

Optimality in quantum data compression using dynamical entropy

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In this article we study lossless compression of strings of pure quantum states of indeterminate-length quantum codes which were introduced by Schumacher and Westmoreland. Past work has assumed that the strings of quantum data are prepared to be encoded in an independent and identically distributed way. We discuss the notion of quantum stochastic ensembles, allowing us to consider strings of quantum states prepared in a general way. For any quantum stochastic ensemble we define an associated quantum dynamical system and prove that the optimal average codeword length via lossless coding is equal to the quantum dynamical entropy of the associated quantum dynamical system.

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I. INTRODUCTION

In the theory of data compression of classical information theory one wishes to encode a symbol set, S , with a code, C , which is a mapping from the symbol set S to the set A^+ of all finite strings (or sequences) of elements from the alphabet A , where A is usually taken to be the binary alphabet $\{0, 1\}$. The set A^+ is frequently referred to as the codebook, and its elements are called codewords. Since we compress long strings (sequences) of messages, concatenation is used to extend the code C to the set S^+ containing all finite strings from the symbol set S . This extension of C is denoted by C^+ , and it is called the extended code. A code C is said to be uniquely decodable if its extended code is an injective function. In that case, the decoding function is the inverse of C^+ . If each symbol x of the symbol set S that we wish to encode is always prepared with the same probability $p(x)$, independent of the string of symbols that have appeared earlier, then the sequence $(X_n)_{n \in \mathbb{N}}$ of random variables which gives us the string of symbols to be encoded is independent and identically distributed (i.i.d.) with values in the symbol set S with probability mass function equal to $(p(x))_{x \in S}$. If X denotes any member of this sequence of random variables, then its Shannon entropy $H(X)$ is defined as

$$H(X) = - \sum_{x \in S} p(x) \log_2 p(x).$$

In Shannon's original works on the subject [1,2], the Noiseless Coding Theorem was proved which states that, for any $\delta > 0$, $(H(X) + \delta)$ -many binary bits per symbol are sufficient in order to encode strings of symbols if each entry of the sequence is prepared in a i.i.d. way, with probability of error tending to zero as the length of the strings tend to infinity. Moreover, Shannon showed that for any $R < H(X)$,

if at most R bits are used per symbol, then the probability of error tends to one as the length of the strings tend to infinity. Thus the Shannon entropy $H(X)$ can be interpreted as the minimum expected number of binary bits per symbol that are necessary in order to encode strings of symbols with arbitrarily small error (i.e., asymptotically lossless coding) given that the elements of the string of symbols are encoded in an i.i.d. way.

The setting of quantum data compression for indeterminate-length quantum codes is similar to the setting of classical data compression. In this case, the symbol set S contains the symbol states which are normalized vectors spanning a Hilbert space H_S . Here we consider only the compression of pure quantum states, therefore we restrict our attention to normalized vectors or pure states. The classical binary alphabet $A = \{0, 1\}$ is replaced by the set of qubits $\mathcal{A} = \{|0\rangle, |1\rangle\}$ which is the standard orthonormal basis of the Hilbert space $\mathcal{H}_{\mathcal{A}} = \mathbb{C}^2$. The classical codebook A^+ is replaced by the free Fock space $H_{\mathcal{A}}^{\oplus} = \bigoplus_{\ell=0}^{\infty} H_{\mathcal{A}}^{\otimes \ell}$. A quantum code is a linear isometry $U : H_S \rightarrow H_{\mathcal{A}}^{\oplus}$, and the corresponding extended code is a map U^+ which is defined on the free Fock space $H_S^{\oplus} = \bigoplus_{\ell=0}^{\infty} H_S^{\otimes \ell}$ by "concatenation"; i.e., tensor products of the values of U in the free Fock space $H_{\mathcal{A}}^{\oplus}$. The quantum code U is called uniquely decodable if U^+ is also an isometry.

The Noiseless Coding Theorem was extended to indeterminate-length quantum codes in 1995 by Schumacher [3]. Schumacher showed that, for any $\delta > 0$, $(S(\rho) + \delta)$ -many qubits per symbol are sufficient in order to encode strings of symbol states if each entry of the sequence is prepared in a i.i.d. way, with probability of error tending to zero as the length of the strings tends to infinity. Here $\rho = \rho_S$ is the ensemble state representing the quantum ensemble S , and $S(\rho)$ is the von Neumann entropy of the density matrix ρ given by

$$S(\rho) = -\text{tr}(\rho \log_2 \rho).$$

Moreover, Schumacher showed that for any $R < S(\rho)$, if at most R qubits are used per symbol, then the probability of error tends to one as the length of the strings tends to infinity.

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Thus the von Neumann entropy $S(\rho)$ can be interpreted as the minimum expected number of qubits per symbol that are necessary in order to encode strings of symbol states with arbitrarily small error (i.e., asymptotically lossless coding) given that the elements of the string of symbol states are prepared in an i.i.d. way.

Indeterminate-length quantum codes were considered by Schumacher and Westmoreland [4], and later by Müller, Rogers, and Nagarajan [5,6] and Bellomo *et al.* [7]. In all three of these papers, the authors prove a version of the quantum Kraft-McMillan Theorem, which states that every uniquely decodable quantum code must satisfy an inequality in terms of the lengths of its eigenstates. Their presentations are very similar to that of the classical Kraft-McMillan Theorem (Theorems 5.2.1 and 5.5.1 of Ref. [8]) except that these authors did not provide a converse statement. In Theorem 2.9, we present a modified version of the quantum Kraft-McMillan Theorem giving a converse statement, thus characterizing the uniquely decodable quantum codes. Our Theorem 2.9 comes in handy when we define an optimal quantum code that corresponds to a given ensemble.

In Sec. II B we introduce the notion of quantum stochastic ensemble and Markov ensemble, allowing us to prepare strings of symbol states for quantum data compression such that the appearance of each symbol in the string may depend on the previous symbols; i.e., the strings of symbol states are not necessarily prepared in an i.i.d. way. Quantum sources that emit sequences of quantum symbols that are not necessarily statistically independent have been considered in the literature [9], and they are well suited for quantum communications. A stochastic ensemble is a sequence $(\mathcal{S}^k)_{k \in \mathbb{N}}$, where $\mathcal{S}^k = \{p(n_1, \dots, n_k), |s_{n_1} \dots s_{n_k}\rangle\}_{n_1, \dots, n_k=1}^N$ for each $k \in \mathbb{N}$ such that p is the probability mass function of a discrete stochastic process \mathbf{X} , $\{|s_n\rangle\}_{n=1}^N$ is a collection of vector states referred to as the symbol states, and $p(n_1, \dots, n_k)$ is the probability that the string of quantum symbols $|s_{n_1} \dots s_{n_k}\rangle$ is encoded, for each $k \in \mathbb{N}$ and $n_1, \dots, n_k \in \{1, \dots, N\}$.

Our main results, Theorems 3.4 and 3.8, give quantum dynamical entropy interpretations for the average minimum codeword length per symbol as the length of strings of symbol states tend to infinity when the coding is assumed to be lossless. These results extend the results of Schumacher [3] and Bellomo *et al.* [7], which state that for an i.i.d. prepared quantum ensemble the optimal codeword length per symbol is equal to the von Neumann entropy of the initial ensemble state for asymptotically lossless coding. In our result we use the quantum Markov chain (QMC) approach to quantum dynamical entropy, which we recall in Sec. III B. The notion of QMC was introduced by Accardi [10], and its use for describing dynamical entropy first appeared in Ref. [11] in terms of the Accardi-Ohya-Watanabe (AOW) entropy. Another QMC approach was introduced by Tuyls [12] for the study of the Alicki-Fannes (AF) entropy, which was introduced in Ref. [13] and often is referred to as ALF entropy to emphasize Lindblad’s contributions. Finally, a generalization of both QMC approaches was given in Ref. [14], where the authors introduced the Kossakowski-Ohya-Watanabe (KOW) entropy. Throughout this article, we will follow mainly the terminology and notations of Refs. [11] and [14].

II. DATA COMPRESSION

In what follows, all codings will be done into strings of bits or strings of qubits for classical and quantum codes, respectively. Therefore all codewords will be strings of elements from a binary alphabet $A = \{0, 1\}$ (in the classical case) or, possibly the superposition of, strings from a quantum binary alphabet $\mathcal{A} = \{|0\rangle, |1\rangle\}$ which is an orthonormal basis of the Hilbert space $H_{\mathcal{A}} = \mathbb{C}^2$ (in the quantum case). The extensions to d -bits or d -qubits can easily be done in both cases.

A. Classical codes and the Kraft Inequality

Let S be a finite or countable set equipped with the power set σ -algebra $\mathcal{P}(S)$, and let X be a random variable with values in S . The set S will be referred to as the *symbol set* that we wish to encode. In the literature, the set S is referred to as the set of objects, the message set, or sometimes even the index set. For any set Y , we will set Y^+ equal to the set $\cup_{\ell=0}^{\infty} Y^\ell$, which is the collection of all possible finite strings from Y , where Y^0 denotes the empty set (or empty string). Last, let $A = \{0, 1\}$ be the binary alphabet. A *code* $C : S \rightarrow A^+$ is a mapping from S to A^+ , the set of finite strings with letters in the binary alphabet A . The range of the code, A^+ , is referred to as the *codebook*, and its elements are the *codewords*. Moreover, for each $x \in S$, we refer to $C(x)$ as the *codeword of the symbol* x . For each $a \in A^+$, we call the length of a [denoted by $\ell(a)$] the unique integer m such that $a \in A^m$.

The *expected length* of a code C on a symbol set S is given by

$$\text{EL}(C) := \sum_{x \in S} p(x)\ell(C(x)) = \mathbb{E}[\ell(C(X))],$$

where $p : S \rightarrow [0, 1]$ is the probability mass function (pmf) of the random variable X and the expectation \mathbb{E} is taken with respect to p .

We extend the code C by concatenation to obtain the *extended code*, also called the extension of C , i.e.,

$$C^+(x_1x_2 \dots x_n) = C(x_1)C(x_2) \dots C(x_n) \\ \text{for all } x_1x_2 \dots x_n \in S^n \text{ and } n \in \mathbb{N},$$

and we define $C^+(\emptyset) = \emptyset$. We call the code C *uniquely decodable* whenever its extension C^+ is injective; i.e., C is uniquely decodable whenever all strings of symbols from S are pairwise distinguishable. In lossless coding we are interested only in uniquely decodable codes.

An extremely useful class of uniquely decodable codes are the so-called instantaneous (or *prefix-free*) codes. A code is said to be prefix-free if no codeword is the prefix of another; i.e., for every distinct pair $x, y \in S$ there is no $a \in A^+$ such that $C(x)a = C(y)$. Prefix-free codes are called instantaneous because the decoder is able to read out each codeword from a string of codewords, instantaneously, as soon as he or she sees that word appears in a string (without waiting for the entire string).

The Kraft-McMillan Inequality is fundamental in classical data compression.

Theorem 2.1. (Kraft-McMillan Inequality, Theorems 5.2.1 and 5.5.1 of Ref. [8]) For any uniquely decodable code over a symbol set S with cardinality $|S| = m \in \mathbb{N}$, the codeword

lengths $\ell_1, \ell_2, \dots, \ell_m$ must satisfy the inequality

$$\sum_{i=1}^m 2^{-\ell_i} \leq 1.$$

Conversely, given a set of codeword lengths that satisfies this inequality, there exists an instantaneous code with these code lengths.

Remark 2.2. The Kraft-McMillan Inequality is sometimes referred to only as the Kraft Inequality. This is due to the fact that Kraft was the first to prove the inequality in Ref. [15], although his original result refers only to instantaneous codes. McMillan later extended Kraft's work to include all uniquely decodable codes in Ref. [16]. Furthermore, it is worth noting that the Kraft-McMillan Inequality can be extended to a countable set of symbols (see Theorem 5.2.2 and the corollary following Theorem 5.5.1 in Ref. [8]). When including countable sets of symbols, the inequality is referred to as the Extended Kraft-McMillan Inequality.

An immediate corollary to the Kraft-McMillan Inequality is the following:

Corollary 2.3. Given any uniquely decodable code with codeword lengths $\ell_1, \ell_2, \dots, \ell_m$, there exists an instantaneous code with these same code lengths.

We call a uniquely decodable code C *optimal* whenever the expected length $\text{EL}(C)$ is minimized; i.e., the optimal uniquely decodable code is given by

$$\begin{aligned} C_{\text{opt}} &:= \operatorname{argmin}_C \{ \text{EL}(C) : C \text{ is uniquely decodable} \} \\ &= \operatorname{argmin}_C \left\{ \text{EL}(C) : \begin{array}{l} \text{the codeword lengths of the} \\ C \text{ satisfy } \sum_i 2^{-\ell_i} \leq 1 \end{array} \right\}, \end{aligned} \quad (1)$$

where the last equality follows from Theorem 2.1. We set $\text{EL}^*(X) := \text{EL}(C_{\text{opt}})$ the *optimal expected length of the random variable* X . The results for the optimal expected length are summarized in the following:

Theorem 2.4. ([8], Theorem 5.4.1) Let X be a random variable with range in the symbol set S . Then the optimal expected length of X satisfies the inequality

$$H(X) \leq \text{EL}^*(X) < H(X) + 1,$$

where $H(X)$ is the Shannon entropy of X , i.e., $H(X) = -\sum_{i \in S} p_i \log_2 p_i$ where $(p_i)_{i \in S}$ is the pmf of X .

Well-known examples of codes which satisfy the inequality of Theorem 2.4 are the so-called Huffman codes and Shannon-Fano codes.

In the above theorem, we are interested only in the compressibility of single codewords. Suppose instead that we wish to compress strings of codewords with code distributions given by a stochastic process $\mathbf{X} = (X_i)_{i=1}^{\infty}$. Then, for each $n \in \mathbb{N}$, Theorem 2.4 holds for the random vector (X_1, X_2, \dots, X_n) , giving

$$\begin{aligned} H(X_1, X_2, \dots, X_n) &\leq \text{EL}^*(X_1, X_2, \dots, X_n) \\ &< H(X_1, X_2, \dots, X_n) + 1. \end{aligned}$$

For each $n \in \mathbb{N}$, we set

$$\text{EL}_n^*(\mathbf{X}) := \frac{1}{n} \text{EL}^*(X_1, X_2, \dots, X_n) \quad (2)$$

to be the *optimal expected codeword length per symbol* for the first n symbols. We can then express the optimal expected codeword length per symbol (over all symbols) in terms of the entropy rate, which is a dynamical entropy for stochastic processes. The *entropy rate* of a stochastic process $\mathbf{X} = (X_n)_{n=1}^{\infty}$ is given by

$$H(\mathbf{X}) = \lim_{n \rightarrow \infty} \frac{1}{n} H(X_1, \dots, X_n),$$

whenever the limit exists. There are many instances when it is known that the above limit exists (e.g., stationary stochastic processes; see Theorem 4.2.1 of Ref. [8]).

Theorem 2.5. (Ref. [8], Theorem 5.4.2) The optimal expected codeword length per symbol for a stochastic process $\mathbf{X} = (X_i)_{i=1}^{\infty}$ satisfies

$$\frac{H(X_1, X_2, \dots, X_n)}{n} \leq \text{EL}_n^*(\mathbf{X}) < \frac{H(X_1, X_2, \dots, X_n)}{n} + \frac{1}{n}.$$

Moreover, if \mathbf{X} is such that the limit defining entropy rate exists (e.g., \mathbf{X} is a stationary stochastic process), then

$$\text{EL}_n^*(\mathbf{X}) \rightarrow H(\mathbf{X}) \quad \text{as } n \rightarrow \infty.$$

In particular, if \mathbf{X} consists of independent identically distributed (i.i.d.) copies of a random variable X , then

$$\text{EL}_n^*(\mathbf{X}) \rightarrow H(X) \quad \text{as } n \rightarrow \infty.$$

This finishes our brief overview of data compression in classical information theory. For a more detailed exposition see Ch. 5 of Ref. [8].

B. Quantum data compression

We begin with the description of indeterminate-length quantum codes, whose preliminary investigation began with Schumacher [17] and Braunstein *et al.* [18], and they were formalized in Ref. [4]. We may think of the codes introduced in the previous section as being varying-length codes; the term indeterminate-length is used to draw attention to the fact that a quantum code must allow for superpositions of codewords, including those superpositions containing codewords with different lengths. We will follow mainly the formalisms in Ref. [7] as opposed to the zero-extended forms of Ref. [4]. A description of the connection between these two formalisms can be found in Ref. [19].

For any Hilbert space H , we will denote by $H^{\oplus} := \bigoplus_{\ell=0}^{\infty} H^{\otimes \ell}$ the free Fock space of H , where $H^{\otimes 0} = \mathbb{C}$. We will denote the scalar $1 \in H^{\otimes 0}$ by $|\emptyset\rangle$ and refer to it as the empty string. Let $\mathcal{S} = \{p_n, |s_n\rangle\}_{n=1}^N$ be an *ensemble of pure states*, or simply ensemble, where $p = \{p_n\}_{n=1}^N$ is the pmf of a random variable X and $|s_n\rangle$ is an element of a d -dimensional Hilbert space H_S , for each $1 \leq n \leq N$, such that $H_S = \text{span}\{|s_n\rangle\}_{n=1}^N$. The collection $\{|s_n\rangle\}_{n=1}^N$ will be referred to as the *symbol states* of the ensemble \mathcal{S} . An (*indeterminate-length*) *quantum code*, U , over a quantum binary alphabet $\mathcal{A} := \{|0\rangle, |1\rangle\}$, which is an orthonormal basis for $H_{\mathcal{A}} = \mathbb{C}^2$, is a linear isometry $U : H_S \rightarrow H_{\mathcal{A}}^{\oplus}$. The *extended quantum code* of U is the linear mapping $U^+ : H_S^{\oplus} \rightarrow H_{\mathcal{A}}^{\oplus}$ given by

$$U^+(|s_1 s_2 \dots s_n\rangle) = U(|s_1\rangle)U(|s_2\rangle) \dots U(|s_n\rangle),$$

for all $|s_1 s_2 \cdots s_n\rangle \in H_S^{\otimes n}$ and $n \in \mathbb{N}$, and we set $U^+(\emptyset) = |\emptyset\rangle$, where concatenation is defined according to Definition 2.3 of Ref. [5] (see also Sec. V of Ref. [6]).

The quantum code U is said to be *uniquely decodable* if the extended quantum code U^+ is an isometry. Throughout this paper, we will restrict ourselves only to the situation where the range of U is a subset of $H_A^{\oplus \ell_{\max}}$ for some $\ell_{\max} \in \mathbb{N}$; i.e., there is a finite upper bound ℓ_{\max} on the length of all codewords.

Remark 2.6. The authors of Ref. [19] allow nonempty strings to map to the empty string. In their paper, the authors send along a classical side channel to give the lengths of the codewords, and so that convention is possible. Without the classical side channel (as is the approach in the present paper) allowing nonempty strings to map to the empty string will cause the quantum code to *not* be uniquely decodable.

Let $\mathcal{S} = \{p_n, |s_n\rangle\}_{n=1}^N$ be an ensemble whose symbol states $\{|s_n\rangle\}_{n=1}^N$ span a Hilbert space H_S of dimension d . Consider a classical uniquely decodable code, C , on a symbol set, $S = \{x_i\}_{i=1}^d$, with d -many symbols. We will construct a corresponding uniquely decodable quantum code, U , from C by identifying the classical binary alphabet $A = \{0, 1\}$ with the quantum binary alphabet $\mathcal{A} = \{|0\rangle, |1\rangle\} \subseteq \mathbb{C}^2$ and the symbol set, S , with any orthonormal basis $\{|e_i\rangle\}_{i=1}^d$ of H_S ; this construction is given in Ref. [7]. Fix an orthonormal basis $\{|e_i\rangle\}_{i=1}^d$ of H_S and define the quantum code $U : H_S \rightarrow H_A^{\oplus}$ by the equation

$$U = \sum_{i=1}^d |C(x_i)\rangle\langle e_i|. \quad (3)$$

It is clear that $|C(x_i)\rangle \in H_A^{\otimes \ell_i} \subseteq H_A^{\oplus}$, where ℓ_i is the length of $C(x_i)$, and that $\{|C(x_i)\rangle\}_{i=1}^d$ is an orthonormal set, so that U is a linear isometry. Furthermore, since C is uniquely decodable, the map $U^\ell : H_S^{\otimes \ell} \rightarrow H_A^{\oplus}$ defined by the equation

$$U^\ell = \sum_{i_1=1}^d \cdots \sum_{i_\ell=1}^d |C(x_{i_1})C(x_{i_2}) \cdots C(x_{i_\ell})\rangle\langle e_{i_1} e_{i_2} \cdots e_{i_\ell}|$$

is a linear isometry for each $\ell \in \mathbb{N}$. Since the extended quantum code $U^+ : H_S^{\oplus} \rightarrow H_A^{\oplus}$ is given by

$$U^+ = \sum_{\ell=0}^{\infty} U^\ell,$$

we see that U^+ is a linear isometry and hence U is uniquely decodable. We will refer to quantum codes constructed from classical ones by Eq. (3) as *classical-quantum encoding schemes* (*c-q schemes*).

Remark 2.7. Notice that the symbol states $\{|s_n\rangle\}_{n=1}^N$ of the ensemble \mathcal{S} are not directly encoded by the $|C(x_i)\rangle$'s unless $N = d$, and there exists a permutation σ of $\{1, \dots, d\}$ such that $|s_{\sigma(i)}\rangle = |e_i\rangle$ for every $i \in \{1, \dots, d\}$. In fact $U|s_n\rangle$ need not belong to $H_A^{\otimes \ell}$ for any $\ell \in \mathbb{N}$, but can in general be in a superposition of different lengths. (Hence the term indeterminate-length quantum codes.)

The Kraft-McMillan Inequality (Theorem 2.1) was initially extended to the quantum domain in Ref. [4] and subsequently in Refs. [5] and [7]. Before presenting (a slightly different) quantum Kraft-McMillan Inequality, we will first introduce

the length observable and quantum codes with length eigenstates. The length observable Λ acting on H_A^{\oplus} is given by

$$\Lambda := \sum_{\ell=0}^{\infty} \ell \Pi_\ell, \quad (4)$$

where Π_ℓ is the orthogonal projection onto the subspace $H_A^{\otimes \ell}$ of H_A^{\oplus} .

We say that a quantum code $U : H_S \rightarrow H_A^{\oplus}$ has *length eigenstates* if U has the form

$$U = \sum_{i=1}^d |\psi_i\rangle\langle e_i| \quad (5)$$

for some orthonormal basis $\{|e_i\rangle\}_{i=1}^d$ of H_S and some sequence $\{|\psi_i\rangle\}_{i=1}^d \subseteq H_A^{\oplus}$ such that, for each $1 \leq i \leq d$, $|\psi_i\rangle \in H_A^{\otimes \ell_i}$ for some $\ell_i \in \mathbb{N}$.

Note that the $|\psi_i\rangle$'s are orthogonal due to U being a linear isometry. It is easy to see that every c-q scheme is a quantum code with length eigenstates. Last, for each $\ell \in \mathbb{N} \cup \{0\}$, we will refer to the elements of the set $\{\psi_i : i \in \{1, \dots, d\}, \psi_i \in H_A^{\otimes \ell}\}$ as the *length ℓ eigenstates* of U , and we will refer to $\{\ell_i\}_{i=1}^d$, where, for each $i = 1, \dots, d$, $\psi_i \in H_A^{\otimes \ell_i}$, as the *length eigenvalues* of U .

Remark 2.8. The quantum versions of the Kraft-McMillan Inequality proved in Sec. II C of Ref. [4] and Theorem 3.6 of Ref. [5] are more general than the same proved in Theorem 1 of Ref. [7], although the formalisms are quite different in all three. Our version of the quantum Kraft-McMillan Inequality, presented below, is a generalization of Ref. [7] but is not quite in the full generality of Sec. II C of Ref. [4] (in the forward direction) because we consider only uniquely decodable codes (as opposed to the more general notion called *condensable codes* considered in Ref. [4]). However, our version does have a converse statement, similar to the classical Kraft-McMillan Inequality, which is missing from the aforementioned quantum versions.

Theorem 2.9. (quantum Kraft-McMillan Inequality) Any uniquely decodable quantum code U with length eigenstates over a binary alphabet must satisfy the inequality

$$\text{tr}(U^\dagger 2^{-\Lambda} U) \leq 1.$$

Conversely, if $U : H_S \rightarrow H_A^{\oplus}$ is a linear isometry with length eigenstates satisfying the above inequality, then there exists a c-q scheme \tilde{U} with the same number of length ℓ eigenstates for each $\ell \in \mathbb{N}$.

The proof of Theorem 2.9 is presented in the Appendix.

We would like to find a quantum code which minimizes the amount of resources required. Unfortunately there are numerous ways to define the length of a codeword for an indeterminate-length quantum code [e.g., base length [19], exponential length (Definition 6 of Ref. [7]), etc.]. Here, we follow Definition 3 of Ref. [7] and define the *length of a codeword* $|\omega\rangle$, which is a normalized vector in H_A^{\oplus} given by $|\omega\rangle = U|s\rangle$ for a unique symbol state $|s\rangle \in \{|s_n\rangle\}_{n=1}^N$, as the expectation with respect to the length observable in Eq. (4). Explicitly, the length of a codeword $|\omega\rangle = U|s\rangle$ will be given by a function $\ell : H_A^{\oplus} \rightarrow \mathbb{R}^+$, defined as follows:

$$\ell(|\omega\rangle) := \langle \omega | \Lambda | \omega \rangle = \langle U s, \Lambda U s \rangle = \langle s, U^\dagger \Lambda U s \rangle. \quad (6)$$

Whenever U has length eigenstates and is given by Eq. (5), we see that Eq. (6) simplifies to

$$\ell(|\omega\rangle) = \sum_{i=1}^d |\langle e_i | s \rangle|^2 \ell_i,$$

where $\{\ell_i\}_{i=1}^d$ denotes the set of length eigenvalues of U .

Again we follow Ref. [7] and, for any ensemble $\mathcal{S} = \{p_n, |s_n\rangle\}_{n=1}^N$, we define the ensemble *state* $\rho_{\mathcal{S}}$ of \mathcal{S} by

$$\rho_{\mathcal{S}} = \sum_{n=1}^N p_n |s_n\rangle \langle s_n| \in S_1(H_S).$$

If U is a quantum code on H_S define the *average codeword length* with respect to the ensemble \mathcal{S} by

$$\text{EL}(U) = \text{tr}(\rho_{\mathcal{S}} U^\dagger \Lambda U).$$

We denote by U_{opt} the *optimal quantum code with length eigenstates* for the ensemble \mathcal{S} if

$$\begin{aligned} U_{\text{opt}} &:= \text{argmin}_U \{ \text{EL}(U) : U \text{ is uniquely decodable with} \\ &\quad \text{length eigenstates} \} \\ &= \text{argmin}_U \{ \text{EL}(U) : \text{tr}(U^\dagger 2^{-\Lambda} U) \leq 1 \} \\ &= \text{argmin}_U \left\{ \text{EL}(U) : U \text{ is a c-q scheme satisfying} \right. \\ &\quad \left. \times \sum_{i=1}^d 2^{-\ell_i} \leq 1 \right\}, \end{aligned}$$

where the second and third equalities follow from Theorem 2.9, and the $\{\ell_i\}_{i=1}^d$ in the third equality denote the length eigenvalues of U . The existence of U_{opt} follows from the existence of C_{opt} in Eq. (1) by the backward direction of Theorem 2.9. The *optimal average codeword length* for the ensemble \mathcal{S} is given by

$$\text{EL}^*(\rho_{\mathcal{S}}) := \text{EL}(U_{\text{opt}}) = \text{tr}(\rho_{\mathcal{S}} U_{\text{opt}}^\dagger \Lambda U_{\text{opt}}). \quad (7)$$

It is shown in Theorem 2 of Ref. [7] that the optimal c-q scheme (and hence optimal quantum code with length eigenstates by the converse of Theorem 2.9) is given by the classical Huffman codes. The bounds on $\text{EL}^*(\rho_{\mathcal{S}})$ in terms of the von Neumann entropy follow immediately.

Theorem 2.10. The minimum average codeword length for an ensemble \mathcal{S} is bounded as follows:

$$S(\rho_{\mathcal{S}}) \leq \text{EL}^*(\rho_{\mathcal{S}}) < S(\rho_{\mathcal{S}}) + 1.$$

Proof. See Theorem 3 of Ref. [7].

Next, we wish to consider the optimal average codeword length per symbol for a collection of ensembles $\{\mathcal{S}^k\}_{k=1}^\infty$, where $\mathcal{S}^k = \{p(n_1, \dots, n_k), |s_1 s_2 \dots s_k\rangle\}_{n_1, \dots, n_k=1}^N$ and probabilities given by the pmf p of a stochastic process \mathbf{X} . We will refer to such collections of ensembles as stochastic ensembles. Note that, by the definitions of a stochastic process, a stochastic ensemble \mathcal{S}^k must be compatible in the following sense:

$$\sum_{n_{k+1}=1}^N p(n_1, \dots, n_k, n_{k+1}) = p(n_1, \dots, n_k),$$

for all $n_1, \dots, n_k \in \{1, \dots, N\}$ and $k \in \mathbb{N}$. Notice that we allow for the possibility that preparations of the ensemble at each time be dependent upon previous preparations. If the

preparations of the ensemble are independent and identically prepared copies of $\mathcal{S} = \{p_n, |s_n\rangle\}_{n=1}^N$; i.e., the stochastic process \mathbf{X} is made up of i.i.d. copies of a random variable X , then $p(n_1, \dots, n_k) = p_{n_1} p_{n_2} \dots p_{n_k}$ and $\rho_{\mathcal{S}^k} = \rho_{\mathcal{S}}^{\otimes k}$, where $\rho_{\mathcal{S}^k} = \sum_{n_1, \dots, n_k=1}^N p(n_1, \dots, n_k) |s_{n_1} \dots s_{n_k}\rangle \langle s_{n_1} \dots s_{n_k}|$. For each $k \in \mathbb{N}$, let

$$\text{EL}_k^*(\rho_{\mathcal{S}^k}) = \frac{1}{k} \text{EL}^*(\rho_{\mathcal{S}^k})$$

be the *optimal average codeword length per symbol* for the first k symbols with respect to the ensemble \mathcal{S}^k , where $\text{EL}^*(\rho_{\mathcal{S}^k})$ is given by Eq. (7). Notice that the optimal average codeword length per symbol is defined analogously to the classical case in Eq. (2). Then, from Theorem 2.10, we have

$$\frac{1}{k} S(\rho_{\mathcal{S}^k}) \leq \text{EL}_k^*(\rho_{\mathcal{S}^k}) < \frac{1}{k} S(\rho_{\mathcal{S}^k}) + \frac{1}{k}. \quad (8)$$

In the following section, we will relate the above quantities to the dynamical entropy of a quantum dynamical system.

III. AN EXPRESSION FOR THE OPTIMAL QUANTUM DATA COMPRESSION RATE USING QUANTUM DYNAMICAL ENTROPY

A. A quantum dynamical system associated with a stationary Markov ensemble

In this article we consider stochastic processes $\mathbf{X} = (X_n)_{n=1}^\infty$ such that for some fixed $N < \infty$, each X_n is a random variable with values in $\{1, \dots, N\}$. If \mathbf{X} is a stochastic process set $p_{\mathbf{X}}$ to be the pmf of \mathbf{X} , i.e., $p_{\mathbf{X}}$ is a probability measure on $\{0, \dots, N\}^\mathbb{N}$, such that for every $k \in \mathbb{N}$ and $n_1, \dots, n_k \in \{1, \dots, N\}$, $p_{\mathbf{X}}(n_1, \dots, n_k) = \Pr[X_1 = n_1, \dots, X_k = n_k]$. We define the associated stochastic ensemble $\{\mathcal{S}^k\}_{k=1}^\infty$ by setting $\mathcal{S}^1 = \{p_{\mathbf{X}}(n), |s_n\rangle\}_{n=1}^N$ whose symbol states span H_S and $\mathcal{S}^k = \{p_{\mathbf{X}}(n_1, \dots, n_k), |s_{n_1} \dots s_{n_k}\rangle\}_{n_1, \dots, n_k=1}^N$ whose symbol states span $H_S^{\otimes k}$ for each $k \in \mathbb{N}$.

Recall that a stochastic process \mathbf{X} is called *stationary* if the measure $p_{\mathbf{X}}$ is invariant with respect to the translation map, i.e., if for every $k \in \mathbb{N}$, $p_{\mathbf{X}} = p_{\mathbf{Y}^{(k)}}$ where the stochastic process $\mathbf{Y}^{(k)} = (Y_n^{(k)})_{n \in \mathbb{N}}$ is defined by $Y_n^{(k)} = X_{n+k}$ for every $n \in \mathbb{N}$. Obviously, if the stochastic process $\mathbf{X} = (X_n)_{n=1}^\infty$ is stationary, then the random variables X_n are identically distributed. The stochastic process \mathbf{X} is called a *Markov process* if $\Pr[X_{k+1} | X_1, \dots, X_k] = \Pr[X_{k+1} | X_k]$ for every $k \in \mathbb{N}$. Hence a stochastic process $\mathbf{X} = (X_n)_{n=1}^\infty$ is a *stationary Markov process* if and only if the random variables X_n are identically distributed and there exists a column stochastic matrix $(p_{i,j})_{i,j=1}^N$ such that for every $k \in \mathbb{N}$ and $n_1, \dots, n_k, n_{k+1} \in \{1, \dots, N\}$, we have

$$\begin{aligned} \Pr[X_{k+1} = n_{k+1} | X_1 = n_1, \dots, X_k = n_k] \\ = \Pr[X_{k+1} = n_{k+1} | X_k = n_k] = p_{n_{k+1}, n_k}. \end{aligned}$$

Equivalently, a stochastic process $\mathbf{X} = (X_n)_{n=1}^\infty$ is a stationary Markov process if and only if the random variables X_n are identically distributed and there exist a transition matrix $P = (p_{n,m})_{n,m=1}^N$ and an initial distribution $p = \{p_n\}_{n=1}^N$ such that the pmf of \mathbf{X} is given by $p_{\mathbf{X}}(n_1, \dots, n_k) = p_{n_1} \prod_{l=2}^k p_{n_l, n_{l-1}}$, for each $k \in \mathbb{N}$ and $1 \leq n_1, \dots, n_k \leq N$, and p is invariant with respect to P ; i.e., $Pp = p$; (the reason that p is called

“initial distribution” is because it coincides with the distribution of the random variable X_1). We will refer to a stochastic ensemble governed by a Markov process \mathbf{X} as the *Markov ensemble governed by \mathbf{X}* . Whenever the Markov process is stationary we will refer to the Markov ensemble as being stationary.

Let $\{\mathcal{S}^k\}_{k=1}^\infty$ be a stationary Markov ensemble governed by a stationary Markov process \mathbf{X} having transition matrix $P = (p_{n,m})$ and initial distribution $p = \{p_n\}_{n=1}^N$. Setting $d = \dim(H_S)$, so that $d^k = \dim(H_S^{\otimes k})$ for each $k \in \mathbb{N}$, the following sequence of ensemble states which represent this collection of ensembles is defined:

$$\rho_{S^1} = \sum_{n=1}^N p_n |s_n\rangle \langle s_n| \in M_d = S_1(H_S)$$

and for each $k \in \mathbb{N}$ with $k \geq 2$,

$$\begin{aligned} \rho_{S^k} &= \sum_{n_1, \dots, n_k=1}^N p_{n_1} \prod_{l=2}^k p_{n_l, n_{l-1}} |s_{n_1} \cdots s_{n_k}\rangle \langle s_{n_1} \cdots s_{n_k}| \\ &= \sum_{n_1=1}^N p_{n_1} |s_{n_1}\rangle \langle s_{n_1}| \otimes \cdots \otimes \sum_{n_k=1}^N p_{n_k, n_{k-1}} |s_{n_k}\rangle \langle s_{n_k}| \in M_d^{\otimes k} \\ &= S_1(H_S^{\otimes k}), \end{aligned}$$

where we used the following notation.

Notation 3.1. If H is a Hilbert space, then $S_1(H)$ will denote the space of trace-class operators on H . In the sequel we will frequently denote by \mathcal{A} a unital C^* -algebra, and $\Sigma(\mathcal{A})$ will denote the set of normal states on \mathcal{A} . Since we assume that $\mathcal{A} = B(H)$, we will identify each normal state, $\omega \in \Sigma(\mathcal{A})$, with its density operator $\rho \in S_1(H)$ through the identification $\omega(\cdot) = \text{tr}(\rho \cdot)$.

We now define a quantum dynamical system associated with the above stationary Markov ensemble. Recall that a *quantum dynamical system* is a triplet $(\mathcal{A}, \Theta, \rho)$ where \mathcal{A} is a unital C^* -algebra, $\Theta : \mathcal{A} \rightarrow \mathcal{A}$ is a positive unital map, and ρ is a density operator on \mathcal{A} . In some situations the map Θ is taken to be a $*$ -automorphism, but we will not adopt this restriction here. The reason that we assume that Θ is positive and unital is because we would like to have that the dual map Θ^\dagger maps the set of density operator of \mathcal{A} (i.e., the set of positive unital functionals on \mathcal{A}) to itself. Throughout this paper we will, for simplicity, ignore the GNS construction, and when we do not specify the C^* -algebra we will assume that it is equal to $\mathcal{A} = B(H)$ for some Hilbert space H .

The quantum dynamical system associated to the above stationary Markov ensemble is defined as follows: Let $\mathcal{A} = B(\mathbb{C}^N)$, $\Theta : B(\mathbb{C}^N) \rightarrow B(\mathbb{C}^N)$ be defined by

$$\Theta(|k\rangle \langle \ell|) = \delta_{k,\ell} \sum_{i=1}^N p_{k,i} |i\rangle \langle i|, \tag{9}$$

and let the density operator $\rho \in S_1(\mathbb{C}^N)$ be defined by

$$\rho = \sum_{n=1}^N p_n |n\rangle \langle n|. \tag{10}$$

It is easy to verify that the map Θ is positive and unital. Indeed,

$$\begin{aligned} \Theta(\mathbb{1}_{\mathbb{C}^N}) &= \Theta\left(\sum_{k=1}^M |k\rangle \langle k|\right) \\ &= \sum_{k,i=1}^N p_{k,i} |i\rangle \langle k| \\ &= \sum_{i=1}^N |i\rangle \langle i| \sum_{k=1}^N p_{k,i} \\ &= \sum_{i=1}^N |i\rangle \langle i| = \mathbb{1}_{\mathbb{C}^N}, \end{aligned}$$

thus Θ is unital. Also it is easy to verify that the dual map $\Theta^\dagger : S_1(\mathbb{C}^N) \rightarrow S_1(\mathbb{C}^N)$ is given by

$$\Theta^\dagger(|m\rangle \langle n|) = \delta_{m,n} \sum_{i=1}^N p_{i,n} |i\rangle \langle n|. \tag{11}$$

Note that throughout the article we use dagger to denote trace-class duality, e.g.,

$$\text{tr}(\Theta^\dagger(|m\rangle \langle n|) |k\rangle \langle \ell|) = \text{tr}(|m\rangle \langle n| \Theta(|k\rangle \langle \ell|)),$$

for all $m, n, k, \ell \in \{1, \dots, N\}$, and star to denote Hermitian conjugate with respect to the underlying Hilbert space, \mathbb{C}^N . We thus obtain that the dual map Θ^\dagger is positive and trace-preserving.

Finally we recall that if H is a Hilbert space, then an *operational partition of unity on H* is a family $\gamma = (\gamma_i)_{i=1}^d$ for some $d \in \mathbb{N}$, satisfying $\gamma_i \in B(H)$, for each i , and $\sum_{i=1}^d \gamma_i^* \gamma_i = \mathbb{1}_H$. Let $\gamma = (\gamma_i)_{i=1}^d$ be the operational partition of unity on the Hilbert space \mathbb{C}^N associated with the above Markov ensemble, defined as follows:

$$\gamma_i := \sum_{n=1}^N \langle e_i | s_n \rangle |n\rangle \langle n|, \quad \text{for all } i = 1, \dots, d, \tag{12}$$

where $\{e_i\}_{i=1}^d$ is a fixed orthonormal basis of H_S . Notice that

$$\begin{aligned} \sum_{i=1}^d \gamma_i^* \gamma_i &= \sum_{i=1}^d \sum_{m,n=1}^N \langle e_i | s_m \rangle \overline{\langle e_i | s_n \rangle} |n\rangle \langle n| m\rangle \langle m| \\ &= \sum_{n=1}^N \left(\sum_{i=1}^d |\langle e_i | s_n \rangle|^2 \right) |n\rangle \langle n| \\ &= \sum_{n=1}^N \|s_n\| |n\rangle \langle n| = \mathbb{1}_{\mathbb{C}^N}, \end{aligned}$$

where the second to last equality follows by Parseval’s identity. Hence γ is indeed an operational partition of unity.

B. Quantum dynamical entropy via quantum Markov chains

In this subsection we recall the definition of quantum Markov chains (QMCs) and dynamical entropy thereon. Fix a quantum dynamical system $(\mathcal{A}, \Theta, \rho)$ with $\mathcal{A} = B(H)$ for some Hilbert space H and fix an operational partition of unity $\gamma = (\gamma_i)_{i=1}^d$ on H . Following p. 413 of Ref. [12] (see also Eq. 3.14 of Ref. [14]), we will consider the *transition expectation*

$$\mathcal{E}_\gamma : M_d \otimes \mathcal{A} = B(\mathbb{C}^d \otimes H) \rightarrow \mathcal{A}$$

given by the equation

$$\begin{aligned} \mathcal{E}_\gamma([a_{i,j}]_{i,j=1}^d) &= \sum_{i,j=1}^d \gamma_i^* a_{i,j} \gamma_j \quad \text{for all } [a_{i,j}]_{i,j=1}^d \\ &= \sum_{i,j=1}^d |e_i\rangle\langle e_j| \otimes a_{i,j} \in M_d \otimes \mathcal{A} \end{aligned} \quad (13)$$

for some fixed orthonormal basis $\{e_i\}_{i=1}^d$ of \mathbb{C}^d . Further we define the transition expectation

$$\mathcal{E}_{\gamma,\Theta} = \Theta \circ \mathcal{E}_\gamma : M_d \otimes \mathcal{A} \rightarrow \mathcal{A}. \quad (14)$$

Its dual map

$$\begin{aligned} \mathcal{E}_{\gamma,\Theta}^\dagger &= \mathcal{E}_\gamma^\dagger \circ \Theta^\dagger : S_1(H) = \Sigma(\mathcal{A}) \rightarrow S_1(\mathbb{C}^d \otimes H) \\ &= \Sigma(M_d \otimes \mathcal{A}) \end{aligned}$$

(which is defined using trace duality), is usually called a lifting because it ‘‘lifts’’ states from \mathcal{A} to $M_d \otimes \mathcal{A}$.

If H is a Hilbert space, \mathcal{A} is the von Neumann algebra $B(H)$ of all bounded operators on H , ρ is a density operator on H , and, for some $d \in \mathbb{N}$, $\mathcal{E} : M_d \otimes \mathcal{A} \rightarrow \mathcal{A}$ is a transition expectation, then the pair $\{\rho, \mathcal{E}\}$ is called a *quantum Markov chain* (QMC). We will be specifically interested in QMCs whose transition expectation is given by Eq. (14). Given a quantum Markov chain, we define the *quantum Markov state* ψ on $M_d^{\otimes \mathbb{N}}$ by the equation

$$\begin{aligned} \psi(a_1 \otimes \cdots \otimes a_n) \\ = \text{tr}(\rho \mathcal{E}(a_1 \otimes \mathcal{E}(a_2 \otimes \mathcal{E}(\cdots \mathcal{E}(a_n \otimes \mathbb{1}_H) \cdots))))), \end{aligned} \quad (15)$$

for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in M_d$. Notice that the assumption that the transition expectation \mathcal{E} is unital implies that ψ is compatible in the sense that

$$\psi(a_1 \otimes \cdots \otimes a_n \otimes \mathbb{1}_{\mathbb{C}^d}) = \psi(a_1 \otimes \cdots \otimes a_n),$$

for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in M_d$. Moreover, it was shown in Proposition 3.7 of Ref. [20] that the state ψ on $M_d^{\otimes \mathbb{N}}$ indeed exists.

The *joint correlations* for ψ are given by the density matrices $\rho_n \in M_d^{\otimes n}$ satisfying

$$\psi(a_1 \otimes \cdots \otimes a_n) = \text{tr}(\rho_n a_1 \otimes \cdots \otimes a_n), \quad (16)$$

for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in M_d$.

Putting the above pieces together, if $\Theta : \mathcal{A} \rightarrow \mathcal{A}$ is a positive, unital map on the von Neumann algebra $\mathcal{A} = B(H)$, ρ is a density operator on H , and $\gamma = (\gamma_i)_{i=1}^d$ is an operational partition of unity of H , then the dynamical entropy of $(\mathcal{A}, \Theta, \rho)$ with respect to γ is given by

$$h(\Theta, \rho, \gamma) = \limsup_{n \rightarrow \infty} \frac{1}{n} S(\rho_n), \quad (17)$$

where $S(\cdot)$ is the von Neumann entropy and the transition expectation is given by Eq. (14). Further, given a subalgebra \mathcal{B} of \mathcal{A} , the *dynamical entropy of $(\mathcal{A}, \Theta, \rho)$ with respect to \mathcal{B}*

is given by

$$h_{\mathcal{B}}(\Theta, \rho) = \sup_{\gamma \subseteq \mathcal{B}} h(\Theta, \rho, \gamma).$$

Remark 3.2. The dynamical entropy above is the generalized AF dynamical entropy as defined by the authors of Ref. [14]. The description we give is very similar to that of the AF dynamical entropy given by Tuyls [12]; however, we do not restrict ourselves to $*$ -automorphisms as does the standard construction of AF dynamical entropy.

C. Computation of the quantum dynamical entropy of the quantum dynamical system defined in Sec. III A

Let $(\mathcal{A}, \Theta, \rho)$ be the quantum dynamical system defined by (9) and (10), and let γ be the operational partition of unity defined by (12). In this subsection, we will use the definitions given in Sec. III B in order to compute the quantum dynamical entropy $h(\Theta, \rho, \gamma)$ and give its interpretation as the optimal compression rate of the quantum Markov ensemble that we consider.

First we define vectors

$$|s'_n\rangle := |s_n\rangle \otimes |n\rangle \in H_S \otimes \mathbb{C}^N \quad \text{for } n = 1, \dots, N$$

(which are orthonormal even though the vectors $(|s_n\rangle)_{n=1}^N \subseteq H_S$ are not necessarily mutually orthogonal), and the state

$$\begin{aligned} \rho' &:= \sum_{n=1}^N p_n |s'_n\rangle\langle s'_n| = \sum_{n=1}^N p_n |s_n\rangle\langle s_n| \otimes |n\rangle\langle n| \in M_d \otimes M_N \\ &= S_1(H_S \otimes \mathbb{C}^N). \end{aligned} \quad (18)$$

Before proceeding with the construction of the quantum Markov chain, we give a technical lemma which will be helpful later.

Lemma 3.3. Let $\{\mathcal{S}^k\}_{k=1}^\infty$ be a stationary Markov ensemble, with symbol states $\{|s_n\rangle\}_{n=1}^N$, which is governed by a stationary Markov process \mathbf{X} with transition matrix $P = (p_{n,m})$. Let $\Theta, \gamma, \mathcal{E}_\gamma$, and $\mathcal{E}_{\gamma,\Theta}$ be defined as above. Then the lifting $\mathcal{E}_{\gamma,\Theta}^\dagger : S_1(\mathbb{C}^N) \rightarrow S_1(H_S) \otimes S_1(\mathbb{C}^N)$ acts on the diagonal states of $S_1(\mathbb{C}^N)$ in the following way:

$$\mathcal{E}_{\gamma,\Theta}^\dagger(|n\rangle\langle n|) = \sum_{m=1}^N p_{m,n} |s'_m\rangle\langle s'_m|,$$

for each $|n\rangle$ in the orthonormal basis of \mathbb{C}^N . Moreover,

$$\mathcal{E}_{\gamma,\Theta}^\dagger(\rho) = \rho',$$

where ρ and ρ' are given by Eqs. (10) and (18), respectively.

The proof of Lemma 3.3 can be found in the Appendix.

Next, we will consider the quantum Markov state ψ given by the chain $\{\rho, \mathcal{E}_{\gamma,\Theta}\}$, where ρ is given in Eq. (10) and $\mathcal{E}_{\gamma,\Theta}$ is as in Eq. (14). Then, for each $k \in \mathbb{N}$ and $a_1, \dots, a_k \in B(H_S) = M_d$, we have

$$\begin{aligned} \psi(a_1 \otimes \cdots \otimes a_k) &= \text{tr}(\rho \mathcal{E}_{\gamma,\Theta}(a_1 \otimes \mathcal{E}_{\gamma,\Theta}(\cdots \mathcal{E}_{\gamma,\Theta}(a_k \otimes \mathbb{1}_{\mathbb{C}^N})))) \quad \text{by Eq. (15)} \\ &= \text{tr}(\mathcal{E}_{\gamma,\Theta}^\dagger(\rho) a_1 \otimes \mathcal{E}_{\gamma,\Theta}(\cdots \mathcal{E}_{\gamma,\Theta}(a_k \otimes \mathbb{1}_{\mathbb{C}^N}))) \end{aligned}$$

$$\begin{aligned}
 &= \text{tr} \left(\sum_{n_1=1}^N p_{n_1} |s'_{n_1}\rangle\langle s'_{n_1}| a_1 \otimes \mathcal{E}_{\gamma, \Theta}(a_2 \otimes \mathcal{E}_{\gamma, \Theta}(\cdots \mathcal{E}_{\gamma, \Theta}(a_k \otimes \mathbb{1}_{\mathbb{C}^N}))) \right) \\
 &= \sum_{n_1=1}^N p_{n_1} \text{tr} (|s_{n_1}\rangle\langle s_{n_1}| a_1) \text{tr} (|n_1\rangle\langle n_1| \mathcal{E}_{\gamma, \Theta}(a_2 \otimes \mathcal{E}_{\gamma, \Theta}(\cdots \mathcal{E}_{\gamma, \Theta}(a_k \otimes \mathbb{1}_{\mathbb{C}^N}))) \\
 &= \sum_{n_1, n_2=1}^N p_{n_1} p_{n_2, n_1} \text{tr} (|s_{n_1}\rangle\langle s_{n_1}| a_1) \text{tr} (|s_{n_2}\rangle\langle s_{n_2}| a_2) \text{tr} (|n_2\rangle\langle n_2| \mathcal{E}_{\gamma, \Theta}(a_3 \otimes \mathcal{E}_{\gamma, \Theta}(\cdots \mathcal{E}_{\gamma, \Theta}(a_k \otimes \mathbb{1}_{\mathbb{C}^N}))) \\
 &\quad \vdots \\
 &= \sum_{n_1, \dots, n_k=1}^N p_{n_1} \prod_{l=2}^k p_{n_l, n_{l-1}} \text{tr} (|s_{n_1}\rangle\langle s_{n_1}| a_1) \cdots \text{tr} (|s_{n_k}\rangle\langle s_{n_k}| a_k),
 \end{aligned}$$

where the “moreover” part of Lemma 3.3 was used in the third equality, the fact $\text{tr}(A \otimes B) = \text{tr}(A) \text{tr}(B)$ was used in the fourth equality, and Lemma 3.3 was used in the fifth equality.

Thus, for each $k \in \mathbb{N}$, the density matrix ρ_k which is defined by Eq. (16) is given by

$$\rho_k = \sum_{n_1, \dots, n_k=1}^N p_{n_1} \prod_{l=2}^k p_{n_l, n_{l-1}} |s_{n_1} \cdots s_{n_k}\rangle\langle s_{n_1} \cdots s_{n_k}| = \rho_{S^k}. \tag{19}$$

Therefore,

$$\begin{aligned}
 h(\Theta, \rho, \gamma) &= \limsup_{k \rightarrow \infty} \frac{1}{k} S(\rho_k) = \limsup_{k \rightarrow \infty} \frac{1}{k} S(\rho_{S^k}) \\
 &= \limsup_{k \rightarrow \infty} \text{EL}_k^*(\rho_{S^k}),
 \end{aligned}$$

where the first equality holds by the definition of the dynamical entropy in Eq. (17) and the last equality follows from Equation (8). We have proved the following result.

Theorem 3.4. Given any stationary Markov ensemble $\{S^k\}_{k=1}^\infty$, the optimal average codeword length per symbol (via lossless coding) converges to the dynamical entropy of the above-described quantum dynamical system $(B(\mathbb{C}^N), \Theta, \rho)$ with respect to the operational partition of unity γ defined in Eq. (12) in the following sense:

$$\limsup_{k \rightarrow \infty} \text{EL}_k^*(\rho_{S^k}) = \limsup_{k \rightarrow \infty} \frac{1}{k} S(\rho_{S^k}) = h(\Theta^*, \rho_0, \gamma).$$

We recover the result of Schumacher [3] and Bellomo *et al.* [7], which states that the optimal codeword length per symbol for an i.i.d. prepared ensemble, $\{S^k\}_{k=1}^\infty$, (via asymptotically lossless coding) is equal to the von Neumann entropy of the initial ensemble state, ρ_{S^1} :

Corollary 3.5. Given a Markov process \mathbf{X} made up of i.i.d. copies of a random variable X , the stationary Markov ensemble $\{S^k\}_{k=1}^\infty$ governed by \mathbf{X} has optimal codeword length per symbol (via lossless coding) given by

$$\lim_{k \rightarrow \infty} \text{EL}_k^*(\rho_{S^k}) = S(\rho_{S^1}).$$

Proof. First notice that \mathbf{X} is governed by the transition matrix $P = (p_{n,m})_{n,m=1}^N$ such that $p_{n,m} = p_n$, for every $1 \leq n, m \leq N$, where $p = (p_n)_{n=1}^N$ is the initial distribution of \mathbf{X} . Therefore

$$\rho_{S^k} = \rho_{S^1}^{\otimes k}, \quad \text{for each } k \in \mathbb{N}.$$

Using the construction from above and Eq. (19), we have that

$$S(\rho_k) = S(\rho_{S^k}) = S(\rho_{S^1}^{\otimes k}) = kS(\rho_{S^1}),$$

where the last inequality follows by additivity of von Neumann entropy [see, e.g., Eq. (2.8) of Ref. [21]]. Therefore, by Theorem 3.4, we have

$$\lim_{k \rightarrow \infty} \text{EL}_k^*(\rho_{S^k}) = \lim_{k \rightarrow \infty} \frac{1}{k} S(\rho_{S^k}) = \lim_{k \rightarrow \infty} \frac{1}{k} kS(\rho_{S^1}) = S(\rho_{S^1}). \quad \blacksquare$$

Next we turn to a similar representation for general stochastic ensembles. We chose to present the case of the stationary Markov ensemble separately since the construction is simpler than in the general case.

D. A quantum dynamical system associated with a general stochastic ensemble

Consider a stochastic process $\mathbf{X} = (X_n)_{n=1}^\infty$ with values in $\{1, \dots, N\}$ for some $N < \infty$ and with pmf p ; i.e., for any $k \in \mathbb{N}$ and any $(n_1, \dots, n_k) \in \{1, \dots, N\}^k$ we have $p(n_1, \dots, n_k) = \Pr[X_1 = n_1, \dots, X_k = n_k]$. Define the associated stochastic ensemble $\{S^k\}_{k=1}^\infty$ by $S^1 = \{p(n), |s_n\rangle\}_{n=1}^N$ whose symbol states span H_S and $S^k = \{p(n_1, \dots, n_k), |s_{n_1} \cdots s_{n_k}\rangle\}_{n_1, \dots, n_k=1}^N$ whose symbol states span $H_S^{\otimes k}$ for each $k \in \mathbb{N}$. Again, setting $d = \dim(H_S)$, so that $d^k = \dim(H_S^{\otimes k})$ for each $k \in \mathbb{N}$, we define the following sequence of ensemble states which represents this stochastic ensemble:

$$\rho_{S^1} = \sum_{n=1}^N p(n) |s_n\rangle\langle s_n| \in M_d = S_1(H_S)$$

and, for each $k \in \mathbb{N}$ with $k \geq 2$, define $\rho_{S^k} \in M_d^{\otimes k} = S_1(H_S^{\otimes k})$ by

$$\begin{aligned} \rho_{S^k} &= \sum_{n_1, \dots, n_k=1}^N p(n_1, \dots, n_k) |s_{n_1} \cdots s_{n_k}\rangle \langle s_{n_1} \cdots s_{n_k}| \\ &= \sum_{n_1=1}^N p(n_1) |s_{n_1}\rangle \langle s_{n_1}| \otimes \cdots \otimes \\ &\quad \times \sum_{n_k=1}^N p(n_k | n_1, \dots, n_{k-1}) |s_{n_k}\rangle \langle s_{n_k}|. \end{aligned}$$

We define a quantum dynamical system associated to the above quantum ensemble as follows: Let $H = (\mathbb{C}^N)^{\oplus} = \bigoplus_{n=0}^{\infty} (\mathbb{C}^N)^{\otimes n}$ be the free Fock space of \mathbb{C}^N . Recall that $(\mathbb{C}^N)^{\otimes 0} = \mathbb{C}$, and we denote by $|\emptyset\rangle$ the vector $1 \in (\mathbb{C}^N)^{\otimes 0}$. We denote $\{|n\rangle : n \in \{1, \dots, N\}\}$ the standard orthonormal basis of \mathbb{C}^N and

$$\begin{aligned} \{|\emptyset\rangle\} \cup \{|\bar{n}\rangle : \bar{n} \in \{1, \dots, N\}^k, k \in \mathbb{N}\} \\ = \{|\bar{n}\rangle : \bar{n} \in \{1, \dots, N\}^+\} \end{aligned}$$

the standard orthonormal basis of H . Before proceeding further we introduce two useful notations on the standard orthonormal basis of H , which will be used later. If $\bar{n} = (n_1, \dots, n_k) \in \{1, \dots, N\}^k$ for some $k \in \mathbb{N}$, then we set

$$\begin{aligned} \text{final}(\bar{n}) = n_k, \quad \text{pruned}(\bar{n}) = |n_1, \dots, n_{k-1}\rangle \text{ if } k \geq 2 \\ \text{and } \text{pruned}(\bar{n}) = |\emptyset\rangle \text{ if } k = 1. \end{aligned}$$

Set \mathcal{A} to denote C^* -subalgebra of $B(H)$ generated by the identity operator and the rank-one operators of the form $|\bar{n}\rangle \langle \bar{m}|$ where $\bar{n}, \bar{m} \in \{1, \dots, N\}^+$. Define a unital map $\Theta : \mathcal{A} \rightarrow \mathcal{A}$ by

$$\begin{aligned} \Theta(|\bar{n}\rangle \langle \bar{m}|) = 0 \quad \text{if at least one of } \bar{n}, \bar{m} \text{ is equal to } \emptyset, \\ \Theta(|\bar{n}\rangle \langle \bar{m}|) = 0 \quad \text{if } \bar{n} \neq \bar{m}, \end{aligned}$$

$$\Theta(|\bar{n}\rangle \langle \bar{n}|) = p(\text{final}(\bar{n}) | \text{pruned}(\bar{n})) | \text{pruned}(\bar{n})\rangle \langle \text{pruned}(\bar{n})|.$$

It is easy to see that the map Θ is positive and (by definition) unital. Finally we define a state

$$\rho = |\emptyset\rangle \langle \emptyset|,$$

on \mathcal{A} and consider the dynamical system $(\mathcal{A}, \Theta, \rho)$.

We also define an operational partition of unity of the Hilbert space H which is associated to the above quantum ensemble. Let $\{e_i\}_{i=1}^d$ be a fixed orthonormal basis of H_S , and let $\gamma = (\gamma_i)_{i=1}^d$ be the operational partition of unity defined by

$$\begin{aligned} \gamma_i(|\bar{n}\rangle) &= \langle e_i | s_{\text{final}(\bar{n})} \rangle |\bar{n}\rangle \text{ for } \bar{n} \in \{1, \dots, N\}^k, \\ \gamma_i(|\emptyset\rangle) &= \langle e_i | s_1 \rangle |\emptyset\rangle, \end{aligned}$$

for each $i \in \{1, \dots, d\}$. Notice that

$$\sum_{i=1}^d \gamma_i^* \gamma_i(|\bar{n}\rangle) = \sum_{i=1}^d |\langle e_i, s_{\text{final}(\bar{n})} \rangle|^2 |\bar{n}\rangle = \|s_{n_{k-1}}\|^2 |\bar{n}\rangle = |\bar{n}\rangle$$

for $\bar{n} \in \bigcup_{k \in \mathbb{N}} \{1, \dots, N\}^k$. Similarly,

$$\sum_{i=1}^d \gamma_i^* \gamma_i(|\emptyset\rangle) = \sum_{i=1}^d |\langle e_i, s_1 \rangle|^2 |\emptyset\rangle = \|s_1\|^2 |\emptyset\rangle = |\emptyset\rangle,$$

and thus γ is indeed an operational partition of unity on H . In order to unify the last three displayed formulas, we define $\text{final}(\emptyset) = 1$, and thus we can write

$$\gamma_i(|\bar{n}\rangle) = \langle e_i | s_{\text{final}(\bar{n})} \rangle |\bar{n}\rangle \quad \text{for all } \bar{n} \in \{1, \dots, N\}^+. \quad (20)$$

Let \mathcal{E}_γ and $\mathcal{E}_{\gamma, \Theta}$ be the transition expectation maps from $M_d \otimes \mathcal{A}$ to \mathcal{A} by Eqs. (13) and (14), respectively.

Before stating the next result we introduce some notation.

Notation 3.6. For each $k \in \mathbb{N}$ and $\bar{n} \in \{1, \dots, N\}^k$ we set

$$\begin{aligned} |s'_{\bar{n}}\rangle &= |s_{\text{final}(\bar{n})}\rangle \otimes |\bar{n}\rangle \in H_S \otimes (\mathbb{C}^N)^{\otimes k} \quad \text{and} \\ |s'_{\emptyset}\rangle &= |s_1\rangle \otimes |\emptyset\rangle \in H_S \otimes (\mathbb{C}^N)^{\otimes 0}. \end{aligned}$$

Also for $\bar{n} = (n_1, \dots, n_k) \in \{1, \dots, N\}^k$ and $\ell \in \{1, \dots, N\}$ we set

$$\bar{n} \circ \ell = (n_1, \dots, n_k, \ell) \text{ and } \emptyset \circ \ell = \ell.$$

We now state a technical lemma which will be used in the proof of the main result.

Lemma 3.7. Let $\{\mathcal{S}^k\}_{k=1}^{\infty}$ be a stochastic ensemble with symbol states $\{|s_n\rangle\}_{n=1}^N$ which is governed by a stochastic process \mathbf{X} with pmf p . Let $H, \mathcal{A}, \Theta, \rho, \gamma, \mathcal{E}_\gamma$, and $\mathcal{E}_{\gamma, \Theta}$ be defined as above. Then the lifting $\mathcal{E}_{\gamma, \Theta}^\dagger : \Sigma(\mathcal{A}) \rightarrow M_d \otimes \Sigma(\mathcal{A})$ acts on the diagonal states of $S_1(H)$ in the following way:

$$\mathcal{E}_{\gamma, \Theta}^\dagger(|\bar{n}\rangle \langle \bar{n}|) = \sum_{k=1}^N p(k | \bar{n}) |s'_{\bar{n} \circ k}\rangle \langle s'_{\bar{n} \circ k}|,$$

for each $|\bar{n}\rangle$ in the standard orthonormal basis of H where we adopt the convention $p(k | \emptyset) := p(k) = \Pr[X_1 = k]$ for $k \in \{1, \dots, N\}$. Moreover,

$$\mathcal{E}_{\gamma, \Theta}^\dagger(\rho) = \sum_{k=1}^N p(k) |s'_k\rangle \langle s'_k|.$$

The proof of Lemma 3.7 can be found in the Appendix.

Next, we will consider the quantum Markov state ψ given by the chain $\{\rho, \mathcal{E}_{\gamma, \Theta}\}$. For each $k \in \mathbb{N}$ and $a_1, \dots, a_k \in B(H_S) = M_d$, we have

$$\psi(a_1 \otimes \cdots \otimes a_k) = \text{tr}(\rho \mathcal{E}_{\gamma, \Theta}(a_1 \otimes \mathcal{E}_{\gamma, \Theta}(\cdots \mathcal{E}_{\gamma, \Theta}(a_k \otimes \mathbb{1}_H)))) \quad \text{by Equation (15)}$$

$$= \text{tr}(\mathcal{E}_{\gamma, \Theta}^\dagger(\rho) a_1 \otimes \mathcal{E}_{\gamma, \Theta}(\cdots \mathcal{E}_{\gamma, \Theta}(a_k \otimes \mathbb{1}_H)))$$

$$= \text{tr}\left(\sum_{n_1=1}^N p(n_1) |s'_{n_1}\rangle \langle s'_{n_1}| a_1 \otimes \mathcal{E}_{\gamma, \Theta}(a_2 \otimes \mathcal{E}_{\gamma, \Theta}(\cdots \mathcal{E}_{\gamma, \Theta}(a_k \otimes \mathbb{1}_H)))\right)$$

$$\begin{aligned}
 &= \sum_{n_1=1}^N p(n_1) \operatorname{tr} (|s_{n_1}\rangle\langle s_{n_1}| a_1) \operatorname{tr} (|n_1\rangle\langle n_1| \mathcal{E}_{\gamma, \Theta}(a_2 \otimes \mathcal{E}_{\gamma, \Theta}(\cdots \mathcal{E}_{\gamma, \Theta}(a_k \otimes \mathbb{1}_H)))) \\
 &= \sum_{n_1, n_2=1}^N p(n_1) p(n_2|n_1) \operatorname{tr} (|s_{n_1}\rangle\langle s_{n_1}| a_1) \operatorname{tr} (|s_{n_2}\rangle\langle s_{n_2}| a_2) \operatorname{tr} (|n_1, n_2\rangle\langle n_1, n_2| \mathcal{E}_{\gamma, \Theta}(a_3 \otimes \mathcal{E}_{\gamma, \Theta}(\cdots \mathcal{E}_{\gamma, \Theta}(a_k \otimes \mathbb{1}_H)))) \\
 &= \sum_{n_1, n_2=1}^N p(n_1, n_2) \operatorname{tr} (|s_{n_1}\rangle\langle s_{n_1}| a_1) \operatorname{tr} (|s_{n_2}\rangle\langle s_{n_2}| a_2) \operatorname{tr} (|n_1, n_2\rangle\langle n_1, n_2| \mathcal{E}_{\gamma, \Theta}(a_3 \otimes \mathcal{E}_{\gamma, \Theta}(\cdots \mathcal{E}_{\gamma, \Theta}(a_k \otimes \mathbb{1}_H)))) \\
 &\quad \vdots \\
 &= \sum_{n_1, \dots, n_k=1}^N p(n_1, \dots, n_k) \operatorname{tr} (|s_{n_1}\rangle\langle s_{n_1}| a_1) \cdots \operatorname{tr} (|s_{n_k}\rangle\langle s_{n_k}| a_k),
 \end{aligned}$$

where the “moreover” part of Lemma 3.7 was used in the third equality, the fact $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \operatorname{tr}(B)$ was used in the fourth equality, and Lemma 3.7 was used in the fifth equality.

Thus, for each $k \in \mathbb{N}$, the density matrix ρ_k which is defined in Eq. (16) is given by

$$\rho_k = \sum_{n_1, \dots, n_k=1}^N p(n_1, \dots, n_k) |s_{n_1} \cdots s_{n_k}\rangle\langle s_{n_1} \cdots s_{n_k}| = \rho_{S^k}.$$

Therefore,

$$\begin{aligned}
 h(\Theta, \rho, \gamma) &= \limsup_{k \rightarrow \infty} \frac{1}{k} S(\rho_k) = \limsup_{k \rightarrow \infty} \frac{1}{k} S(\rho_{S^k}) \\
 &= \limsup_{k \rightarrow \infty} \operatorname{EL}_k^*(\rho_{S^k}),
 \end{aligned}$$

where the first equality holds by the definition of the dynamical entropy in Eq. (17) and the last equality follows from Eq. (8). We have proved the following theorem.

Theorem 3.8. Given any stochastic ensemble $\{\mathcal{S}^k\}_{k=1}^\infty$, the optimal average codeword length per symbol (via lossless coding) converges to the dynamical entropy of the above-described quantum dynamical system $(\mathcal{A}, \Theta, \rho)$ with respect to the operational partition of unity γ defined by Eq. (20) in the following sense:

$$\limsup_{k \rightarrow \infty} \operatorname{EL}_k^*(\rho_{S^k}) = \limsup_{k \rightarrow \infty} \frac{1}{k} S(\rho_{S^k}) = h(\Theta, \rho, \gamma).$$

It should be noted that Theorem 3.4 can be considered a corollary of Theorem 3.8. However, we have presented it separately since the construction is simpler in the case of Markov ensembles.

E. Examples

Examples of quantum sources that produce not-necessarily statistically independent quantum symbols have been considered in the literature. In Ref. [9] examples of quantum sources that produce statistically independent symbols (called “Bernoulli sources”) as well as quantum sources producing

not-necessarily statistically independent quantum symbols are considered. In Ref. [19] the authors consider quantum Morse codes as an example of quantum communication since quantum data compression can be viewed as a special case of noiseless quantum communication. In communications, either classical or quantum, the assumption of statistical independence of the symbols to be communicated gives a serious restriction to the content of information which is communicated. Thus the need of considering quantum sources emitting not-necessarily statistical independent quantum symbols naturally arises.

We have already shown in Corollary 3.5 that we can recover the result of Refs. [3] and [7], which states that the optimal codeword length per symbol for an i.i.d. prepared ensemble (i.e., Bernoulli sources) is equal to the von Neumann entropy of the initial ensemble state. First we illustrate that we can recover Theorem 2.5 from Theorem 3.8.

Example 3.9 (Classical-Quantum Codes). Let $S = \{n\}_{n=1}^d$ be a classical symbol set of cardinality d for some $d \in \mathbb{N}$, $C : S \rightarrow A^+$ be a uniquely decodable code into strings from the binary alphabet A , and $\mathbf{X} = (X_n)_{n=1}^\infty$ be a stochastic process governing the frequency of symbols from the symbol set S . Let H_S be a d -dimensional Hilbert space spanned by an orthonormal basis $\{|s_n\rangle\}_{n=1}^d$ and define the stochastic ensemble as usual by $\mathcal{S}^k = \{p(n_1, \dots, n_k), |s_{n_1} \cdots s_{n_k}\rangle\}_{n_1, \dots, n_k=1}^d$, where p denotes the pmf of \mathbf{X} . Then since the $|s_n\rangle$'s are orthonormal, it is easy to see that the ensemble states ρ_{S^k} are diagonal, for each $k \in \mathbb{N}$. Hence the optimal average codeword length per symbol for the stochastic ensemble, given by $h(\mathcal{A}, \Theta, \rho)$ in Theorem 3.8, is exactly equal to the entropy rate of the stochastic process (see Theorem 2.5).

Next we illustrate here the usefulness of Theorem 3.4 on non-Bernoulli sources with two simple examples. For each of the two examples, the Hilbert space H_S has dimension $d = 2$ and an orthonormal basis $\{|e_i\rangle\}_{i=1}^2$.

Example 3.10. For the second example consider the normalized nonorthogonal symbols $|s_1\rangle = |e_1\rangle$, $|s_2\rangle = |e_2\rangle$, $|s_3\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle)$ (i.e., the Bell state $|+\rangle$), and $|s_4\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle - |e_2\rangle)$ (i.e., the Bell state $|-\rangle$), which span $H_S = \mathbb{C}^2$ (i.e., we consider $N = 4$ in the setting described in Sec. II B).

Consider the transition matrix

$$P = (p_{i,j})_{i,j=1}^4 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

where $p_{i,j}$ represents the conditional probability that the quantum source emits $|s_i\rangle$ right after it emits $|s_j\rangle$. A (nonunique)

i.e.,

$$\Theta((a_{i,j})_{i,j=1}^4) = \begin{pmatrix} \frac{1}{2}(a_{1,1} + a_{2,2}) & \frac{1}{2}(a_{1,1} + a_{2,2}) & 0 & 0 \\ \frac{1}{2}(a_{1,1} + a_{2,2}) & \frac{1}{2}(a_{1,1} + a_{2,2}) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(a_{1,1} + a_{2,2}) & \frac{1}{2}(a_{3,3} + a_{4,4}) \\ 0 & 0 & \frac{1}{2}(a_{1,1} + a_{2,2}) & \frac{1}{2}(a_{3,3} + a_{4,4}) \end{pmatrix}$$

and

$$\rho = \frac{1}{4}|1\rangle\langle 1| + \frac{1}{4}|2\rangle\langle 2| + \frac{1}{4}|3\rangle\langle 3| + \frac{1}{4}|4\rangle\langle 4| = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Consider the operational partition of unity $\gamma = (\gamma_i)_{i=1}^2$ for \mathbb{C}^4 given as in Eq. (12). Theorem 3.4 states that the optimal average codeword length per symbol via lossless coding is equal to the dynamical entropy of Θ with respect to the partition γ when measured using the state ρ , i.e., $h(\Theta, \rho, \gamma)$. We can compute the joint correlations $(\rho_n)_{n=1}^\infty$ of this dynamical system using Eq. (19) to see that

$$\rho_1 = \frac{1}{4} \sum_{n=1}^4 |s_n\rangle\langle s_n| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

and in general

$$\rho_k = \frac{1}{2^{k+1}} \sum_{n_1, \dots, n_k=1}^2 |s_{n_1}\rangle\langle s_{n_1}| \otimes \dots \otimes |s_{n_k}\rangle\langle s_{n_k}| \\ + \frac{1}{2^{k+1}} \sum_{m_1, \dots, m_k=1}^2 |s_{m_1}\rangle\langle s_{m_1}| \otimes \dots \otimes |s_{m_k}\rangle\langle s_{m_k}|.$$

It is easy to verify that

$$\sum_{n_1, \dots, n_k=1}^2 |s_{n_1}\rangle\langle s_{n_1}| \otimes \dots \otimes |s_{n_k}\rangle\langle s_{n_k}| \\ = \sum_{m_1, \dots, m_k=3}^4 |s_{m_1}\rangle\langle s_{m_1}| \otimes \dots \otimes |s_{m_k}\rangle\langle s_{m_k}| = \mathbb{1}_{\mathbb{C}^{2^k}},$$

by applying these sums to bases of \mathbb{C}^{2^k} that are formed by taking the tensor products of $|s_i\rangle$'s. Thus $\rho_k = \frac{1}{2^k} \mathbb{1}_{\mathbb{C}^{2^k}}$ and hence $S(\rho_k) = k$, for each $k \in \mathbb{N}$. Therefore by Theorem 3.4 we obtain that the optimal average compression rate per symbol for the above quantum ensemble is equal to 1 qubit.

Example 3.11. Consider the normalized nonorthogonal symbol states $|s_1\rangle = |e_1\rangle$, $|s_2\rangle = -\frac{1}{2}|e_1\rangle + \frac{\sqrt{3}}{2}|e_2\rangle$ and $|s_3\rangle =$

fixed probability distribution of P is equal to the column vector $(\frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4})^T$. Consider the quantum dynamical system $(\mathcal{A}, \Theta, \rho)$ where $\mathcal{A} = B(\mathbb{C}^4)$, $\Theta : \mathcal{A} \rightarrow \mathcal{A}$ is given by

$$\Theta(|k\rangle\langle \ell|) = \delta_{k,\ell} \sum_{i=1}^3 p_{\ell,i} |i\rangle\langle i|,$$

$-\frac{1}{2}|e_1\rangle - \frac{\sqrt{3}}{2}|e_2\rangle$ which span H_S (i.e., we consider $N = 3$ in the setting described in Sec. II B). Consider the transition matrix

$$P = (p_{i,j})_{i,j=1}^3 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

where $p_{i,j}$ represents the conditional probability that the quantum source emits $|s_i\rangle$ right after it emits $|s_j\rangle$. The unique fixed probability distribution of P is equal to the column vector $(\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3})^T$. Consider the quantum dynamical system $(\mathcal{A}, \Theta, \rho)$ where $\mathcal{A} = B(\mathbb{C}^3)$, $\Theta : \mathcal{A} \rightarrow \mathcal{A}$ is given by

$$\Theta(|k\rangle\langle \ell|) = \delta_{k,\ell} \sum_{i=1}^3 p_{\ell,i} |i\rangle\langle i|,$$

i.e.,

$$\Theta((a_{i,j})_{i,j=1}^3) \\ = \begin{pmatrix} \frac{1}{2}(a_{2,2} + a_{3,3}) & 0 & 0 \\ 0 & \frac{1}{2}(a_{1,1} + a_{3,3}) & 0 \\ 0 & 0 & \frac{1}{2}(a_{1,1} + a_{3,3}) \end{pmatrix}$$

and

$$\rho = \frac{1}{3}|1\rangle\langle 1| + \frac{1}{3}|2\rangle\langle 2| + \frac{1}{3}|3\rangle\langle 3| = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Consider the operational partition of unity $\gamma = (\gamma_i)_{i=1}^2$ for \mathbb{C}^3 given by

$$\gamma_1 = \langle e_1|s_1\rangle|1\rangle\langle 1| + \langle e_1|s_2\rangle|2\rangle\langle 2| + \langle e_1|s_3\rangle|3\rangle\langle 3| \\ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

and

$$\begin{aligned} \gamma_2 &= \langle e_2 | s_1 \rangle |1\rangle \langle 1| + \langle e_2 | s_2 \rangle |2\rangle \langle 2| + \langle e_2 | s_3 \rangle |3\rangle \langle 3| \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}. \end{aligned}$$

Theorem 3.4 states that the optimal average codeword length per symbol via lossless coding is equal to the dynamical entropy of Θ with respect to the partition γ when measured using the state ρ , i.e., $h(\Theta, \rho, \gamma)$. We can compute the joint correlations $(\rho_n)_{n=1}^\infty$ of this dynamical system using Eq. (19) to see that

$$\rho_1 = \frac{1}{3} |s_1\rangle \langle s_1| + \frac{1}{3} |s_2\rangle \langle s_2| + \frac{1}{3} |s_3\rangle \langle s_3| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

and in general

$$\begin{aligned} \rho_k &= \frac{1}{3 \cdot 2^{k-1}} \sum_{n_1=1}^3 \sum_{n_2, \dots, n_k=1}^2 |s_{n_1}\rangle \langle s_{n_1}| \otimes |s_{n_2}\rangle \langle s_{n_2}| \\ &\quad \times |s_{n_3}\rangle \langle s_{n_3}| \otimes \cdots \otimes |s_{n_k}\rangle \langle s_{n_k}|, \end{aligned}$$

where

$$n'_k = \sum_{l=1}^k n_l \pmod{3},$$

and we adopt the convention that the mod 3 function takes values in the set $\{1, 2, 3\}$. Using MATLAB we can obtain the following approximate values of the von Neumann entropies of the above matrices:

k	1	2	3	4	5	6
$\frac{1}{k} S(\rho_k)$	1	0.9528	0.9306	0.9169	0.9076	0.9008
k	7	8	9	10	11	12
$\frac{1}{k} S(\rho_k)$	0.8957	0.8918	0.8886	0.8861	0.8839	0.8822

The above decreasing numbers indicate that the optimal average compression rate per symbol for the above quantum stochastic ensemble is strictly less than 1 qubit.

IV. CONCLUDING REMARKS

In this paper, we developed further the theory of quantum data compression for indeterminate length quantum codes, building on the previous work of Schumacher and Westmoreland [4] and Bellomo *et al.* [7]. We presented the quantum Kraft Inequality with an additional converse statement which was not present in previous works; this additional converse statement makes the statement of the quantum Kraft Inequality more reminiscent of its classical counterpart. We also introduce the notion of stochastic ensembles and, in particular, stationary Markov ensembles, which, to the best of our knowledge, have not been considered elsewhere. The main contributions of this work are Theorems 3.4 and 3.8, which give a dynamical entropy interpretation of the optimal compression rate for stationary Markov and identically distributed

stochastic ensembles, respectively, extending the results of Schumacher [3] and Bellomo *et al.* [7], where the quantum symbol states to be encoded were prepared in an i.i.d. way. In doing so, we give a quantum Markov chain representation of a particular open quantum random walk. An interesting direction for future study is the development of quantum data compression on the symmetric Fock space, which is commonly used to model photons. We hope to develop this theory further in future work.

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APPENDIX: PROOFS OF AUXILIARY RESULTS

Proof of Theorem 2.9. For the forward direction we adapt the proof of Sec. II C of Ref. [4] to our formalism. Let U be a uniquely decodable quantum code with length eigenstates of the form

$$U = \sum_{i=1}^d |\psi_i\rangle \langle e_i|$$

and let $\{\ell_i\}_{i=1}^d$ be the length eigenvalues of U . For each $n, N \in \mathbb{N}$, let

$$\begin{aligned} C_n^N &= \{|\psi\rangle \in H_A^{\otimes N} : |\psi\rangle \\ &= |\psi_{i_1}\rangle |\psi_{i_2}\rangle \cdots |\psi_{i_n}\rangle \text{ for some } i_1, \dots, i_n \in \{1, \dots, d\}\} \end{aligned}$$

be the collection of length N strings consisting of n -many codewords and let

$$\begin{aligned} d_\ell &= |\{i \in \{1, \dots, d\} : \psi_i \in H_A^{\otimes \ell}\}| \\ &= |\{i \in \{1, \dots, d\} : \ell_i = \ell\}| \end{aligned}$$

be the number of length ℓ eigenstates of U , for each $\ell \in \mathbb{N}$. Then, by the unique decodability of U , each element of C_n^N has a unique representation as a string of n codewords and the elements of C_n^N are pairwise orthogonal, and hence we have

$$|C_n^N| = \sum_{\ell_1 + \dots + \ell_n = N} d_{\ell_1} d_{\ell_2} \cdots d_{\ell_n} \leq 2^N.$$

Thus

$$\begin{aligned} &2^{-N} \sum_{\ell_1 + \dots + \ell_n = N} d_{\ell_1} d_{\ell_2} \cdots d_{\ell_n} \\ &= \sum_{\ell_1 + \dots + \ell_n = N} (2^{-\ell_1} d_{\ell_1}) (2^{-\ell_2} d_{\ell_2}) \cdots (2^{-\ell_n} d_{\ell_n}) \leq 1. \end{aligned}$$

Set $\ell_{\max} = \max_{1 \leq i \leq d} \{\ell_i\}$ so that $N \leq n \ell_{\max}$. Summing the above inequality over N we obtain

$$\begin{aligned} &\sum_{\ell_1, \ell_2, \dots, \ell_n=1}^{\ell_{\max}} (2^{-\ell_1} d_{\ell_1}) (2^{-\ell_2} d_{\ell_2}) \cdots (2^{-\ell_n} d_{\ell_n}) \\ &= \left(\sum_{\ell=1}^{\ell_{\max}} (2^{-\ell} d_\ell) \right)^n \leq n \ell_{\max}. \end{aligned}$$

Notice that the left-hand side of this inequality is exponential whereas the right-hand side is linear. This implies that the left-hand side is bounded above by one. Hence we must have that

$$\text{tr}(U^\dagger 2^{-\Lambda} U) = \sum_{\ell=1}^{\ell_{\max}} 2^{-\ell} \text{tr}(U^\dagger \Pi_\ell U) = \sum_{\ell=1}^{\ell_{\max}} 2^{-\ell} d_\ell \leq 1. \quad (\text{A1})$$

Notice that the inequality in Eq. (A1) is simply a restatement of the classical Kraft-McMillan Inequality.

Conversely, suppose that U is a linear isometry with length eigenstates satisfying the quantum Kraft-McMillan Inequality, and define $\{\ell_i\}_{i=1}^d$, ℓ_{\max} and $\{d_\ell\}_{\ell=1}^{\ell_{\max}}$ as above. Then

$$\sum_{\ell=1}^{\ell_{\max}} 2^{-\ell} d_\ell = \sum_{\ell=1}^{\ell_{\max}} 2^{-\ell} \text{tr}(U^\dagger \Pi_\ell U) = \text{tr}(U^\dagger 2^{-\Lambda} U) \leq 1,$$

and hence the classical Kraft-McMillan inequality is also valid. Thus, by the converse of the classical Kraft-McMillan Theorem, one can find a classical uniquely decodable code C which has exactly d_ℓ -many codewords of length ℓ , for each $\ell \in \mathbb{N}$. The c-q scheme \tilde{U} constructed from this classical code C has the desired properties. ■

Proof of Lemma 3.3. Since $\mathcal{E}_{\gamma, \Theta} = \Theta \circ \mathcal{E}_\gamma$ we have that $\mathcal{E}_{\gamma, \Theta}^\dagger = \mathcal{E}_\gamma^\dagger \circ \Theta^\dagger$.

Next we consider the lifting $\mathcal{E}_\gamma^\dagger : S_1(\mathbb{C}^N) \rightarrow S_1(H_C) \otimes S_1(\mathbb{C}^N)$, which we claim is given by the formula

$$\mathcal{E}_\gamma^\dagger(\sigma) = [\gamma_i \sigma \gamma_j^*]_{i,j=1}^d = \sum_{i,j=1}^d |e_i\rangle\langle e_j| \otimes \gamma_i \sigma \gamma_j^*, \quad (\text{A2})$$

where we have identified $S_1(H_S)$ with M_d given the matrix representation with respect to the fixed orthonormal basis $\{|e_i\rangle\}_{i=1}^d$ used in Eqs. (13) and (12). Indeed, for $[a_{i,j}]_{i,j=1}^d \in B(H_S) \otimes B(\mathbb{C}^N)$ and $\sigma \in S_1(\mathbb{C}^N)$, we have

$$\begin{aligned} & \text{tr}(\sigma \mathcal{E}_\gamma([a_{i,j}]_{i,j=1}^d)) \\ &= \text{tr}\left(\sigma \sum_{i,j=1}^d \gamma_i^* a_{i,j} \gamma_j\right) = \sum_{i,j=1}^d \text{tr}(\sigma \gamma_i^* a_{i,j} \gamma_j) \\ &= \sum_{i,j=1}^d \text{tr}(\gamma_j \sigma \gamma_i^* a_{i,j}) = \sum_{i,j=1}^d \text{tr}(\gamma_i \sigma \gamma_j^* a_{j,i}) \\ &= \text{tr}\left(\sum_{i,j=1}^d |e_i\rangle\langle e_j| \otimes \gamma_i \sigma \gamma_j^* [a_{i,j}]_{i,j=1}^d\right), \end{aligned}$$

which proves the validity of Eq. (A2). Then, for each $|m\rangle\langle m| \in S_1(\mathbb{C}^N)$, we have

$$\begin{aligned} & \mathcal{E}_\gamma^\dagger(|m\rangle\langle m|) \\ &= \sum_{i,j=1}^d |e_i\rangle\langle e_j| \otimes \gamma_j |m\rangle\langle m| \gamma_i^* \\ &= \sum_{i,j=1}^d |e_i\rangle\langle e_j| \otimes \langle e_i, s_m | m \rangle \langle m | s_m, e_j \rangle \quad \text{by Eq. (12)} \end{aligned}$$

$$\begin{aligned} &= \left| \sum_{i=1}^d \langle e_i, s_m \rangle e_i \right| \left\langle \sum_{j=1}^d \langle e_j, s_m \rangle e_j \right| \otimes |m\rangle\langle m| \\ &= |s_m\rangle\langle s_m| \otimes |m\rangle\langle m| = |s'_m\rangle\langle s'_m|. \quad (\text{A3}) \end{aligned}$$

Combining Eqs. (9) and (A3), for each $|n\rangle\langle n| \in S_1(\mathbb{C}^N)$, we have

$$\begin{aligned} \mathcal{E}_{\gamma, \Theta}^\dagger(|n\rangle\langle n|) &= \mathcal{E}_\gamma^\dagger\left(\sum_{m=1}^N p_{m,n} |m\rangle\langle m|\right) \quad \text{by Eq. (11)} \\ &= \sum_{m=1}^N p_{m,n} |s'_m\rangle\langle s'_m| \quad \text{by Eq. (A3)}. \quad (\text{A4}) \end{aligned}$$

For the moreover statement, we have

$$\begin{aligned} \mathcal{E}_{\gamma, \Theta}^\dagger(\rho) &= \sum_{n=1}^N p_n \mathcal{E}_{\gamma, \Theta}^\dagger(|n\rangle\langle n|) \quad \text{by Eq. (10)} \\ &= \sum_{n,m=1}^N p_n p_{m,n} |s'_m\rangle\langle s'_m| \quad \text{by Eq. (A4)} \\ &= \sum_{m=1}^N p_m |s'_m\rangle\langle s'_m| = \rho' \quad \text{since } \mathbf{X} \text{ is stationary.} \end{aligned}$$

Proof of Lemma 3.7. It is easy to see that for each $\bar{m}, \bar{n} \in \{1, \dots, N\}^+$ (i.e., $|\bar{m}\rangle, |\bar{n}\rangle$ belong in the standard orthonormal basis of H), we have

$$\Theta^\dagger(|\bar{m}\rangle\langle \bar{n}|) = \delta_{\bar{m}, \bar{n}} \sum_{k=1}^N p(k|\bar{n}) |\bar{n} \circ k\rangle\langle \bar{n} \circ k|. \quad (\text{A5})$$

Next we consider the lifting $\mathcal{E}_\gamma^\dagger : S_1(H) \rightarrow S_1(H_S) \otimes S_1(H)$, which [by Eq. (A2)] is given by the formula

$$\mathcal{E}_\gamma^\dagger(\sigma) = [\gamma_i \sigma \gamma_j^*]_{i,j=1}^d = \sum_{i,j=1}^d |e_i\rangle\langle e_j| \otimes \gamma_i \sigma \gamma_j^*,$$

where we have identified $S_1(H_S)$ with M_d given the matrix representation with respect to the fixed orthonormal basis $\{|e_i\rangle\}_{i=1}^d$ of the Hilbert space H_S . Then, for each $\bar{n} \in \{1, \dots, N\}^+$, we have

$$\begin{aligned} \mathcal{E}_\gamma^\dagger(|\bar{n}\rangle\langle \bar{n}|) &= \sum_{i,j=1}^d |e_i\rangle\langle e_j| \otimes \gamma_i |\bar{n}\rangle\langle \bar{n}| \gamma_j^* \\ &= \sum_{i,j=1}^d |e_i\rangle\langle e_j| \otimes \langle e_i | s_{\text{final}(\bar{n})} | \bar{n} \rangle \langle \bar{n} | s_{\text{final}(\bar{n})} | e_j \rangle \\ &= \left| \sum_{i=1}^d \langle e_i | s_{\text{final}(\bar{n})} \rangle e_i \right| \left\langle \sum_{j=1}^d \langle e_j | s_{\text{final}(\bar{n})} \rangle e_j \right| \otimes |\bar{n}\rangle\langle \bar{n}| \\ &= |s_{\text{final}(\bar{n})}\rangle\langle s_{\text{final}(\bar{n})}| \otimes |\bar{n}\rangle\langle \bar{n}| = |s'_\bar{n}\rangle\langle s'_\bar{n}|, \quad (\text{A6}) \end{aligned}$$

where we used Eq. (20) in the second equality.

Combining Eqs. (A5) and (A6), for each $\bar{n} \in \{1, \dots, N\}^+$, we have

$$\begin{aligned} \mathcal{E}_{\gamma, \Theta}^\dagger(|\bar{n}\rangle\langle\bar{n}|) &= \mathcal{E}_\gamma^\dagger \left[\sum_{k=1}^N p(k|\bar{n})|\bar{n} \circ k\rangle\langle\bar{n} \circ k| \right] \\ &= \sum_{k=1}^N p(k|\bar{n})|s'_{\bar{n} \circ k}\rangle\langle s'_{\bar{n} \circ k}|. \end{aligned} \quad (\text{A7})$$

For the moreover statement, we have

$$\begin{aligned} \mathcal{E}_{\gamma, \Theta}^\dagger(\rho) &= \sum_{k=1}^N p(k|\emptyset)|s'_k\rangle\langle s'_k| \quad \text{by Eq. (A7)} \\ &= \sum_{k=1}^N p(k)|s'_k\rangle\langle s'_k|, \end{aligned}$$

where we again used the convention that $p(k|\emptyset) = p(k)$, for all $k \in \{1, \dots, N\}$, in the last equality. ■

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