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Optimal verification of two-qubit pure states

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In a recent work [Pallister, Linden, and Montanaro, Phys. Rev. Lett. **120**, 170502 (2018)], Pallister *et al.* proposed an optimal strategy to verify nonmaximally entangled two-qubit pure states under the constraint that the accessible measurements are locally projective and nonadaptive. Their good result leads naturally to the following question: What is the optimal strategy among general local operations and classical communication (LOCC) measurements? In this paper, we answer this problem completely for two-qubit pure states. To be specific, we give the optimal strategy for each of the following available classes of measurements: (i) local operations and one-way classical communication (one-way LOCC) measurements; (ii) local operations and two-way classical communication (two-way LOCC) measurements; and (iii) separable measurements. Surprisingly, our results reveal that for the two-qubit pure state verification problem two-way LOCC measurements remarkably outperform one-way LOCC measurements and have the same power as the separable measurements.

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I. INTRODUCTION

On the way to the quantum era, quantum devices for generating particular states have been extensively studied and widely used [1–4]. As such, it becomes necessary to verify that these devices truly work as they are specified reliably and efficiently with measurements that are accessible. A standard approach is to estimate the output states with quantum state tomography [5–12]. However, this method is both time consuming and computationally difficult; even verifying a few-qubit photonic state is already experimentally challenging [13,14]. State discrimination can be used only when the true state is restricted to a limited set of known candidates [15], which is far from the practical verification setting. State detection [16] and state estimation are not applicable for this purpose [17,18], either. Only state verification can guarantee the quality of the generated state in the practical setting. Various studies have been designed for this task [19–23], using only local measurements. Though these methods achieve considerable efficiency, no optimal method except for the maximally entangled state [24–27] is known so far although the optimality gives the ultimate performance for this problem.

Given the intrinsic difficulty in state verification, in this paper we focus on verifying the nonmaximally entangled two-qubit pure states, in hopes of gaining deeper understanding of the verification problem. In fact, nonmaximally entangled states are generally easier to prepare experimentally and are still useful for various quantum protocols. Moreover, pure nonmaximally entangled states are more useful for several purposes in quantum information theory than maximally entangled states (e.g., detection of nonlocality [28–30]), making their verification important both from the theoretical and the experimental points of view. Importantly, we construct optimal strategies when different classes of measurements are available: one-way local operations and classical communication (LOCC) measurements, two-way LOCC measurements, and separable measurements. We find that for the problem under consideration two-way LOCC measurements achieve the same performance as separable measurements, while outperforming one-way LOCC measurements dramatically.

Before presenting the results, we review the notations. We denote by $|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle \equiv (|0\rangle - |1\rangle)/\sqrt{2}$ the eigenstates of the Pauli X operator, and we denote by $|\top\rangle \equiv (|0\rangle + i|1\rangle)/\sqrt{2}$ and $|\bot\rangle \equiv (|0\rangle - i|1\rangle)/\sqrt{2}$ the eigenstates of the Pauli Y operator. When measuring a qubit with a Pauli operator, the outcome is written as $(-1)^i$ where $i \in \{0, 1\}$. We denote by $\mathcal{H}$ the two-qubit composite system and by $\mathbb{1}$ the identity operator on $\mathcal{H}$. We call a positive operator T with $0 \leqslant T \leqslant \mathbb{1}$ a one-way LOCC (local operations and only one-way classical communication) positive operator-valued measure (POVM) element on $\mathcal{H}$ if the two-outcome POVM $\{T, \mathbb{1} - T\}$ can be implemented by one-way LOCC. We also define a two-way LOCC (local operations and two-way classical communication) POVM element and a separable POVM element in the same way by using the two-way LOCC and the separable operations, respectively. Interested readers might refer to [31] for details on these operations. We write the set of one-way LOCC from Alice to Bob, one-way LOCC from Bob to Alice, two-way LOCC, and separable POVM elements as $\mathcal{T}_{\rightarrow}$, $\mathcal{T}_{\leftarrow}$, $\mathcal{T}_{\leftrightarrow}$, and $\mathcal{T}_{\text{sep}}$. These classes satisfy the relation $\mathcal{T}_{\rightarrow}(\mathcal{T}_{\leftarrow}) \subseteq \mathcal{T}_{\leftrightarrow} \subseteq \mathcal{T}_{\text{sep}}$. The condition $T \in \mathcal{T}_c$ is equivalent to the condition $\mathbb{1} - T \in \mathcal{T}_c$, where $c \in \{\rightarrow, \leftarrow, \leftrightarrow, \text{sep}\}$. For a positive operator Ω on $\mathcal{H}$, $\lambda_i(\Omega)$ denotes the ith eigenvalue of

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Ω and $\lambda_i^{\downarrow}(\Omega)$ denotes the ith largest eigenvalue of Ω, where $i = 1, 2, 3, 4$.

II. TWO-QUBIT PURE STATE VERIFICATION

Consider a quantum device that is designed to produce the two-qubit pure state

$$|\Psi\rangle = \sqrt{1-\lambda}|00\rangle + \sqrt{\lambda}|11\rangle, \tag{1}$$

where $\lambda \in [0, 1/2]$. However, it might work incorrectly and actually output states $\sigma_1, \sigma_2, \cdots, \sigma_N$ in N runs. It is guaranteed that the fidelity $\langle\Psi|\sigma_j|\Psi\rangle$ either is 1 or satisfies $\langle\Psi|\sigma_j|\Psi\rangle \leqslant 1-\epsilon$ for all j for some $\epsilon > 0$. The task is to determine which is the case. The conclusion is useful if we assume the next state σ_{N+1} has the same behavior as the previous ones.

To achieve this task, we perform two-outcome measurements from a set of accessible measurements to test the state. Each two-outcome measurement $\{T_l, 1 - T_l\}$ is specified by an operator T_l, which corresponds to passing the test, and is performed with probability p_l. We require that the target state $|\Psi\rangle$ always passes the test, that is, $T_l|\Psi\rangle = |\Psi\rangle$ for all T_l. In the bad case, the maximal probability that σ_j passes the test is given by [23,32]

$$\max_{\langle\Psi|\sigma_j|\Psi\rangle\leqslant 1-\epsilon} \mathrm{Tr}(\Omega\sigma_j) = 1 - [1-\lambda_2^{\downarrow}(\Omega)]\epsilon,$$

where $\Omega = \sum_l p_l T_l$ is called a strategy. After N runs, σ_j in the bad case can pass all tests with probability at most $[1-[1-\lambda_2^{\downarrow}(\Omega)]\epsilon]^N$. Hence to achieve confidence $1-\delta$, it suffices to have [32]

$$N \geqslant \frac{\ln\delta}{\ln[1-[1-\lambda_2^{\downarrow}(\Omega)]\epsilon]} \approx \frac{1}{[1-\lambda_2^{\downarrow}(\Omega)]\epsilon}\ln\frac{1}{\delta}. \tag{2}$$

The optimal strategy is obtained by minimizing the second largest eigenvalue $\lambda_2^{\downarrow}(\Omega)$. If there is no restriction on the accessible measurements, the optimal strategy is given by the measurement $\{|\Psi\rangle\langle\Psi|, \mathbb{1} - |\Psi\rangle\langle\Psi|\}$, under which $\Omega = |\Psi\rangle\langle\Psi|$, $\lambda_2^{\downarrow}(\Omega) = 0$, and $N \approx \epsilon^{-1}\ln\delta^{-1}$. This efficiency cannot be improved if collective measurements are allowed [23]. However, it is difficult to perform such measurements experimentally when $|\Psi\rangle$ is entangled. It is thus meaningful to devise efficient (or even optimal) strategies based on measurements satisfying reasonable constraints. Owari and Hayashi [33] studied the case where the incorrect states are the maximally mixed state, with the target to minimize the trace of Ω. They derived optimal strategies when one-way LOCC and separable measurements are available, and showed that two-way LOCC measurements remarkably improve the performance compared to one-way LOCC measurements. Recently, Pallister, Linden, and Montanaro [32] proposed an optimal strategy Ω_{PLM} to verify $|\Psi\rangle$, under the constraint that the accessible measurements must be locally projective and nonadaptive. The strategy Ω_{PLM} [34] has the second largest eigenvalue:

$$\lambda_2^{\downarrow}(\Omega_{\mathrm{PLM}}) = \frac{2+2\sqrt{\lambda(1-\lambda)}}{4+2\sqrt{\lambda(1-\lambda)}}. \tag{3}$$

Note that the set of accessible measurements in [32] forms a strict subset of $\mathcal{T}_{\rightarrow}$.

These interesting results lead to the following question: What is the optimal strategy when general LOCC measurements, i.e., adaptive choices of local measurements, are available? In this paper, we investigate this problem comprehensively. We derive optimal strategies for verifying $|\Psi\rangle$ when the following different classes of measurements are available: $\mathcal{T}_{\rightarrow}$, $\mathcal{T}_{\leftarrow}$, $\mathcal{T}_{\leftrightarrow}$, and $\mathcal{T}_{\mathrm{sep}}$. In the following, we say a strategy Ω is in $\mathcal{T}_c$ and written $\Omega \in \mathcal{T}_c$ if $\Omega = \sum_l p_l T_l$ and $T_l \in \mathcal{T}_c$ for all l, and a strategy Ω is *optimal* in $\mathcal{T}_c$ if $\Omega \in \mathcal{T}_c$ and for arbitrary $\Omega' \in \mathcal{T}_c$ satisfying $\Omega'|\Psi\rangle = |\Psi\rangle$, $\lambda_2^{\downarrow}(\Omega) \leqslant \lambda_2^{\downarrow}(\Omega')$.

Here we discuss some general properties of arbitrary strategy Ω. Consider the product of local unitaries $U_\theta \otimes U_{-\theta}$, where $U_\theta = |0\rangle\langle 0| + e^{i\theta}|1\rangle\langle 1|$ and $\theta \in [0, 2\pi]$. Let $|\Psi^\perp\rangle = \sqrt{\lambda}|00\rangle - \sqrt{1-\lambda}|11\rangle$, then $\{|\Psi\rangle, |\Psi^\perp\rangle, |01\rangle, |10\rangle\}$ are the four eigenstates of $U_\theta \otimes U_{-\theta}$. Using this property we can simplify the form of Ω by *averaging*, where the averaged strategy is defined as

$$\Omega_a := \frac{1}{2\pi}\int_0^{2\pi} (U_\theta \otimes U_{-\theta})\Omega(U_\theta \otimes U_{-\theta})^\dagger d\theta.$$

Since the second largest eigenvalues of Ω_a and Ω are the matrix norms of $P^\perp\Omega_a P^\perp$ and $P^\perp\Omega P^\perp$ with $P^\perp := \mathbb{1} - |\Psi\rangle\langle\Psi|$, we have $\lambda_2^{\downarrow}(\Omega_a) \leqslant \lambda_2^{\downarrow}(\Omega)$. That is, averaging over θ cannot make the strategy worse. As $\mathbb{1} \geqslant \Omega_a \geqslant |\Psi\rangle\langle\Psi|$ and the vectors $|01\rangle$ and $|10\rangle$ of $U_\theta \otimes U_{-\theta}$ have different eigenvalues from those of $|\Psi\rangle$ and $|\Psi^\perp\rangle$, after averaging Ω_a can be expressed as

$$\Omega_a = |\Psi\rangle\langle\Psi| + \lambda_2|\Psi^\perp\rangle\langle\Psi^\perp| + \lambda_3|01\rangle\langle 01| + \lambda_4|10\rangle\langle 10| \tag{4}$$

for some $\lambda_2, \lambda_3, \lambda_4 \in [0, 1)$. We also consider the fact that $|\Psi\rangle$ is invariant under qubit swapping. Using the swapping operation s for the roles of Alice and Bob, the resulting strategy $\overline{\Omega}_a := \frac{1}{2}\Omega_a + \frac{1}{2}s(\Omega_a)$ has performance at least as good as that of Ω_a and admits the form

$$\overline{\Omega}_a = |\Psi\rangle\langle\Psi| + \lambda_2|\Psi^\perp\rangle\langle\Psi^\perp| + \lambda_3(|01\rangle\langle 01| + |10\rangle\langle 10|) \tag{5}$$

for some $\lambda_2, \lambda_3 \in [0, 1)$. We should be careful when using the swapping invariance property, as to implement the strategy $\overline{\Omega}_a$ an extra step of messaging is required.

The above-discussed framework is nonadversarial in the sense that the malicious device produces incorrect states randomly and independently. One may also consider the adversarial scenario where the malicious device may produce an arbitrary state ρ on the whole system $\mathcal{H}^{N+1}$. The task is then to ensure that the reduced state on one system has fidelity larger than $1-\epsilon$ by performing N tests on other systems. We remark that minimizing the second largest eigenvalue leads to the optimization of the strategy even in this scenario [23, Sec. III.E].

III. STRATEGY USING ONE-WAY LOCC MEASUREMENTS

First we propose a strategy in $\mathcal{T}_{\rightarrow}$. Then we show it is optimal when only $\mathcal{T}_{\rightarrow}$ are available.

Let $|v_\pm\rangle = \sqrt{1-\lambda}|0\rangle \pm \sqrt{\lambda}|1\rangle$. Alice performs the X measurement on the target state and sends outcome i to Bob.

If $i = 0$, Bob performs measurement $\{|v_+\rangle\langle v_+|, \mathbb{1} - |v_+\rangle\langle v_+|\}$ and accepts if the outcome is v_+. If $i = 1$, Bob performs measurement $\{|v_-\rangle\langle v_-|, \mathbb{1} - |v_-\rangle\langle v_-|\}$ and accepts if the outcome is v_-. The corresponding POVM element T_1 (passing the test) has the form

$$T_1 = |+\rangle\langle+| \otimes |v_+\rangle\langle v_+| + |-\rangle\langle-| \otimes |v_-\rangle\langle v_-|.$$

We define two other POVM elements T_2 and T_3 similarly to T_1 but with the X measurement replaced by the Y and Z measurements on Alice's side, respectively. These two elements read

$$\begin{aligned} T_2 &= |\top\rangle\langle\top| \otimes |w_-\rangle\langle w_-| + |\bot\rangle\langle\bot| \otimes |w_+\rangle\langle w_+|, \\ T_3 &= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|, \end{aligned} \tag{6}$$

where $|w_\pm\rangle = \sqrt{1-\lambda}|0\rangle \pm i\sqrt{\lambda}|1\rangle$. It holds that $T_j|\Psi\rangle = |\Psi\rangle$ and $T_j \in \mathcal{T}_\rightarrow$ for $j = 1, 2, 3$.

The one-way strategy goes as follows. In each round, Alice chooses a measurement from $\{T_1, T_2, T_3\}$ with *a priori* probability $\{\frac{1-p}{2}, \frac{1-p}{2}, p\}$ to test the state, where $p \in [0, 1]$ is a free parameter. The strategy has the form

$$\begin{aligned} \Omega_\rightarrow &= \frac{1-p}{2}T_1 + \frac{1-p}{2}T_2 + pT_3 \\ &= |\Psi\rangle\langle\Psi| + p|\Psi^\perp\rangle\langle\Psi^\perp| + (1-p)\lambda|01\rangle\langle 01| \\ &\quad + (1-p)(1-\lambda)|10\rangle\langle 10|. \end{aligned}$$

Minimizing $\lambda_2^\downarrow(\Omega_\rightarrow)$ with respect to $p \in [0, 1]$, we get $p = \frac{1-\lambda}{2-\lambda}$ and

$$\begin{aligned} \Omega_\rightarrow = {} & |\Psi\rangle\langle\Psi| + \lambda_2^\rightarrow|\Psi^\perp\rangle\langle\Psi^\perp| + \lambda_3^\rightarrow|01\rangle\langle 01| \\ & + \lambda_2^\rightarrow|10\rangle\langle 10|, \end{aligned}$$

where $\lambda_2^\rightarrow = \frac{1-\lambda}{2-\lambda}$ and $\lambda_3^\rightarrow = \frac{\lambda}{2-\lambda}$. Obviously, $\Omega_\rightarrow \in \mathcal{T}_\rightarrow$.

Now we show the optimality of $\Omega_\rightarrow$. Let $|t, s\rangle := \sqrt{t}|0\rangle + e^{is}\sqrt{1-t}|1\rangle$, where $t \in [0, 1]$ and $s \in [0, 2\pi]$. When a one-way LOCC strategy Ω detects $|\Psi\rangle$ with certainty, the strategy is composed of Alice's POVM $\int 2|t, s\rangle\langle t, s|P_{TS}(dtds)$ with some probability distribution P_{TS} and Bob's two-outcome measurements $\{|t, s, B\rangle\langle t, s, B|, \mathbb{1} - |t, s, B\rangle\langle t, s, B|\}$, where $|t, s, B\rangle$ is the normalized vector of $\sqrt{t(1-\lambda)}|0\rangle + e^{-is}\sqrt{(1-t)\lambda}|1\rangle$. Then, the strategy Ω is written as

$$\Omega = 2\int |t, s\rangle\langle t, s| \otimes |t, s, B\rangle\langle t, s, B|P_{TS}(dtds). \tag{7}$$

Following the averaging argument in Eq. (4), we get Ω_a, obtained from Ω. For the analysis of Ω_a, we treat the variable t in Eq. (7) as the random variable T subject to P_{TS}, and focus on the expectation $\mathbb{E}_T$ under the marginal distribution P_T. To guarantee that Alice's measurement in Ω_a is a POVM, $\mathbb{E}_T[T] = \frac{1}{2}$ needs to hold. In Appendix A we show that Ω_a satisfies Eq. (4) with

$$\lambda_2(\Omega_a) = 1 - \Xi, \lambda_3(\Omega_a) = \Xi\lambda, \lambda_4(\Omega_a) = \Xi(1-\lambda),$$

where $\Xi := 2\mathbb{E}_T[\frac{T(1-T)}{T+\lambda-2\lambda T}] \geqslant 0$. As $\lambda_3(\Omega_a) \leqslant \lambda_4(\Omega_a)$, $\lambda_2^\downarrow(\Omega_a)$ is minimized when $\lambda_2(\Omega_a) = \lambda_4(\Omega_a)$. Solving the equation, we get $\Xi = \frac{1}{2-\lambda}$ and $\lambda_2^\downarrow(\Omega_a) = \frac{1-\lambda}{2-\lambda}$. This concludes the optimality of $\Omega_\rightarrow$.

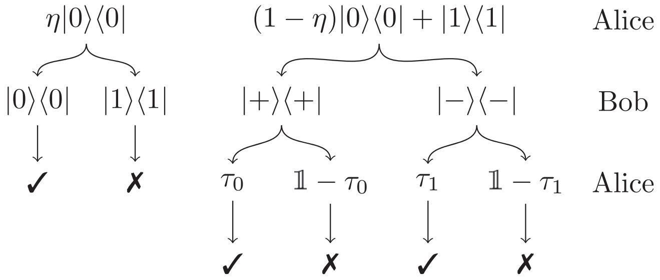

FIG. 1. The two-way measurement $\{T_1^{A\rightarrow B}, \mathbb{1} - T_1^{A\rightarrow B}\}$. Alice first performs measurement $\{\delta|0\rangle\langle 0|, (1-\delta)|0\rangle\langle 0| + |1\rangle\langle 1|\}$ and sends the outcome to Bob. Conditioned on the outcome, Bob adopts different measurements on his postmeasurement state and sends the outcome to Alice if necessary. Alice then performs the corresponding two-outcome measurement $\{\tau_j, \mathbb{1} - \tau_j\}$ to detect the final state she holds.

Switching the role between Alice and Bob, we get a symmetric version $\Omega_\leftarrow$ of $\Omega_\rightarrow$. Consider the new strategy $\widehat{\Omega}_\leftrightarrow = (\Omega_\rightarrow + \Omega_\leftarrow)/2$. Minimizing the second largest eigenvalue of $\widehat{\Omega}_\leftrightarrow$ with respect to p gives

$$\widehat{\Omega}_\leftrightarrow = |\Psi\rangle\langle\Psi| + \tfrac{1}{3}(\mathbb{1} - |\Psi\rangle\langle\Psi|). \tag{8}$$

This two-way two-step strategy outperforms $\Omega_\rightarrow$ in the small regime of λ. More details on $\widehat{\Omega}_\leftrightarrow$ can be found in Appendix B.

IV. STRATEGY USING TWO-WAY LOCC MEASUREMENTS

First we describe two measurements both detecting $|\Psi\rangle$ correctly. They are inspired by the two-way LOCC test given in [33]. Then we show that an appropriate convex combination of these measurements achieves optimality even if separable measurements are available. In what follows, we assume $\eta = 1 - \sqrt{\frac{\lambda}{1-\lambda}}$ and $p = \frac{\lambda}{1+\sqrt{\lambda(1-\lambda)}}$.

Consider the following measurement procedure.

(1) Alice performs measurement $\{M_0 \equiv \eta|0\rangle\langle 0|, M_1 \equiv (1-\eta)|0\rangle\langle 0| + |1\rangle\langle 1|\}$ and sends the measurement outcome M_i to Bob.

(2) Conditioning on i, Bob does the following. If $i = 0$, Bob performs Z measurement and accepts when the outcome is zero. If $i = 1$, Bob performs X measurement and sends outcome $j \in \{0, 1\}$ to Alice.

(3) Conditioning on j, Alice performs measurement $\{\tau_j, \mathbb{1} - \tau_j\}$ to check the state she holds, where τ_j is the postmeasurement state on Alice's system when the input state is $|\Psi\rangle$. If she detects τ_j, she accepts.

The corresponding POVM element $T_1^{A\rightarrow B}$ (passing the test) has the form

$$\begin{aligned} T_1^{A\rightarrow B} = {} & \eta|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |\widetilde{v}_+\rangle\langle\widetilde{v}_+| \otimes |+\rangle\langle +| \\ & + |\widetilde{v}_-\rangle\langle\widetilde{v}_-| \otimes |-\rangle\langle -|, \end{aligned}$$

where $|\widetilde{v}_\pm\rangle = \sqrt{(1-\eta)A}|0\rangle \pm \sqrt{B}|1\rangle$, $A := \frac{(1-\lambda)(1-\eta)}{1-\eta+\lambda\eta}$, and $B := \frac{\lambda}{1-\eta+\lambda\eta}$. Note that $|\widetilde{v}_\pm\rangle$ are not normalized. See Fig. 1 for illustration of this measurement. Note that $T_1^{A\rightarrow B}|\Psi\rangle = |\Psi\rangle$ and $T_1^{A\rightarrow B} \in \mathcal{T}_\leftrightarrow$. The superscript $A \rightarrow B$ of T_1 indicates that T_1 begins with Alice sending the outcome to Bob. A symmetric element $T_1^{B\to A}$ is obtained by switching the role between Alice and Bob. We define another POVM element $T_2^{A\to B}$ analogous to $T_1^{A\to B}$ but with the X measurement replaced by the Y measurement on Bob's side, which reads

$$\begin{aligned}T_2^{A\to B} = &\eta|0\rangle\langle 0|\otimes|0\rangle\langle 0| + |\widetilde{w}_-\rangle\langle\widetilde{w}_-|\otimes|\top\rangle\langle\top| \\ &+ |\widetilde{w}_+\rangle\langle\widetilde{w}_+|\otimes|\bot\rangle\langle\bot|,\end{aligned}$$

where $|\widetilde{w}_\pm\rangle = \sqrt{(1-\eta)A}|0\rangle \pm i\sqrt{B}|1\rangle$. By construction, $T_2^{A\to B}|\Psi\rangle = |\Psi\rangle$ and $T_2^{A\to B}\in\mathcal{T}_\leftrightarrow$.

Our two-way strategy is given by the following procedure. In each round, Alice chooses a measurement from $\{T_1^{A\to B}, T_2^{A\to B}, T_1^{B\to A}, T_2^{B\to A}, T_3\}$ with *a priori* distribution $\{\frac{1-p}{4}, \frac{1-p}{4}, \frac{1-p}{4}, \frac{1-p}{4}, p\}$ to verify the state, where T_3 is defined in Eq. (6). If $T_i^{A\to B}$ is chosen, Alice executes the measurement; if $T_i^{B\to A}$ is chosen, Alice sends notification to Bob to ask Bob to execute the measurement. The corresponding strategy is given by

$$\begin{aligned}\Omega_\leftrightarrow &= \frac{p-1}{4}\left(T_1^{A\to B} + T_2^{A\to B} + T_1^{B\to A} + T_2^{B\to A}\right) + pT_3 \\ &= |\Psi\rangle\langle\Psi| + \lambda^*(\mathbb{1} - |\Psi\rangle\langle\Psi|),\end{aligned} \tag{9}$$

where

$$\lambda^* = \frac{\sqrt{\lambda(1-\lambda)}}{1+\sqrt{\lambda(1-\lambda)}}.$$

By construction, $\Omega_\leftrightarrow|\Psi\rangle = |\Psi\rangle$ and $\Omega_\leftrightarrow\in\mathcal{T}_\leftrightarrow$. In Appendix C, we show how the magic values of η and p are chosen.

Our strategy $\Omega_\leftrightarrow$ can be implemented by two-way LOCC, using up to three-step classical communication. This makes it possible for experimental implementation. When $\lambda = 0$, $|\Psi\rangle = |00\rangle$, the optimal strategy of which is provably given by the measurement $\{|00\rangle\langle 00|, \mathbb{1} - |00\rangle\langle 00|\}$. Our strategy $\Omega_\leftrightarrow$ reduces exactly to this optimal measurement when $\lambda = 0$, which means our two-way strategy is globally optimal for $|00\rangle$. However, all other strategies – $\Omega_{\rm PLM}$, $\Omega_\rightarrow$, $\Omega_\leftarrow$, and $\widehat{\Omega}_\leftrightarrow$ – do not share this property.

Now we show the optimality of $\Omega_\leftrightarrow$ among strategies using separable measurements. Since a two-way LOCC measurement is a separable measurement, this optimality also shows the optimality using two-way LOCC measurements. The proof is divided into two parts: first we prove all optimal strategies in $\mathcal{T}_{\rm sep}$ are homogeneous, then we construct explicitly an optimal homogeneous strategy in $\mathcal{T}_{\rm sep}$.

A strategy Ω for $|\Psi\rangle$ is *homogeneous* if it has the form

$$\Omega = |\Psi\rangle\langle\Psi| + \kappa(\mathbb{1} - |\Psi\rangle\langle\Psi|), \tag{10}$$

where $\kappa\in[0,1]$. As examples, the strategies $\widehat{\Omega}_\leftrightarrow$ and $\Omega_\leftrightarrow$ are homogeneous. It turns out that the optimal strategies using separable measurements are always homogeneous. Following the arguments in Eq. (5), we know optimal strategies Ω in $\mathcal{T}_{\rm sep}$ can always be written as Eq. (5) for some $\lambda_2, \lambda_3\in[0,1)$. Assuming on the contrary $\lambda_2 \neq \lambda_3$, we then construct homogeneous strategies with smaller second largest eigenvalues than that of Ω, which in turn violates the optimality of Ω. In Theorem 1 of [33], the authors proposed a separable test of the form

$$T_4 = |\Psi\rangle\langle\Psi| + \sqrt{\lambda(1-\lambda)}(|01\rangle\langle 01| + |10\rangle\langle 10|).$$

In case $\lambda_2 > \lambda_3$, we consider a convex combination between Ω and T_4 such that the combination is homogeneous. The new strategy has a smaller second largest eigenvalue than that of Ω. We can show the opposite case in the same way using T_3 defined in Eq. (6) instead of T_4.

We are left to derive an optimal homogeneous strategy in $\mathcal{T}_{\rm sep}$. We are actually interested in the following optimization problem:

$$\begin{aligned}&\min\kappa \\ &\text{such that } 0\leqslant\kappa\leqslant 1,\ \Omega = |\Psi\rangle\langle\Psi| + \kappa(\mathbb{1} - |\Psi\rangle\langle\Psi|),\ \Omega\in\mathcal{T}_{\rm sep}.\end{aligned}$$

As the separability condition is equivalent to the positive partial transpose condition for two-qubit operators [35,36], this problem can be analytically solved. Denote by Ω^{T_B} the partial transpose of Ω on system B. The eigenvalues of Ω^{T_B} are

$$\begin{aligned}&\lambda_1 = 1-\lambda+\lambda\kappa,\quad \lambda_2 = \lambda+\kappa-\lambda\kappa, \\ &\lambda_3 = \kappa + (1-\kappa)\sqrt{\lambda(1-\lambda)},\quad \lambda_4 = \kappa - (1-\kappa)\sqrt{\lambda(1-\lambda)}.\end{aligned}$$

As $\lambda_1, \lambda_2, \lambda_3 \geqslant 0$ for $\lambda\in[0,1/2]$ and $\kappa\in[0,1]$, the condition $\Omega^{T_B}\geqslant 0$ is then equivalent to $\lambda_4\geqslant 0$, resulting in

$$\kappa \geqslant \kappa^* := \frac{\sqrt{\lambda(1-\lambda)}}{1+\sqrt{\lambda(1-\lambda)}}.$$

The optimal homogeneous strategy then has the form

$$\Omega_{\rm sep} = |\Psi\rangle\langle\Psi| + \kappa^*(\mathbb{1} - |\Psi\rangle\langle\Psi|).$$

Together with the fact that optimal strategies in $\mathcal{T}_{\rm sep}$ are always homogeneous, we completely solve the problem of verifying $|\Psi\rangle$ using separable measurements. Moreover, as $\lambda_2^\downarrow(\Omega_\leftrightarrow) = \lambda_2^\downarrow(\Omega_{\rm sep})$, the optimality can be achieved by two-way LOCC measurements.

V. COMPARISON

In Fig. 2, we compare the number of measurements required to verify $|\Psi\rangle$ within $\epsilon = 0.01$ and $\delta = 0.1$ using the four described strategies: $\Omega_{\rm PLM}$, $\Omega_\rightarrow$, $\widehat{\Omega}_\leftrightarrow$, and $\Omega_\leftrightarrow$, as a function of λ, which is the Schmidt coefficient of $|\Psi\rangle$. The number of measurements is computed using Eq. (2). One can see that our proposed strategies give remarkable improvements over $\Omega_{\rm PLM}$ for the full range of λ, which witnesses the advantage of adaptivity in state verification: allowing conditional measurements can markedly improve the verification efficiency. Intuitively, one might expect that the more entangled the $|\Psi\rangle$ the harder to verify it using local measurements. The two-way strategies $\widehat{\Omega}_\leftrightarrow$ and $\Omega_\leftrightarrow$ justify this intuition. However, the one-way strategy $\Omega_\rightarrow$, though it achieves optimality when $|\Psi\rangle$ is maximally entangled, has inefficient performance in the small regime of λ, where $|\Psi\rangle$ is less entangled. This is due to the fact that in the one-way case the symmetric role between Alice and Bob cannot be utilized. The strict gaps among $\Omega_\rightarrow$, $\widehat{\Omega}_\leftrightarrow$, and $\Omega_\leftrightarrow$ reveal the power of classical communication in state verification: with just an extra step of messaging, one can significantly boost the performance.

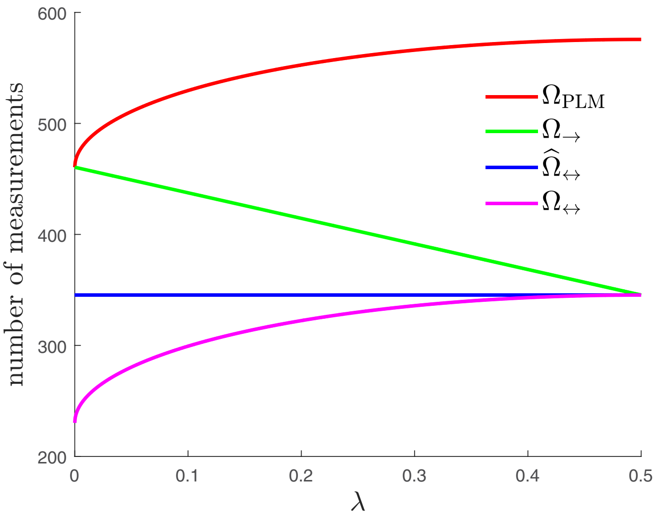

FIG. 2. The number of measurements required to verify the two-qubit pure state $|\Psi\rangle$ using various strategies – $\Omega_{\rm PLM}$, $\Omega_{\rightarrow}$, $\widehat{\Omega}_{\leftrightarrow}$, and $\Omega_{\leftrightarrow}$ – as a function of λ. Here, $\epsilon = 0.01$ and $\delta = 0.1$. These parameters are chosen in accordance with Fig. 1 of [32] for better comparison.

VI. CONCLUSION

In this paper, we studied the two-qubit pure state verification problem in depth. We constructed optimal strategies when the following classes of measurement are accessible: (i) one-way LOCC, (ii) two-way LOCC, and (iii) separable measurements. Our proposed strategies are dramatically more efficient than all known candidates based on local measurements and are comparable to the optimal strategy when there is no restriction on the accessible measurement at all. Our results revealed that for this problem the two-way LOCC measurement remarkably outperforms the one-way LOCC measurement and achieves the same performance as the separable measurement. In principle, the technique used here to construct strategies for verifying two-qubit pure states can be generalized to pure states of more qubits and higher dimensions. However, it might be rather difficult to get the optimal strategies.

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Note added. Recently, we became aware of related works by Li *et al.* [37] and Yu *et al.* [38].

APPENDIX A: OPTIMAL STRATEGY USING ONE-WAY LOCC MEASUREMENTS

In this section we give more details on the proof that our proposed one-way LOCC strategy is optimal. As is shown in the main text, a general one-way LOCC strategy for verifying $|\Psi\rangle$ can be written as

$$\Omega = 2\int |t,s\rangle\langle t,s| \otimes |t,s,B\rangle\langle t,s,B| P_{TS}(dtds). \tag{A1}$$

To analyze Ω, we treat the variable t in Eq. (A1) as the random variable T subject to the marginal distribution P_T and use $\mathbb{E}_T$ to denote the expectation under P_T. The constraint that Alice's measurement in Ω must be a POVM induces

$$\Omega^A = \mathrm{Tr}_B\,\Omega = 2\begin{pmatrix} \mathbb{E}_T[T] & * \\ * & \mathbb{E}_T[1-T] \end{pmatrix} = \mathbb{1}.$$

Focusing on the diagonal terms, we have the condition $\mathbb{E}[T] = \frac{1}{2}$. Here, we do not use the condition for the off-diagonal terms. By letting $\Xi := 2\mathbb{E}_T[\frac{T(1-T)}{D}]$ with $D := T(1-\lambda) + (1-T)\lambda$, the condition gives the following two relations:

$$\begin{aligned} &2\mathbb{E}_T\left[\frac{T^2}{D}\right](1-\lambda) + \lambda\Xi \\ &\quad = 2\mathbb{E}_T\left[\frac{T[T(1-\lambda)+(1-T)\lambda]}{D}\right] = 1, \\ &2\mathbb{E}_T\left[\frac{(1-T)^2}{D}\right]\lambda + (1-\lambda)\Xi \\ &\quad = 2\mathbb{E}_T\left[\frac{T[T(1-\lambda)+(1-T)\lambda]}{D}\right] = 1. \end{aligned}$$

We then get Ω_a from Ω, using the averaging technique described in the main text. When expressed in the standard basis, Ω_a satisfies

$$\begin{aligned} \Omega_a &= \begin{pmatrix} 1-\lambda\Xi & 0 & 0 & \Xi\sqrt{\lambda(1-\lambda)} \\ 0 & \Xi\lambda & 0 & 0 \\ 0 & 0 & \Xi(1-\lambda) & 0 \\ \Xi\sqrt{\lambda(1-\lambda)} & 0 & 0 & 1-(1-\lambda)\Xi \end{pmatrix} \\ &= |\Psi\rangle\langle\Psi| + \lambda_2|\Psi^\perp\rangle\langle\Psi^\perp| + \lambda_3|01\rangle\langle 01| + \lambda_4|10\rangle\langle 10|, \end{aligned}$$

with $\lambda_3 = \langle 01|\Omega_a|01\rangle = \Xi\lambda$, $\lambda_4 = \langle 10|\Omega_a|10\rangle = \Xi(1-\lambda)$, and

$$\begin{aligned} \lambda_2 &= \langle\Psi^\perp|\Omega_a|\Psi^\perp\rangle \\ &= [1-\lambda\Xi]\lambda - 2\lambda(1-\lambda)\Xi + [1-(1-\lambda)\Xi](1-\lambda) \\ &= 1 - [\lambda^2 + 2\lambda(1-\lambda) + (1-\lambda)^2]\Xi \\ &= 1 - \Xi. \end{aligned}$$

APPENDIX B: STRATEGY USING TWO-WAY TWO-STEP LOCC MEASUREMENTS

Here we explain in detail the two-way two-step LOCC strategy $\widehat{\Omega}_{\leftrightarrow}$, given in Eq. (8) of the main text. We first describe its construction and then prove its optimality when only two-step classical communication is allowed. Considering the symmetric role between Alice and Bob, we construct from $\Omega_{\rightarrow}$ a strategy which outperforms $\Omega_{\rightarrow}$ in the small regime of λ. The strategy $\Omega_{\rightarrow}$ is implemented by Alice sending measurement outcomes to Bob and Bob performing conditional measurements. We then get a symmetric version $\Omega_{\leftarrow}$ of $\Omega_{\rightarrow}$ by switching the role between Alice and Bob. The new strategy goes as follows. In each round, Alice first tosses a fair coin: if it is heads up, they use $\Omega_{\rightarrow}$; if it is tails up, they use $\Omega_{\leftarrow}$. The corresponding strategy then has the form

$$\begin{aligned}\widehat{\Omega}_{\leftrightarrow} &= \frac{1}{2}\Omega_{\rightarrow} + \frac{1}{2}\Omega_{\leftarrow}\\ &= |\Psi\rangle\langle\Psi| + p|\Psi^{\perp}\rangle\langle\Psi^{\perp}| + \frac{1-p}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|).\end{aligned}$$

Minimizing the second largest eigenvalue of $\widehat{\Omega}_{\leftrightarrow}$ with respect to $p \in [0, 1]$, we get $p = \frac{1}{3}$ and

$$\widehat{\Omega}_{\leftrightarrow} = |\Psi\rangle\langle\Psi| + \tfrac{1}{3}(\mathbb{1} - |\Psi\rangle\langle\Psi|).$$

We remark that, differently from $\Omega_{\rightarrow}$ and $\Omega_{\leftarrow}$, $\widehat{\Omega}_{\leftrightarrow}$ must be implemented by two-way two-step LOCC. This is due to the symmetrization technique we used to construct $\widehat{\Omega}_{\leftrightarrow}$ from $\Omega_{\rightarrow}$ and $\Omega_{\leftarrow}$. Alice and Bob need an extra step of classical communication to agree on which strategy ($\Omega_{\rightarrow}$ or $\Omega_{\leftarrow}$) is used in the current round. Comparing the performance of $\Omega_{\rightarrow}$ ($\Omega_{\leftarrow}$) and $\widehat{\Omega}_{\leftrightarrow}$, one sees the power of classical communication in verification: with just one extra bit of messaging, $\widehat{\Omega}_{\leftrightarrow}$ outperforms $\Omega_{\rightarrow}$ ($\Omega_{\leftarrow}$) significantly.

We can actually prove that the strategy $\widehat{\Omega}_{\leftrightarrow}$ is the best we can hope for when only two-step classical communication is allowed. Any two-step strategy can be written as a convex combination of one-way LOCC strategies from Alice to Bob and one-way LOCC strategies from Bob to Alice. In Theorem 3 of [33] it was proved that, for any one-way LOCC strategy Ω satisfying $\mathbb{1} \geqslant \Omega \geqslant |\Psi\rangle\langle\Psi|$, $\operatorname{Tr}\Omega \geqslant 2$ holds. Hence, the second largest eigenvalue of Ω is no smaller than $\frac{1}{3}$, concluding the optimality of $\widehat{\Omega}_{\leftrightarrow}$.

APPENDIX C: OPTIMIZATION OF STRATEGY USING TWO-WAY LOCC MEASUREMENTS

When constructing the strategy $\Omega_{\leftrightarrow}$ in the main text, we prefix two magic variables $\eta = 1 - \sqrt{\frac{\lambda}{1-\lambda}}$ and $p = \frac{\lambda}{1+\sqrt{\lambda(1-\lambda)}}$. Here we show that they are actually chosen so that the second largest eigenvalue of $\Omega_{\leftrightarrow}$ is minimized. From now on we assume $\eta \in [0, 1]$ and $p \in [0, 1]$ are two free parameters to be optimized. By construction, $\Omega_{\leftrightarrow}$ is given by

$$\Omega_{\leftrightarrow} = \frac{p-1}{4}\left(T_1^{A\rightarrow B} + T_2^{A\rightarrow B} + T_1^{B\rightarrow A} + T_2^{B\rightarrow A}\right) + pT_3.$$

It can be shown that $\Omega_{\leftrightarrow}$ admits the following spectral decomposition:

$$\begin{aligned}\Omega_{\leftrightarrow} = {}& |\Psi\rangle\langle\Psi| + \lambda_2(\eta, p)|\Psi^{\perp}\rangle\langle\Psi^{\perp}|\\ &+ \lambda_3(\eta, p)(|01\rangle\langle 01| + |10\rangle\langle 10|),\end{aligned}$$

where

$$\begin{aligned}\lambda_2(\eta, p) &= \frac{p(1-\eta) + \lambda\eta}{1 - \eta + \lambda\eta},\\ \lambda_3(\eta, p) &= \frac{(1-p)[\lambda + (1-\lambda)(1-\eta)^2]}{2(1 - \eta + \lambda\eta)}.\end{aligned}$$

Our target is to minimize $\lambda_2^{\downarrow}(\Omega_{\leftrightarrow})$, the second largest eigenvalue of $\Omega_{\leftrightarrow}$, over the free parameters η and p for fixed λ. This optimization problem then is given by

$$\lambda_2^{\downarrow}(\Omega_{\leftrightarrow}) := \min_{\eta\in[0,1],p\in[0,1]} \max\{\lambda_2(\eta, p), \lambda_3(\eta, p)\}.$$

$\lambda_2^{\downarrow}(\Omega_{\leftrightarrow})$ is minimized for fixed λ when the derivatives with respect to η and p vanish. As $\lambda_2(\eta, p)$ is monotonically increasing with p while $\lambda_3(\eta, p)$ is monotonically decreasing with p in the range $p \in [0, 1]$, and $\lambda_2(0, \eta) < \lambda_3(0, \eta)$ for $\eta \in [0, 1]$, $\lambda_2^{\downarrow}(\Omega_{\leftrightarrow})$ is minimized when $\lambda_2(\eta, p) = \lambda_3(\eta, p)$. Solving this equation with respect to p, we get

$$\begin{aligned}p^* &= \frac{-\lambda\eta^2 + \eta^2 - 2\eta + 1}{2\lambda\eta - \lambda\eta^2 + \eta^2 - 4\eta + 3},\\ \lambda_2(\eta) = \lambda_3(\eta) &= \frac{2\lambda\eta - \lambda\eta^2 + \eta^2 - 2\eta + 1}{2\lambda\eta - \lambda\eta^2 + \eta^2 - 4\eta + 3},\end{aligned}$$

where p^* is the solution of the equation, which is also the optimal choice of p. We minimize $\lambda_2(\eta)$ with respect to η. The partial derivative is given by

$$\frac{\partial\lambda_2}{\partial\eta} = -\frac{2(1-\lambda)(\eta - \eta_-)(\eta - \eta_+)}{(2\lambda\eta - \lambda\eta^2 + \eta^2 - 4\eta + 3)^2},$$

where $\eta_{\pm} = 1 \pm \sqrt{\frac{\lambda}{1-\lambda}}$. Solving the equation $\partial\lambda_2/\partial\eta = 0$ in the range $\eta \in [0, 1]$ gives the optimal choice $\eta^* = \eta_- = 1 - \sqrt{\frac{\lambda}{1-\lambda}}$. Substituting η^* into p^*, we get $p^* = \frac{\lambda}{1+\sqrt{\lambda(1-\lambda)}}$ expressed in terms of λ solely. Substituting in the optimal choices of η^* and p^* gives the desired optimal strategy.

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