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Editors' Suggestion

Epistemically restricted phase-space representation, weak momentum value, and reconstruction of the quantum wave function

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A phase-space distribution associated with a quantum state was previously proposed, which incorporates a specific epistemic restriction parametrized by a global random variable on the order of Planck constant, transparently manifesting quantum uncertainty in phase space. Here we show that the epistemically restricted phase-space (ERPS) distribution can be determined via weak measurement of momentum followed by postselection on position. In the ERPS representation, the phase and amplitude of the wave function are neatly captured respectively by the position-dependent (conditional) average and the variance of the epistemically restricted momentum fluctuation. They are in turn respectively determined by the real and imaginary parts of the weak momentum value, permitting a reconstruction of wave function using weak momentum value measurement, and an interpretation of momentum weak value in terms of epistemically restricted momentum fluctuations. The ERPS representation thus provides a transparent and rich framework to study the deep conceptual links between quantum uncertainty embodied in epistemic restriction, quantum wave function, and weak momentum measurement with position postselection, which may offer useful insight to better understand their meaning.

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I. INTRODUCTION

It is remarkable that ever since the inception of quantum mechanics, and despite its uncontested pragmatical successes, physicists still have no consensus on the meaning of pure quantum state or wave function, the key element of the theory. Is wave function a real-physical thing independent of measurement or is it a mathematical tool which represents some form of information? A better intuition about wave function may offer fresh insight into a deeper understanding of quantum paradoxes [1] and may hold the key to identifying the elusive physical resources underlying the power of quantum information protocols relative to their classical counterparts [2–4]. A powerful and rich method that could aid our (classical) intuition to grasp the quantum states and is useful to assess the quantum-classical correpondence and contrast is mapping the quantum states onto quasiprobability distributions over phase space [5–8]. Quasiprobability distributions are the quantum mechanical "analog" of classical phase-space distribution. The Wigner function, the earliest and the most well-known quasiprobability distribution, satisfies most of the intuitive requirements for a "proper" probability distribution over classical phase space, but it may take on negative values. Remarkably, the Wigner function can be operationally determined via standard strong (projective) measurements, leading to a method to reconstruct the underlying quantum state [9–13]. The intuition and insight provided by the quasiprobability representation has recently led to an important application in the field of quantum computation to devise efficient classical simulations and estimations of a certain class of quantum computational algorithms [14–17].

However, it is still not fully understood how two of the most distinctive features of the microscopic world, namely quantum uncertainty and entanglement, are deeply and transparently manifested in the quasiprobability phase-space representation. In addition, the mathematical structure of a complex quantum wave function is not transparently reflected in the associated quasiprobability distributions. The construction of quasiprobability is formal, guided more by mathematical ingenuity rather than coming from deep thinking about quantum uncertainty and entanglement [6–8]. Moreover, there are infinitely many quasiprobability representations, and the choice seems arbitrary. Can an insightful phase-space representation of quantum mechanics be singled out or motivated uniquely by requiring the microscopic world to obey the quantum uncertainty relation and to directly and transparently reflect the structure of a quantum wave function? Partly motivated by this conceptual problem, a phase-space representation for quantum mechanics was proposed in Ref. [18]. In the phase-space representation, a quantum wave function is associated with a phase-space distribution which explicitly incorporates a specific epistemic restriction parameterized by a global random variable on the order of Planck constant, transparently manifesting a quantum uncertainty relation in a classical phase space.

Meanwhile, in the past decades, there has been a lot of interest in an intriguing concept of weak value measurement over pre- and postselected ensembles [1,19–21]. To

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an extent, this measurement protocol reflects our ordinary intuition about measurement in everyday life [22], wherein one first gently measures a physical quantity without much disturbing the system and then follows with a postselection on a subensemble of interest. Yet, the outcomes of weak value measurement may go beyond the range of values obtained via the standard strong measurement, fueling hot debates about its meaning [23–30]. Weak value measurement offers access into a rich microscopic regime unreachable by the standard value and has found many interesting applications in quantum metrology [31,32], to study the conundra in quantum foundation [1,33–39], and recently for measuring the quantum state directly [40–43].

In the present work, we show that the above two seemingly different concepts, i.e., the "epistemically restricted phase space" (ERPS) representation [18] and the weak value measurement [19], are deeply interrelated; clarification of their relationship may offer fresh insight into their meaning. Namely, first, the ERPS distribution can be defined operationally using the weak measurement of momentum followed by postselection on position, denoted as the weak momentum value. Moreover, in the ERPS representation, the phase and the amplitude of the wave function are directly captured respectively by the position-dependent (conditional) average and the variance of the epistemically (statistically) restricted momentum fluctuation, which are, in turn, respectively determined by the real and imaginary parts of the weak momentum value. This observation permits a simple reconstruction of the wave function via the weak momentum value measurement and offers an interpretation of the complex weak momentum value in terms of the epistemically restricted momentum fluctuation. We further speculate on the possible deep relations between the epistemically restricted momentum field, Wiseman's naively observable average momentum field [22], and Hall-Johansen's best estimation of momentum given position [23,24,44], mediated operationally by the weak momentum measurement with position postselection. We expect that these deep links between different fundamental concepts may shed new light on the elusive meaning of wave function, to be further elaborated in the future [45].

II. EPISTEMICALLY RESTRICTED PHASE-SPACE REPRESENTATION FOR QUANTUM MECHANICS

Consider a general system of N spatial degrees of freedom with the configuration $q = (q_1, \dots, q_N)$. First, given a preparation characterized by a pure quantum state $|\psi\rangle$, we write the associated wave function $\psi(q) \doteq \langle q|\psi\rangle$ in polar form as

$$\psi(q) = \sqrt{\rho(q)}e^{iS(q)/\hbar}, \tag{1}$$

so that the amplitude $\rho(q)$ and the phase $S(q)$, both are real-valued functions, are given by $\rho(q) = |\psi(q)|^2$ and $S(q) = \frac{\hbar}{2i}[\ln\psi(q) - \ln\psi^*(q)]$. Following Born, we interpret $\rho(q)$ as the probability density that the system has a configuration q. We then define a conditional probability of the momentum $p = (p_1, \dots, p_N)$ given the conjugate positions q, associated with the wave function $\psi(q)$ via its phase and amplitude, as [18]

$$\mathbb{P}_\psi(p|q,\xi) = \prod_{n=1}^{N}\delta\left(p_n - \left(\partial_{q_n}S + \frac{\xi}{2}\frac{\partial_{q_n}\rho}{\rho}\right)\right). \tag{2}$$

Here ξ is a global-nonseparable variable with the dimension of action, fluctuating randomly on a microscopic timescale with the probability density $\chi(\xi)$, so that its average and variance are constant in space and time, given by

$$\overline{\xi} \doteq \int d\xi\,\xi\,\chi(\xi) = 0 \;\;\&\;\; \overline{\xi^2} = \hbar^2. \tag{3}$$

The conditional probability distribution of phase space associated with a preparation characterized by a quantum state $|\psi\rangle$ thus reads

$$\mathbb{P}_\psi(p,q|\xi) = \mathbb{P}_\psi(p|q,\xi)\rho(q) = \prod_{n=1}^{N}\delta\left(p_n - \left(\partial_{q_n}S + \frac{\xi}{2}\frac{\partial_{q_n}\rho}{\rho}\right)\right)\rho(q). \tag{4}$$

Clearly, unlike the Wigner function, the phase-space distribution $\mathbb{P}_\psi(p,q|\xi)$ is, by construction, always nonnegative. One can then show that the quantum expectation value of any quantum observable $\hat{O}$ up to second order in the momentum operator $\hat{p}$ over a quantum state $|\psi\rangle$ can be expressed as the conventional statistical average of the classical quantity $O(p,q)$ over the phase-space distribution of Eq. (4) [18], i.e.,

$$\langle\psi|\hat{O}|\psi\rangle = \int dq d\xi dp O(p,q)\mathbb{P}_\psi(p,q|\xi)\chi(\xi) \doteq \langle O\rangle_\psi\,. \tag{5}$$

Here, the Hermitian operator $\hat{O}$ must take the same form as that obtained by applying the standard canonical quantization scheme to $O(p,q)$ with a specific ordering of operators. Moreover, for $O(p,q)$ with cross terms between momenta of different degrees of freedom, i.e., p_ip_j, $i \neq j$, the nonseparability of ξ is indeed indispensable. See the Methods section in Ref. [18] for a proof.

Reading Eq. (5) from the right-hand side to the left-hand side, i.e., as a reconstruction of the quantum expectation value from the classical statistical phase-space average, one can transparently see that the form of the Hermitian quantum observable $\hat{O}$, including its operator ordering, is mathematically related to the definition of the wave function ψ given in Eq. (1). As an example of operator ordering, consider a system with one spatial dimension and a general classical physical quantity up to second order in momentum: $O(p,q) = A(q)p^2 + B(q)p + C(q)$, where $A(q)$, $B(q)$, and $C(q)$ are real-valued functions of q. Then, it is straightforward to show that, imposing the equality of Eq. (5), the correspondence rule between the classical quantity $O(p,q)$ and the associated Hermitian operator $\hat{O}$ takes the following ordering: $O(p,q) = A(q)p^2 + B(q)p + C(q) \mapsto \hat{p}A(\hat{q})\hat{p} + \frac{1}{2}[\hat{p}B(\hat{q}) + B(\hat{q})\hat{p}] + C(\hat{q}) \doteq \hat{O}$ (see the Methods section in Ref. [18]). It is, however, not clear how the ordering of operators in the above phase-space representation is related to the different orderings of operators in various quasiprobability representations [6,7].

We note that while the above-proposed phase-space distribution and the so-called Q (or more general, Husimi) quasiprobability distribution function are both non-negative for arbitrary wave functions, the correspondence rule between the quantum observable and the associated classical quantity in the two phase-space representations are different. To see this, consider a simple example where the quantum observable has the form $\hat{O} = \hat{q}^2$. Then, within our phase-space representation, the associated classical quantity takes the form $O = q^2$, as in Wigner-Weyl correspondence. On the other hand, within the Q phase-space formalism, the associated classical quantity takes the form $O = q^2 - \hbar/2$ [7]. In this sense, the ERPS representation thus inherits the desirable properties of Wigner and Q phase-space representations: namely, the ERPS distribution is non-negative as the Q function, and the correspondence rule between the Hermitian operator and the classical quantity is intuitive as in the Wigner-Weyl correspondence rule.

A few further notes on the physical interpretation of the phase-space representation are in order. First, the δ functional form of the conditional probability of Eq. (2) enables us to write p explicitly as a function of (q, ξ) as

$$p_n(q; \xi, \psi) = \partial_{q_n} S + \frac{\xi}{2} \frac{\partial_{q_n} \rho}{\rho}, \tag{6}$$

$n = 1, \dots, N$. It describes a momentum field fluctuating randomly due to the fluctuation of ξ. As is argued in Ref. [18], Eq. (6) can be interpreted as a fundamental epistemic or statistical restriction [46–49] on the allowed ensemble of trajectories. Namely, while in conventional classical statistical mechanics we can prepare an ensemble of trajectories with a desired distribution of positions $\rho(q)$ using an arbitrary momentum field, in the above phase-space model, the allowed form of momentum field must depend on the targeted $\rho(q)$ as prescribed by Eq. (6). Conversely, given a momentum field $p(q; \xi, \psi)$, unlike in conventional classical statistical mechanics, it is no longer possible to assign each trajectory in the momentum field an arbitrary weight $\rho(q)$. Hence, the allowed forms of distribution of positions $\rho(q)$ depend on, and thus is restricted fundamentally by, the underlying momentum field, satisfying Eq. (6). For this reason, we refer to $\mathbb{P}_\psi(p, q|\xi)$ of Eq. (4) as the epistemically restricted phase-space (ERPS) distribution associated with the wave function $\psi(q)$, and we refer to $p(q; \xi, \psi)$ defined in Eq. (6) as the epistemically restricted momentum field.

We have also argued in Ref. [18] that the epistemic restriction of Eq. (2) or (6), together with Eq. (3), transparently embodies the quantum uncertainty relation in classical phase space. For example, Eqs. (6) and (3) directly imply that the standard deviations of position and momentum, respectively denoted by σ_q and σ_p, must satisfy the Heisenberg-Kennard uncertainty relation $\sigma_q \sigma_p \geqslant \hbar/2$. In this sense, the epistemic restriction of Eq. (6) could be regarded as a "local," i.e., position-dependent, and thus "stronger," manifestation of the quantum uncertainty relation in classical phase space. Moreover, the unitary quantum dynamics, namely the Schrödinger equation, is uniquely singled out by requiring the statistically restricted ensemble of trajectories satisfying Eqs. (6) and (3) to further respect the conservation of average energy and trajectories (probability current) [18]. This includes those that describe quantum dynamical interactions between subsystems, generating quantum entanglement.

As an example of the ERPS representation, consider a spatially one-dimensional system prepared in a Gaussian wave function: $\psi_{\rm G}(q) = (\frac{1}{2\pi\sigma_q^2})^{\frac{1}{4}} \exp(-\frac{(q-q_o)^2}{4\sigma_q^2} + \frac{i}{\hbar} p_o q)$. What does the ERPS distribution look like? First, as per Eq. (1), we have $S_{\rm G}(q) = p_o q$ and $\rho_{\rm G}(q) = (\frac{1}{2\pi\sigma_q^2})^{\frac{1}{2}} \exp(-\frac{(q-q_o)^2}{2\sigma_q^2})$. Inserting it into Eq. (4), the ERPS distribution takes the form $\mathbb{P}_{\psi_{\rm G}}(p, q|\xi) = \delta(p - p_o + \frac{\xi}{2}\frac{(q-q_o)}{\sigma_q^2})(\frac{1}{2\pi\sigma_q^2})^{\frac{1}{2}} \exp(-\frac{(q-q_o)^2}{2\sigma_q^2})$. Note that the marginal distribution of position is given just by a Gaussian distribution, i.e., $\mathbb{P}_{\psi_{\rm G}}(q) = \int dp d\xi \mathbb{P}_{\psi_{\rm G}}(p, q|\xi)\chi(\xi) = \rho_{\rm G}(q) = (\frac{1}{2\pi\sigma_q^2})^{\frac{1}{2}} \exp(-\frac{(q-q_o)^2}{2\sigma_q^2})$. On the other hand, computing the marginal distribution of momentum, assuming that the distribution of ξ has the form $\chi(\xi) = \frac{1}{2}\delta(\xi - \hbar) + \frac{1}{2}\delta(\xi + \hbar)$ satisfying Eq. (3), we straightforwardly obtain a Gaussian distribution of momentum, i.e., $\mathbb{P}_{\psi_{\rm G}}(p) = \int d\xi dq \mathbb{P}_{\psi_{\rm G}}(p, q|\xi)\chi(\xi) = \int d\xi (\frac{2\sigma_q^2}{\pi\xi^2})^{\frac{1}{2}} \exp[-\frac{2\sigma_q^2}{\xi^2}(p - p_o)^2] \chi(\xi) = (\frac{2\sigma_q^2}{\pi\hbar^2})^{\frac{1}{2}} \exp[-\frac{2\sigma_q^2}{\hbar^2}(p - p_o)^2]$, with the standard deviation $\sigma_p^2 = \hbar^2/(4\sigma_q^2)$. Hence, for Gaussian wave functions, both the marginal distributions of position and momentum are equal to the corresponding quantum probabilities obtained in standard (strong) measurement, i.e., $\mathbb{P}_{\psi_{\rm G}}(q) = |\langle q|\psi_{\rm G}\rangle|^2$ and $\mathbb{P}_{\psi_{\rm G}}(p) = |\langle p|\psi_{\rm G}\rangle|^2$, as for the Wigner function. Note however that, unlike the Wigner function, this result cannot be extended to general wave functions. Namely, while we always have $\mathbb{P}_\psi(q) = \rho(q) = |\langle q|\psi\rangle|^2$ for a general wave function, in general we have $\mathbb{P}_\psi(p) \neq |\langle p|\psi\rangle|^2$, as is the case for, e.g., a Q function [7]. Hence, p in $\mathbb{P}_\psi(p, q|\xi)$ is not in general equal to the outcome of the momentum measurement.

Further, unlike general quasiprobability representations which, by construction, are devised to express the quantum expectation value of arbitrary Hermitian operators into phase-space integration evocative of the classical statistical average at the cost of allowing negative quasiprobability [50,51], in the ERPS representation, Eq. (5) applies only for Hermitian operators $\hat{O}$ up to second order in the momentum operator. Moreover, note that quasiprobability distributions are defined (bi)linearly in terms of wave function. By contrast, as explicitly seen in Eq. (4), the ERPS distribution $\mathbb{P}_\psi(p, q|\xi)$ is obtained by nonlinearly mapping the wave function ψ [52]. In this sense, we have thus given up some requirements of the quasiprobability representations and traded them for the following main conceptual advantages of ERPS representation: the quantum uncertainty relation is transparently manifested in the form of epistemic restriction in classical phase space, the ERPS distributions associated with wave functions are non-negative and the classical quantities associated with quantum observables have intuitive forms, and the Schrödinger equation can be directly derived by imposing the conservation of the average energy and the probability current. We expect that, like the quasiprobability representation, the ERPS representation may offer new insight to devise efficient classical simulation and/or estimation of a certain class of quantum computational algorithms [14–17]. An effort along this direction is reported in a different work [53].

Besides satisfying all the axioms for a true probability, the ERPS distribution of Eq. (4) also satisfies the following two intuitive requirements. First, consider a composite of two subsystems with the configuration $q = (q_1, q_2)$ and the conjugate momentum $p=(p_1,p_2)$. Assume that the preparations of the two subsystems are independent of each other so that the total wave function of the composite is factorizable, $\psi(q_1,q_2)=\psi_1(q_1)\psi_2(q_2)$. Noting Eq. (1), in this case, the total phase of the composite wave function is decomposable, $S(q_1,q_2)=S_1(q_1)+S_2(q_2)$, and the amplitude is factorizable, $\rho(q_1,q_2)=\rho(q_1)\rho(q_2)$. Inserting these into Eq. (4), the ERPS distribution associated with a pair of independent preparations is thus conditionally separable, i.e., $P_\psi(p,q|\xi)=P_\psi(p|q,\xi)\rho(q)=P_{\psi_1}(p_1,q_1|\xi)P_{\psi_2}(p_2,q_2|\xi)$. Note however that, due to the nonseparability of ξ, the distribution of (p,q,ξ) is nonfactorizable: $P_\psi(p,q,\xi)=P_\psi(p,q|\xi)\chi(\xi)\neq P_{\psi_1}(p_1,q_1,\xi)P_{\psi_2}(p_2,q_2,\xi)$. Next, let us apply the usual unitary phase-space shift operator, $|\psi\rangle\mapsto\hat{U}_D|\psi\rangle$, where $\hat{U}_D=e^{\frac{i}{\hbar}q_0\hat{p}-\frac{i}{\hbar}p_0\hat{q}}$. In this case, the wave function transforms as $\psi(q)\mapsto\psi(q-q_0)e^{-ip_0(q-q_0)/\hbar-ip_0q_0/2\hbar}$. Noting Eq. (1), this means that the phase transforms as $S(q)\mapsto S(q-q_0)-p_0q+p_0q_0/2$ and the amplitude as $\rho(q)\mapsto\rho(q-q_0)$. Inserting into Eq. (4), the ERPS distribution thus transforms as $P_\psi(p,q|\xi)\mapsto P_\psi(p-p_0,q-q_0|\xi)$; hence, it is covariant under the phase-space shift transformation.

Finally, since we work within the conventional statistical theory, the ERPS representation for pure states (wave functions) discussed above can be naturally extended to an incoherent mixture (convex combination) of pure states. For illustration, consider a mixed state $\hat{\varrho}=\sum_{l=1}^{L}r_l|\psi_l\rangle\langle\psi_l|$, where $0\leqslant r_l\leqslant 1$, $\sum_{l=1}^{L}r_l=1$. Then, the ERPS distribution associated with $\hat{\varrho}$ must take the form $\mathbb{P}_{\hat{\varrho}}(p,q|\xi)\doteq\sum_{l=1}^{L}r_l\mathbb{P}_{\psi_l}(p,q|\xi)$, where each $\mathbb{P}_{\psi_l}(p,q|\xi)$ is given by Eq. (4). The conventional statistical formula to compute the average value of Eq. (5) still applies: we only need to use $\mathbb{P}_{\hat{\varrho}}(p,q|\xi)$ in place of $\mathbb{P}_\psi(p,q|\xi)$. Namely, we have $\mathrm{Tr}\{\hat{\varrho}\hat{O}\}=\sum_l r_l\langle\psi_l|\hat{O}|\psi_l\rangle=\sum_l r_l\int dqd\xi dpO(p,q)\mathbb{P}_{\psi_l}(p,q|\xi)\chi(\xi)=\int dqd\xi dpO(p,q)\sum_l r_l\mathbb{P}_{\psi_l}(p,q|\xi)\chi(\xi)=\int dqd\xi dpO(p,q)\mathbb{P}_{\hat{\varrho}}(p,q|\xi)\chi(\xi)\doteq\langle O\rangle_{\hat{\varrho}}$.

III. ERPS DISTRIBUTION FROM WEAK MEASUREMENT OF MOMENTUM WITH POSTSELECTION ON THE CONJUGATE POSITION

We have nonlinearly mapped the wave function to obtain the ERPS distribution of Eq. (4). This raises the question of whether such a nonlinear mapping can be implemented experimentally. We show in this section that the ERPS distribution can indeed be defined operationally using weak value measurement [19]. For simplicity, consider a system of one spatial dimension; assume that it is prepared in a pure quantum state $|\psi\rangle$. Suppose we make a weak momentum measurement and followed by postselection on a subensemble passing through a position q implemented by making a strong projective measurement onto $|q\rangle\langle q|$. We thereby obtain the complex weak momentum value at q, denoted by $p^{\mathrm{w}}(q;\psi)$, as

$$p^{\mathrm{w}}(q;\psi)\doteq\frac{\langle q|\hat{p}|\psi\rangle}{\langle q|\psi\rangle}. \tag{7}$$

Below we refer to $p^{\mathrm{w}}(q;\psi)$ simply as the weak momentum value. Writing the wave function in polar form as in Eq. (1), the real and imaginary parts of the weak momentum value are straightforwardly given by

$$\mathrm{Re}\{p^{\mathrm{w}}(q;\psi)\}=\partial_qS\quad\text{and}\quad\mathrm{Im}\{p^{\mathrm{w}}(q;\psi)\}=-\frac{\hbar}{2}\frac{\partial_q\rho}{\rho}. \tag{8}$$

It has been shown in general that both the real and the imaginary parts of the weak value can be inferred respectively from the "average" shift of the position and the momentum of the measuring device pointer [20,21].

Noting Eq. (8), the epistemically restricted random momentum field of Eq. (6) can thus be written in terms of the weak momentum value as

$$p(q;\xi,\psi)=\mathrm{Re}\{p^{\mathrm{w}}(q;\psi)\}-\frac{\xi}{\hbar}\mathrm{Im}\{p^{\mathrm{w}}(q;\psi)\}. \tag{9}$$

Equation (9) can immediately be extended to N degrees of freedom so that the conditional distribution of momentum of Eq. (2) can be written as $\mathbb{P}_\psi(p|q,\xi)=\prod_{n=1}^{N}\delta(p_n-[\mathrm{Re}\{p_n^{\mathrm{w}}(q;\psi)\}-\frac{\xi}{\hbar}\mathrm{Im}\{p_n^{\mathrm{w}}(q;\psi)\}])$. The ERPS distribution thus reads, in terms of the weak momentum value, as

$$\begin{aligned}\mathbb{P}_\psi(p,q|\xi)&=\mathbb{P}_\psi(p|q,\xi)\rho(q)\\&=\prod_{n=1}^{N}\delta\Big(p_n-\Big[\mathrm{Re}\big\{p_n^{\mathrm{w}}(q;\psi)\big\}\\&\quad-\frac{\xi}{\hbar}\mathrm{Im}\big\{p_n^{\mathrm{w}}(q;\psi)\big\}\Big]\Big)\rho(q).\end{aligned} \tag{10}$$

Hence, the ERPS distribution can indeed be operationally obtained using weak measurement of momentum followed by postselection on the conjugate position, combined with the introduction of the (hypothetical) global random variable ξ satisfying Eq. (3). It is thus more than just a mathematical artifact. Such a weak momentum value measurement has already been performed as reported in Ref. [34].

The above observation conversely suggests the following statistical interpretation of the meaning of the complex weak momentum value within the ERPS representation. For simplicity, below we work again in one spatial dimension. First, in the ERPS representation, from Eqs. (9) and (3), the real part of the weak momentum value is equal to the position-dependent (conditional) average of the epistemically restricted random momentum field $p(q;\xi,\psi)$ of the system over the fluctuation of ξ, i.e.,

$$\begin{aligned}\overline{p}(q;\psi)&\doteq\int d\xi\,p(q;\xi,\psi)\chi(\xi)=\int d\xi\,dp\,p\mathbb{P}_\psi(p|q,\xi)\chi(\xi)\\&=\partial_qS(q)=\mathrm{Re}\{p^{\mathrm{w}}(q;\psi)\},\end{aligned} \tag{11}$$

where in the last equality we have used Eq. (8). On the other hand, the square of the imaginary part of the weak momentum value is equal to the position-dependent variance of the restricted random momentum field of the system, i.e.,

$$\begin{aligned}\overline{\overline{p}}(q;\psi)&\doteq\int d\xi[p(q;\xi,\psi)-\overline{p}(q;\psi)]^2\chi(\xi)\\&=\int d\xi\,dp(p-\partial_qS)^2\mathbb{P}_\psi(p|q,\xi)\chi(\xi)\\&=\frac{\hbar^2}{4}\left(\frac{\partial_q\rho}{\rho}\right)^2=(\mathrm{Im}\{p^{\mathrm{w}}(q;\psi)\})^2.\end{aligned} \tag{12}$$

Here, to get the second equality we have used Eq. (11), the third is due to Eqs. (6) and (3), and the last equality is implied by Eq. (8). We note that a different statistical interpretation of weak value, within a classical model with an epistemic restriction taking the form of the Heisenberg uncertainty relation reproducing Gaussian quantum mechanics [47], is reported in Ref. [29].

To summarize, the operational protocol of weak momentum measurement with postselection on the conjugate position, combined with the introduction of a global random variable ξ satisfying Eq. (3), defines the ERPS distribution $\mathbb{P}_\psi(p, q|\xi)$, in a specific way. Moreover, the real part of the weak momentum value is manifested in the position-dependent (conditional) average of the statistically restricted momentum fluctuation $\overline{p}(q;\psi)$ as in Eq. (11), and up to its sign, the imaginary part of the weak momentum value is manifested in the variance of the the statistically restricted momentum fluctuation $\overline{\overline{p}}(q;\psi)$ as in Eq. (12), offering a statistical meaning of the complex weak momentum value. In particular, the above phase-space picture naturally leads to conjecture that the randomness in each single repetition of the weak momentum measurement is due to the random fluctuation of the epistemically restricted momentum field of Eq. (6), with an average and a variance that correspond respectively to the real and imaginary parts of the weak momentum value (see also next section on Wiseman's naive scheme to observe average momentum and its relation with the weak momentum value). This conjecture can be checked in experiment if we could probe the outcome of each single shot of weak measurement. Another interesting important question is whether a phase-space distribution with an epistemic restriction can be singled out operationally via the weak momentum value, uniquely, by imposing some further physically intuitive requirements, such as separability for independent preparations and covariance under phase-space shift transformation discussed at the end of the previous section.

IV. DISCUSSION: WAVE-FUNCTION TOMOGRAPHY, WISEMAN'S AVERAGE MOMENTUM FIELD, AND BEST ESTIMATION OF MOMENTUM GIVEN INFORMATION ON POSITION

Equation (8) suggests a simple method for the reconstruction of the quantum wave function $\psi(q) = \sqrt{\rho(q)}\exp[iS(q)/\hbar]$ from weak measurement of momentum with postselection on the conjugate position. First, integrating the left-hand equation in Eq. (8), we obtain the phase of the associated wave function up to a constant or a global phase as $S(q) = \int^q dq'\mathrm{Re}\{p^{\mathrm{w}}(q';\psi)\}$. On the other hand, the amplitude of the original wave function can be computed by integrating the right-hand equation in Eq. (8) as $\rho(q) = \mathcal{C}\exp[-\frac{2}{\hbar}\int^q dq'\mathrm{Im}\{p^{\mathrm{w}}(q';\psi)\}]$, where $\mathcal{C}$ is a normalization constant. Moreover, noting Eqs. (11) and (12), the reconstructed phase and amplitude of the wave function, and thus the whole quantum information concealed in a wave function, are respectively captured neatly by the position-dependent (conditional) average and the variance of the statistically restricted momentum field defined in Eq. (6). The ERPS representation thus reveals a deep conceptual link between the epistemic restriction transparently manifesting quantum uncertainty relation in phase space and the abstract mathematical structure of a complex quantum wave function, mediated by the operationally well-defined notion of weak measurement of momentum followed by postselection on the conjugate position. This conceptual link is further elaborated in a different work to devise an epistemic interpretation of a quantum wave function and quantum uncertainty [45]; see the last paragraph of this section.

Note that in the scheme for a direct measurement of a wave function reported in Ref. [40], i.e., by weakly measuring $|q\rangle\langle q|$ followed by a postselection on a subensemble with vanishing momentum (via strong momentum measurement), the real and imaginary parts of the associated weak value correspond directly to the real and imaginary parts of the quantum wave function, whose physical interpretations are not easy to grasp. By contrast, in our reconstruction scheme above, the real and imaginary parts of the weak momentum value correspond directly to the phase and the amplitude of the wave function, whose physical interpretations are more transparent. In Ref. [54], it is shown that the Dirac-Kirkwood quasiprobability distribution [7,55,56] can be defined operationally using weak measurement of $|q\rangle\langle q|$ followed by a strong projection over $|p\rangle\langle p|$ (without postselection). However, while the Dirac-Kirkwood quasiprobability distribution appears formally to satisfy the Bayes rule [57,58], its transparent interpretation is hampered by the fact that it is complex valued. Moreover, unlike the ERPS distribution, its relation with the quantum uncertainty relation is not immediate.

Next, in Ref. [22], Wiseman considered an intuitive but seemingly "naive" definition of position-dependent (conditional) average momentum, i.e., in a way that would make sense to classical physicists, and interpreted it operationally in terms of weak value measurement. Namely, one makes a weak momentum measurement followed by position postselection, yielding a random value, and takes the average over infinite repetitions of such a measurement procedure. Wiseman [22] then showed that the average momentum field thus defined is equal to the real part of the weak momentum value given by the left-hand equation in Eq. (8). For example, for a particle of mass m in a scalar potential, we have $mv_{\mathrm{W}} \doteq p_{\mathrm{W}}(q) = \mathrm{Re}\{p^{\mathrm{w}}(q;\psi)\} = \partial_q S(q)$, where $p_{\mathrm{W}}(q)$ and $v_{\mathrm{W}}(q)$ are the Wiseman's average momentum and velocity fields. This can be integrated in time to construct "average trajectories" which in turn coincide with the Bohmian trajectories. Such average trajectories, which were calculated numerically for the first time in Ref. [59] in the context of Bohmian mechanics, have been observed by Steinberg's group in a beautiful double-slit experiment reported in Ref. [34]. Within the ERPS representation, as shown in Eq. (11), Wiseman's naively observable average momentum field is just the position-dependent (conditional) average of the statistically restricted momentum field defined in Eq. (6) over the fluctuations of ξ; i.e., we have $p_{\mathrm{W}}(q) = \mathrm{Re}\{\frac{\langle q|\hat{p}|\psi\rangle}{\langle q|\psi\rangle}\} = \partial_q S(q) = \overline{p}(q;\psi)$. Our interpretation of weak momentum value is thus very similar to that of Wiseman's. Moreover, as also stated at the end of the previous section, we may naturally speculate that the randomness in each single repetition of weak momentum measurement in Wiseman's naive scheme is due to the random fluctuation of ξ in Eq. (6). Note, however, that Wiseman's starting point is standard quantum mechanics, while we started from a statistical phase-space model to reconstruct quantum mechanics from scratch using the notion of epistemic restriction parametrized by a global variable ξ fluctuating randomly on the order of the Planck constant [18].

Further, in Ref. [44], working within the standard formalism of quantum mechanics, Hall argued that $\partial_q S(q) = \mathrm{Re}\{\frac{\langle q|\hat{p}|\psi\rangle}{\langle q|\psi\rangle}\}$ can be interpreted as the best classical estimate of quantum momentum compatible with the conjugate position q, minimizing a suitably defined quantum mean-squared error. This idea has been further discussed by Johansen in Ref. [23] to interpret weak value in terms of theory of best estimation and is largely expanded by Hall in Ref. [24]. Noting this, within the ERPS representation, we might interpret the decomposition of the momentum field in Eq. (6) epistemically (as opposed to physical decomposition) as follows. Assume that a preparation characterized by the wave function $\psi(q) = \sqrt{\rho(q)}e^{\frac{i}{\hbar}S(q)}$ generates a random momentum field $p(q;\xi,\psi)$. Then, given information on the conjugate position q, the first term on the right-hand side of the decomposition in Eq. (6), i.e., $\partial_q S(q)$, is interpreted as the best estimate of the momentum field, and the second term, $\frac{\xi}{2}\frac{\partial_q\rho(q)}{\rho(q)} = p(q;\xi,\psi) - \partial_q S(q)$, is interpreted as the estimation error. It is interesting to ask if this interpretation could link the Cramer-Rao inequality limiting an unbiased estimation [60] to the Heisenberg quantum uncertainty relation. If this epistemic interpretation of the decomposition of Eq. (6) is tenable, along with the Copenhagen spirit, we might argue within the ERPS model of Ref. [18] that the quantum wave function $\psi(q) = \sqrt{\rho(q)}e^{\frac{i}{\hbar}S(q)}$ does not represent an agent (observer)-independent objective physical reality [61], but rather a mathematical tool which conveniently summarizes the agent's best estimate of momentum given information on the conjugate position, in the presence of an epistemic restriction. This idea is elaborated in detail in a different work [45].

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