the previous sensors.

VII. DISCUSSION

Although we have mainly discussed the scaling of $\delta\omega_{\rm deph}$, the quantitative upper bound of $\delta\omega_{\rm deph}$ can be obtained by evaluating the formula (D1) in the Appendix.

Let us discuss the relation with the quantum Fisher information (QFI). For a given state, the QFI gives the *lower* bound of $\delta\omega$ as $\delta\omega \geqslant 1/\sqrt{\rm QFI}$, i.e., the Cramer-Rao inequality [9]. The equality is satisfied by *some* optimal positive-operator valued measure (POVM) operators. However, such operators are generally *unknown* for mixed states, and so is the physical measurement process to construct the POVM. Hence, practically, the QFI gives $\delta\omega > 1/\sqrt{\rm QFI}$, which does not ensure the ultimate scaling even when ${\rm QFI} = \Theta(N^2)$. In comparison, we have derived the *upper* bound of $\delta\omega$ as $\delta\omega \leqslant \Theta(N^{-1})$ or $\Theta(N^{-3/4})$ for states with $q = 2$ assuming a *known* measurement: the simple Ramsey-type protocol and reading out with the projection $\hat{\eta}$. That is, the way of achieving the ultimate scaling sensitivity is explicitly given.

In addition, the dynamical aspects in the presence of noise are not clear enough for the QFI because in the Cramer-Rao inequality, the QFI is of the state *after* the noisy time evolution, which is not directly related to the QFI of the initial state. By contrast, we have obtained the upper bound of $\delta\omega$ in terms of q of the *initial* cat state, which is actually prepared in experiments. Such a practical bound is derived because q is directly connected to the equation of motion.

VIII. CONCLUSION

Summing up, we have shown that the sensitivity of generalized cat states composed of N spins can achieve the Heisenberg scaling $\delta\omega = \Theta(N^{-1})$ if they are used to measure a magnetic field without dephasing. Moreover, even in the presence of independent dephasing, we obtained the ultimate scaling $\delta\omega_{\rm deph} = \Theta(N^{-3/4})$ beyond the standard quantum limit. For example, the sensitivity of a generalized cat state converted from a thermal equilibrium state at finite temperature is found to be a few orders of magnitude better than the previous sensors, implying that the difficulty of state preparation could be drastically lifted. Providing a wide class that includes such a peculiar state, our work paves the way to broaden the applications of quantum metrology.

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APPENDIX A: DERIVATION OF (6)

If only a single qubit dephases, the Hamiltonian is

$$\hat{H}_0 + \hat{H}_{\rm int1}(t), \tag{A1}$$

where

$$\hat{H}_0 = \omega \sum_{l=1}^{N} \hat{a}(l) = \omega\hat{A}, \tag{A2}$$

$$\hat{H}_{\rm int1}(t) = \lambda f_l(t)\hat{a}(l). \tag{A3}$$

Since $[\hat{H}_0, \hat{H}_{\rm int1}(t)] = 0$, the interaction picture is convenient:

$$\hat{\rho}^I(t) = e^{i\hat{H}_0 t}\hat{\rho}(t)e^{-i\hat{H}_0 t}, \tag{A4}$$

$$\frac{d\hat{\rho}^I(t)}{dt} = -i[\hat{H}_{\rm int1}(t), \hat{\rho}^I(t)]. \tag{A5}$$

Then we have

$$\hat{\rho}^I(t_{\rm int}) = \hat{\rho}(0) + \sum_{n=1}^{\infty}(-i\lambda)^n \int_0^{t_{\rm int}}\int_0^{t_1}\cdots\int_0^{t_{n-1}} dt_1 dt_2\cdots dt_n\{\hat{H}_{\rm int1}(t_1), [\hat{H}_{\rm int1}(t_2), \cdots[\hat{H}_{\rm int1}(t_n), \hat{\rho}(0)]]\}. \tag{A6}$$

Taking the average over the ensemble of the noise, we obtain

$$\hat{\rho}^I(t_{\rm int}) - \hat{\rho}(0) = \sum_{n=1}^{\infty}(-i\lambda)^n \overline{f_l(t_1)f_l(t_2)\cdots f_l(t_n)}\int_0^{t_{\rm int}}\int_0^{t_1}\cdots\int_0^{t_{n-1}} dt_1 dt_2\cdots dt_n[\hat{a}(l), \hat{\rho}(0)]_n. \tag{A7}$$

Here, we define $[\hat{O}_1, \hat{O}_2]_k$ as $[\hat{O}_1, \hat{O}_2]_{k+1} = [\hat{O}_1, [\hat{O}_1, \hat{O}_2]_k]$ and $[\hat{O}_1, \hat{O}_2]_0 = \hat{O}_2$. Since we assume $\overline{f_j(t)f_k(t')} = \delta_{j,k}$ and the $m(>2)$th cumulants are zero for Gaussian noise, $\overline{f_l(t_1)f_l(t_2)\cdots f_l(t_n)}$ can be decomposed into

$$\overline{f_l(t_1)f_l(t_2)\cdots f_l(t_{2n})} = \sum_{\text{all combination}} \overline{f(t_1')f(t_2')}\,\overline{f(t_3')f(t_4')}\cdots\overline{f(t_{2n-1}')f(t_{2n}')} \tag{A8}$$

$$= (2n-1)(2n-3)\cdots 3\cdot 1 = (2n-1)!! \tag{A9}$$

and

$$\overline{f_l(t_1)f_l(t_2)\cdots f_l(t_{2n+1})} = \sum_{\text{all combination}} \overline{f(t_1')f(t_2')}\,\overline{f(t_3')f(t_4')}\cdots\overline{f(t_{2n-1}')f(t_{2n}')}\,\overline{f(t_{2n+1}')} \tag{A10}$$

$$= 0. \tag{A11}$$

Therefore, we have

$$\hat{\rho}^I(t_{\rm int}) - \hat{\rho}(0) = \sum_{n=1}^{\infty}(-i\lambda)^{2n}(2n-1)!!\int_0^{t_{\rm int}}\int_0^{t_1}\cdots\int_0^{t_{n-1}} dt_1 dt_2\cdots dt_{2n}[\hat{a}(l), \hat{\rho}(0)]_{2n} \tag{A12}$$

$$= \sum_{n=1}^{\infty}(-\lambda^2)^n(2n-1)!!t_{\rm int}^n\frac{1}{(2n)!}[\hat{a}(l), \hat{\rho}(0)]_{2n} \tag{A13}$$

$$= \sum_{n=1}^{\infty}(-\lambda^2 t_{\rm int})^n\frac{1}{2^n n!}[\hat{a}(l), \hat{\rho}(0)]_{2n}. \tag{A14}$$

By assuming $\hat{a}(l)^2 = \hat{1}$, which holds for $\pm\hat{\sigma}_{x,y,z}$, we can simplify the commutation:

$$[\hat{a}(l), \hat{\rho}(0)]_{2n} = \frac{2^{2n}}{2}[\hat{\rho}(0) - \hat{a}(l)\hat{\rho}(0)\hat{a}(l)]. \tag{A15}$$

This gives us

$$\hat{\rho}^I(t_{\text{int}}) - \hat{\rho}(0) = \sum_{n=1}^{\infty}(-\lambda^2 t_{\text{int}})^n \frac{1}{2^n n!}\frac{2^{2n}}{2}[\hat{\rho}(0) - \hat{a}(l)\hat{\rho}(0)\hat{a}(l)] \tag{A16}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-2\lambda^2 t_{\text{int}})^n}{n!}[\hat{\rho}(0) - \hat{a}(l)\hat{\rho}(0)\hat{a}(l)] \tag{A17}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-2\lambda^2 t_{\text{int}})^n}{n!}[\hat{\rho}(0) - \hat{a}(l)\hat{\rho}(0)\hat{a}(l)] - \left[\frac{\hat{\rho}(0) - \hat{a}(l)\hat{\rho}(0)\hat{a}(l)}{2}\right] \tag{A18}$$

$$= \frac{e^{-2\lambda^2 t_{\text{int}}}}{2}[\hat{\rho}(0) - \hat{a}(l)\hat{\rho}(0)\hat{a}(l)] - \frac{\hat{\rho}(0) - \hat{a}(l)\hat{\rho}(0)\hat{a}(l)}{2}, \tag{A19}$$

$$\hat{\rho}^I(t_{\text{int}}) = \hat{\rho}(0) + \frac{e^{-2\lambda^2 t_{\text{int}}} - 1}{2}\hat{\rho}(0) + \frac{1 - e^{-2\lambda^2 t_{\text{int}}}}{2}\hat{a}(l)\hat{\rho}(0)\hat{a}(l) \tag{A20}$$

$$= \frac{1 + e^{-2\lambda^2 t_{\text{int}}}}{2}\hat{\rho}(0) + \frac{1 - e^{-2\lambda^2 t_{\text{int}}}}{2}\hat{a}(l)\hat{\rho}(0)\hat{a}(l). \tag{A21}$$

When N spins dephase, i.e., $\hat{H}_{\text{int}}(t) = \sum_{l=1}^{N}\lambda f_l(t)\hat{a}(l)$, $\hat{\rho}^I(t_{\text{int}})$ can be expressed as

$$\hat{\rho}(t_{\text{int}})^I = \epsilon_N[\epsilon_{N-1}\cdots\epsilon_1(\hat{\rho}(0))], \tag{A22}$$

where

$$\epsilon_j(\hat{\rho}(0)) = \frac{1 + e^{-\lambda^2 t_{\text{int}}^2}}{2}\hat{\rho}(0) + \frac{1 - e^{\lambda^2 t_{\text{int}}^2}}{2}\hat{a}(j)\hat{\rho}(0)\hat{a}(j) \tag{A23}$$

since $[\hat{a}(l), \hat{a}(k)] = 0$ for arbitrary pair of (l, k). Explicitly expressing, we have

$$\rho(t_{\text{int}})^I = \left(\frac{1 + e^{-\lambda^2 t_{\text{int}}^2}}{2}\right)^N \hat{\rho}(0) + \left(\frac{1 + e^{-\lambda^2 t_{\text{int}}^2}}{2}\right)^{N-1}\left(\frac{1 - e^{-\lambda^2 t_{\text{int}}^2}}{2}\right)\sum_{j=1}^{N}\hat{a}(j)\hat{\rho}(0)\hat{a}(j)$$
$$+\cdots+\left(\frac{1 - e^{-\lambda^2 t_{\text{int}}^2}}{2}\right)^N \hat{a}_N\hat{a}_{N-1}\cdots\hat{a}_1\hat{\rho}(0)\hat{a}_1\cdots\hat{a}_{N-1}\hat{a}_N, \tag{A24}$$

$$\hat{\rho}(t) = e^{-i\hat{H}_0 t_{\text{int}}}\left[\left(\frac{1 + e^{-\lambda^2 t_{\text{int}}^2}}{2}\right)^N \hat{\rho}(0) + \left(\frac{1 + e^{-\lambda^2 t_{\text{int}}^2}}{2}\right)^{N-1}\left(\frac{1 - e^{-\lambda^2 t_{\text{int}}^2}}{2}\right)\sum_{j=1}^{N}\hat{a}(j)\hat{\rho}(0)\hat{a}(j)\right.$$
$$\left.+\cdots+\left(\frac{1 - e^{-\lambda^2 t_{\text{int}}^2}}{2}\right)^N \hat{a}_N\hat{a}_{N-1}\cdots\hat{a}_1\hat{\rho}(0)\hat{a}_1\cdots\hat{a}_{N-1}\hat{a}_N\right]e^{i\hat{H}_0 t_{\text{int}}}. \tag{A25}$$

For

$$\hat{\rho}' := \hat{\rho}(t_{\text{int}}) - \left(\frac{1 + e^{-\lambda^2 t_{\text{int}}^2}}{2}\right)^N e^{-i\hat{H}_0 t_{\text{int}}}\hat{\rho}(0)e^{i\hat{H}_0 t_{\text{int}}} \tag{A26}$$

$$= e^{-i\hat{H}_0 t_{\text{int}}}\left[\left(\frac{1 + e^{-\lambda^2 t_{\text{int}}^2}}{2}\right)^{N-1}\left(\frac{1 - e^{-\lambda^2 t_{\text{int}}^2}}{2}\right)\sum_{j=1}^{N}\hat{a}(j)\hat{\rho}(0)\hat{a}(j)\right.$$
$$\left.+\cdots+\left(\frac{1 - e^{-\lambda^2 t_{\text{int}}^2}}{2}\right)^N \hat{a}_N\hat{a}_{N-1}\cdots\hat{a}_1\hat{\rho}(0)\hat{a}_1\cdots\hat{a}_{N-1}\hat{a}_N\right]e^{i\hat{H}_0 t_{\text{int}}} \tag{A27}$$

$$=: e^{-i\hat{H}_0 t_{\text{int}}}\hat{\rho}_0' e^{i\hat{H}_0 t_{\text{int}}}, \tag{A28}$$

we have

$$\left|\frac{d\mathrm{Tr}(\hat{\rho}'\hat{\eta})}{d\omega}\right| = \left|\frac{d}{d\omega}\sum_{k=0}^{\infty}\frac{(i\omega t_{\mathrm{int}})^k}{k!}\mathrm{Tr}(\rho_0'[\hat{A},\hat{\eta}]_k)\right| \tag{A29}$$

$$\leqslant 2\|\hat{A}\|t_{\mathrm{int}}e^{2\omega t_{\mathrm{int}}\|\hat{A}\|}\|\rho_0'\| \tag{A30}$$

$$= 2\|\hat{A}\|t_{\mathrm{int}}e^{2\omega t_{\mathrm{int}}\|\hat{A}\|} \times \left\|\left(\frac{1+e^{-\lambda^2 t_{\mathrm{int}}^2}}{2}\right)^{N-1}\left(\frac{1-e^{-\lambda^2 t_{\mathrm{int}}^2}}{2}\right)\sum_{j=1}^{N}\hat{a}(j)\hat{\rho}(0)\hat{a}(j)\right.$$

$$\left. + \cdots + \left(\frac{1-e^{-\lambda^2 t_{\mathrm{int}}^2}}{2}\right)^{N}\hat{a}_N\hat{a}_{N-1}\cdots\hat{a}_1\hat{\rho}(0)\hat{a}_1\cdots\hat{a}_{N-1}\hat{a}_N\right\| \tag{A31}$$

$$= 2\|\hat{A}\|t_{\mathrm{int}}e^{2\omega t_{\mathrm{int}}\|\hat{A}\|}\left[\left(\frac{1+e^{-\lambda^2 t_{\mathrm{int}}^2}}{2}\right)^{N-1}\left(\frac{1-e^{-\lambda^2 t_{\mathrm{int}}^2}}{2}\right)\binom{N}{1} + \cdots + \left(\frac{1-e^{-\lambda^2 t_{\mathrm{int}}^2}}{2}\right)^{N}\binom{N}{N}\right] \tag{A32}$$

$$= 2\|\hat{A}\|t_{\mathrm{int}}e^{2\omega t_{\mathrm{int}}\|\hat{A}\|}\left[1-\left(\frac{1+e^{-\lambda^2 t_{\mathrm{int}}^2}}{2}\right)^{N}\right]. \tag{A33}$$

Here, we used the following formulas:

$$e^{i\omega\hat{A}t_{\mathrm{int}}}\hat{\eta}e^{-i\omega\hat{A}t_{\mathrm{int}}} = \sum_{k=0}^{\infty}\frac{(i\omega t_{\mathrm{int}})^k}{k!}[\hat{A},\hat{\eta}]_k, \tag{A34}$$

$$|\mathrm{Tr}(\hat{\rho}[\hat{A},\hat{\eta}]_k)| \leqslant 2^k\|\hat{A}\|^k. \tag{A35}$$

The derivation of (A34) is as follows:

$$e^{i\omega\hat{A}t}\hat{\eta}e^{-i\omega\hat{A}t} = \sum_{m,m',\nu,\nu'} e^{i\omega\hat{A}t}|m,\nu\rangle\langle m,\nu|\hat{\eta}|m',\nu'\rangle\langle m',\nu'|e^{-i\omega\hat{A}t} \tag{A36}$$

$$= \sum_{m,m',\nu,\nu'} e^{i\omega A_m t}|m,\nu\rangle\langle m,\nu|\hat{\eta}|m',\nu'\rangle\langle m',\nu'|e^{-i\omega A_{m'} t} \tag{A37}$$

$$= \sum_{m,m',\nu,\nu'} e^{i\omega(A_m-A_{m'})t}|m,\nu\rangle\langle m,\nu|\hat{\eta}|m',\nu'\rangle\langle m',\nu'|, \tag{A38}$$

$$\sum_{k=0}^{\infty}\frac{(i\omega t)^k}{k!}[\hat{A},\hat{\eta}]_k = \sum_{k=0}^{\infty}\frac{(i\omega t)^k}{k!}\sum_{m,m',\nu,\nu'}|m,\nu\rangle\langle m,\nu|[\hat{A},\hat{\eta}]_k|m',\nu'\rangle\langle m',\nu'| \tag{A39}$$

$$= \sum_{k=0}^{\infty}\frac{(i\omega t)^k}{k!}\sum_{m,m',\nu,\nu'}|m,\nu\rangle\sum_{k'=0}^{k}(-1)^{k'}\langle m,\nu|\hat{A}^{k-k'}\hat{\eta}\hat{A}^{k'}|m',\nu'\rangle\langle m',\nu'| \tag{A40}$$

$$= \sum_{k=0}^{\infty}\frac{(i\omega t)^k}{k!}\sum_{m,m',\nu,\nu'}|m,\nu\rangle\sum_{k'=0}^{k}(-1)^{k'}A_m^{k-k'}A_{m'}^{k'}\langle m,\nu|\hat{\eta}|m',\nu'\rangle\langle m',\nu'| \tag{A41}$$

$$= \sum_{k=0}^{\infty}\frac{(i\omega t)^k}{k!}\sum_{m,m',\nu,\nu'}(A_m-A_{m'})^k|m,\nu\rangle\langle m,\nu|\hat{\eta}|m',\nu'\rangle\langle m',\nu'| \tag{A42}$$

$$= \sum_{m,m',\nu,\nu'} e^{i\omega t(A_m-A_{m'})}|m,\nu\rangle\langle m,\nu|\hat{\eta}|m',\nu'\rangle\langle m',\nu'|, \tag{A43}$$

where $\hat{A}|m,\nu\rangle = A_m|m,\nu\rangle$ and ν labels the degeneracy.

So the denominator of the sensitivity is

$$\sqrt{T/t_{\mathrm{int}}}\left|\frac{d\mathrm{Tr}[\hat{\eta}\hat{\rho}(t_{\mathrm{int}})]}{d\omega}\right| \geqslant \sqrt{T/t_{\mathrm{int}}}\left\{\left|\frac{d\mathrm{Tr}[\hat{\eta}e^{-i\omega\hat{A}t_{\mathrm{int}}}\hat{\rho}(0)e^{i\omega\hat{A}t_{\mathrm{int}}}]}{d\omega}\right|\left(\frac{1+e^{-\lambda^2 t_{\mathrm{int}}^2}}{2}\right)^{N} - \left|\frac{d\mathrm{Tr}(\hat{\eta}\hat{\rho}')}{d\omega}\right|\right\} \tag{A44}$$

$$\geqslant \sqrt{T/t_{\mathrm{int}}}\left\{\left|\frac{dP}{d\omega}\right|\left(\frac{1+e^{-\lambda^2 t_{\mathrm{int}}^2}}{2}\right)^{N} - 2\|\hat{A}\|t_{\mathrm{int}}e^{2\omega t_{\mathrm{int}}\|\hat{A}\|}\left[1-\left(\frac{1+e^{-\lambda^2 t_{\mathrm{int}}^2}}{2}\right)^{N}\right]\right\}, \tag{A45}$$

where

$$\left|\frac{dP}{d\omega}\right| \geqslant \left|\left|\omega t_{\text{int}}^2 \text{Tr}\{\hat{\rho}(0)[\hat{A}, [\hat{A}, \hat{\eta}]]\}\right| - \left|it_{\text{int}} \text{Tr}\{\hat{\rho}(0)[\hat{A}, \hat{\eta}]\}\right|\right| - 2t_{\text{int}}\|\hat{A}\|(e^{2\omega t_{\text{int}}\|\hat{A}\|} - 1 - 2\omega t_{\text{int}}\|\hat{A}\|). \tag{A46}$$

Using the result of the case where there is no noise, we obtain (6)

$$\delta\omega_{\text{deph}}\sqrt{T} \leqslant (N\sqrt{t_{\text{int}}})^{-1}\left\{p_1 p_2^2 \left(\frac{1+e^{-2\lambda^2 t_{\text{int}}^2}}{2}\right)^N - 2e^{2\omega t_{\text{int}}\|\hat{A}\|}\frac{\|\hat{A}\|}{N}\left[1 - \left(\frac{1+e^{-2\lambda^2 t_{\text{int}}^2}}{2}\right)^N\right]\right\}^{-1}. \tag{A47}$$

APPENDIX B: THE SCALING OF THE UNCERTAINTY OF THE ESTIMATION

In the standard setup of the quantum metrology, generalized cat states always give the scalings either $\delta\omega = \Theta(N^{-3/4})$ with a finite dephasing rate or $\delta\omega = \Theta(N^{-1})$ with a zero dephasing rate. (For convenience, we express the uncertainty as $\delta\omega$ regardless of the existence of dephasing in this section.) We do not obtain the intermediate scaling such as $\delta\omega = \Theta(N^k)$ with $-1 < k < -3/4$ even with a small dephasing. In this section, we explain the reason by considering a GHZ state $\frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$ of N qubits as an example.

When we try to estimate ω of $\hat{H} = \sum_{j=1}^{N} \frac{\omega}{2}\hat{\sigma}_z^{(j)}$, we (1) prepare the GHZ state, (2) let the state evolve for time t_{int}, (3) read out, and (4) repeat from (1) to (3) for T/t_{int} times (assuming the state preparation and the readout are done instantaneously). Here, T is the total measurement time which we can freely fix at some finite value. In the presence of non-Markovian dephasing, the uncertainty $\delta\omega$ is calculated as [20]

$$\delta\omega = \frac{e^{\frac{N t_{\text{int}}^2}{(T_2)^2}}}{N\sqrt{T t_{\text{int}}}}, \tag{B1}$$

where T_2 is the coherence time of a single qubit determined by the physical system. Our aim is to minimize $\delta\omega\sqrt{T}$ by tuning t_{int}, and to see how it scales with N.

For finite T_2, $\delta\omega\sqrt{T}$ has the minimum value $\frac{\sqrt{2}\exp(1/4)}{N^{3/4}\sqrt{T_2}}$ at $t_{\text{int}} = T_2/2\sqrt{N}$. As we can see from Fig. 2, the minimum value of $\delta\omega\sqrt{T}$ moves to the right as T_2 increases. In the limit of no dephasing, i.e., $T_2 \to \infty$, $\delta\omega\sqrt{T}$ no longer has a minimum value. Instead, we find $\delta\omega\sqrt{T} \to \frac{1}{N\sqrt{t_{\text{int}}}}$, which scales as N^{-1} for $t_{\text{int}} = \Theta(N^0)$.

The intuitive reason why $\delta\omega\sqrt{T}$ has a minimum value with $T_2 < \infty$ is that while larger t_{int} gives more phase accumulation, which contributes to a better sensitivity, the amplitude of the state maintaining useful coherence for sensing diminishes with the increase of t_{int} because of the noise. When there is no noise, on the other hand, the latter does not occur. Hence the sensitivity keeps improving with the increase of t_{int} when $T_2 \to \infty$.

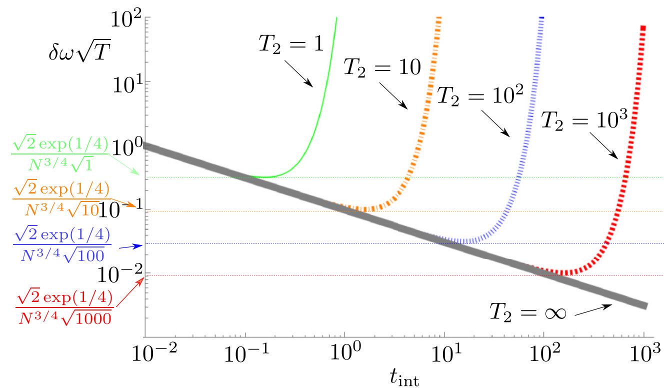

FIG. 2. Log-log plot of $\delta\omega\sqrt{T}$ against t_{int} for $N = 10$. From the left, green, orange (dot-dashed), blue (dotted), and red (dashed) curves correspond to $T_2 = 1$, $T_2 = 10$, $T_2 = 10^2$, $T_2 = 10^3$, respectively. The gray (thick) line corresponds to $T_2 \to \infty$. The minimum value varies in accordance with T_2, but it always scales as $N^{-3/4}$ as long as T_2 is finite. However, when $T_2 \to \infty$, $\delta\omega\sqrt{T}$ takes the form $1/N\sqrt{t_{\text{int}}}$ and keeps decreasing (without minimum values), giving another scaling N^{-1} for the optimal uncertainty.

Although we describe the case with the GHZ state as an example, the same conclusion can be drawn with the field sensor with the generalized cat states.

Therefore, for the reason described above, we do not obtain the intermediate scaling such as $\delta\omega = \Theta(N^k)$ with $-1 < k < -3/4$.

APPENDIX C: CONSTRUCTION OF $\hat{\eta}$

In this section, we explain how to judge whether a given state is useful in metrology and show how to construct a projection $\hat{\eta}$ for a given state. For an arbitrary $\hat{\rho}$, we can judge whether it is helpful in sensing ω of $\omega\hat{A}$ as follows: Find the eigenvalue and eigenstate of $[\hat{A}, [\hat{A}, \hat{\rho}]]$. If the sum of the positive eigenvalues is $\Theta(N^2)$, then it is a generalized cat state of $\hat{A}$, i.e., there exists a projection operator satisfying $\text{Tr}\{\hat{\rho}[\hat{A}, [\hat{A}, \hat{\eta}]]\} = \Theta(N^2)$.

The projection operator $\hat{\eta}$ for the Ramsey-type measurement with the ultimate scaling can be constructed using the eigenstates:

$$\hat{\eta} = \sum_{e_n > 0} |n\rangle\langle n|, \tag{C1}$$

where $\hat{\rho}|n\rangle = e_n|n\rangle$.

Let us give an example. Let $|\psi_\lambda\rangle$ be the following state similar to the Schrödinger's cat state, but differs by the λth spin,

$$|\psi_\lambda\rangle := \frac{1}{\sqrt{2}}|\downarrow\rangle^{\otimes(\lambda-1)}|\uparrow\rangle|\downarrow\rangle^{\otimes(N-\lambda)} + \frac{1}{\sqrt{2}}|\uparrow\rangle^{\otimes(\lambda-1)}|\downarrow\rangle|\uparrow\rangle^{\otimes(N-\lambda)} \quad (\lambda = 1, 2, \ldots, N). \tag{C2}$$

Then, let $\hat{\rho}_{ex}$ be a mixed state of $|\psi_\lambda\rangle$'s:

$$\hat{\rho}_{\text{ex}} := \frac{1}{N}\sum_{\lambda=1}^{N} |\psi_\lambda\rangle\langle\psi_\lambda|. \tag{C3}$$

The eigenstates with positive eigenvalues of $[\hat{M}_z, [\hat{M}_z, \hat{\rho}_{\rm ex}]]$ are $|\psi_\lambda\rangle$'s, and the sum of the eigenvalues is $2(N-2)^2$. Hence the mixed state $\hat{\rho}_{ex}$ can be proven to achieve the ultimate scaling in measuring $\hat{M}_z$ with a projection $\hat{\eta} = N\hat{\rho}$ after Ramsey-type protocol.

APPENDIX D: DERIVATION OF UPPER BOUND

A numerical upper bound of $\delta\omega_{\rm deph}\sqrt{T}$ is obtained through calculating $\mathrm{Tr}(\hat{\rho}[\hat{A}, \hat{\eta}])$ and $\mathrm{Tr}(\hat{\rho}[\hat{A}, [\hat{A}, \hat{\eta}]])$ numerically, and then minimizing

$$(N\sqrt{t_{\rm int}})^{-1}\left\{U\left(\frac{1+e^{-2\lambda^2 t_{\rm int}^2}}{2}\right)^N - 2e^{2\omega t_{\rm int}\|\hat{A}\|}\left[1-\left(\frac{1+e^{-2\lambda^2 t_{\rm int}^2}}{2}\right)^N\right]\right\}^{-1} \tag{D1}$$

by tuning $t_{\rm int}$, where

$$U := \left|\frac{|\omega t_{\rm int}\mathrm{Tr}(\hat{\rho}[\hat{A}, [\hat{A}, \hat{\eta}]])|}{N} - \frac{|i\mathrm{Tr}(\hat{\rho}[\hat{A}, \hat{\eta}])|}{N}\right| - 2\frac{\|\hat{A}\|}{N}(e^{2\omega t_{\rm int}\|\hat{A}\|} - 1 - 2\omega t_{\rm int}\|\hat{A}\|). \tag{D2}$$

We then find $t_{\rm int} \propto 1/\sqrt{N}$ gives the optimal uncertainty $\delta\omega_{\rm deph} = \Theta(N^{3/4})$.

APPENDIX E: RELATION BETWEEN QFI AND q

We would also like to comment that we revealed the unknown general relation between QFI and q. Fröwis and Dür claim that the QFI can characterize the macroscopicity of quantum states [47,51]; if the QFI is of the order of $\Theta(N^2)$, they consider the quantum state as macroscopic. The relationship between QFI and q for general mixed states was an open question. Here, we showed $1/\sqrt{\mathrm{QFI}} \leqslant \delta\omega \leqslant \Theta(N^{-1})$ for $q = 2$ states, assuring the lower bound of QFI to be large. Connecting two criteria defined from different aspects, our results contribute to the further understanding of physics.

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