Unitarity corridors to exceptional points

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Phenomenological quantum Hamiltonians $H^{(N)}(\lambda) = J^{(N)} + \lambda V^{(N)}(\lambda)$ representing a general real N^2-parametric perturbation of an exceptional-point-related unperturbed Jordan-block Hamiltonian $J^{(N)}$ are considered. Tractable as non-Hermitian (in a preselected, unphysical Hilbert space) as well as, simultaneously, Hermitian (in another, "physical" Hilbert space), these matrices may represent a unitary, closed quantum system if and only if the spectrum is real. At small λ we show that the parameters are then confined to a "stability corridor" $\mathcal{S}$ of the access to the extreme dynamical exceptional-point $\lambda \to 0$ regime. The corridors are narrow and N-dependent: they are formed by multiscale perturbations which are small in physical Hilbert space, i.e., which are such that $\lambda V^{(N)}_{j+k,j}(\lambda) = O(\lambda^{(k+1)/2})$ at $k = 1, 2, \ldots, N-1$ and all j.

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I. INTRODUCTION

One of the most characteristic distinguishing features of many innovative *non-Hermitian* (e.g., $\mathcal{PT}$-symmetric [1]) representations $H \neq H^\dagger$ of quantum Hamiltonians is that they can vary with parameters which are *allowed to reach* Kato's exceptional-point values (EPs [2]). The phenomenological appeal of such a limiting transition $g \to g^{EP}$ in $H(g)$ is currently being discovered in a broad range of open quantum systems [3–5] as well as in many less known applications of the theory to various closed quantum systems [6–10]. In the former, open-system setting the spectrum of $H(g)$ is, in general, complex. The $H(g)$-generated quantum time evolution is nonunitary. This gives rise to a number of rather unexpected and interesting time-evolution patterns (for example, at $g = g^{EP}$ one could stop the light [11]) which mainly attracted attention among experimentalists [12–15].

In the latter, closed-system-oriented research, in contrast, the mainstream efforts are currently being concentrated upon the study of many fundamental, not yet fully resolved theoretical questions [16]. One of the most important ones concerns the very relevance of the spectrum. Indeed, under small perturbations, "the location of the eigenvalues may be ...fragile" [17] so that people started believing that also "in quantum mechanics with non-Hermitian operators ...a central role" is to be given to "the mathematical concept of the pseudospectrum" [18].

Our present message is in fact mainly inspired by the necessity of a decisive rejection of the latter claims. The point is that the very definition of the "smallness" of perturbation λV only carries a well-defined physical meaning in the mathematical descriptions of nonunitary *alias* open quantum systems. The claims of "fragility" are then firmly based on the rigorous Roch-Silberman theorem [19] "relating the pseudospectra to the stability of the spectrum under small perturbations" [18]. The use of pseudospectra related to the perturbations with bounded norm $||V|| = O(1)$ and with a small coupling $\lambda < \epsilon$ then results, naturally, in the observation of many "unexpected wild properties of operators familiar from $\mathcal{PT}$-symmetric quantum mechanics" (cited, again, from [18]).

All such claims are mathematically correct of course. It is only necessary to add that they exclusively apply to the open quantum systems. In the case of closed quantum systems the relationship between mathematics and physics is more subtle. We are initially introducing our Hamiltonians H as non-Hermitian in a conventional Hilbert space (in our comprehensive review [20] we proposed to denote this space by dedicated symbol $\mathcal{H}^{(F)}_{\text{(friendly)}}$). In this space the norm $||V||$ and pseudospectra are defined [17]. Naturally, as long as $H \neq H^\dagger$, such a space has to be reclassified as auxiliary and manifestly unphysical. As a consequence, it is necessary to construct another, amended, phenomenologically relevant norm. Only such a norm can be used in the formulations of testable physical predictions concerning the closed quantum systems [21].

In what follows we intend to contribute to the clarification of the misunderstanding. By means of a detailed analysis of a few schematic examples we intend to demonstrate that one must treat the concept of a "sufficiently small perturbation" (entering also the definition of pseudospectrum) with extreme care. We will remind the readers that in quantum mechanics of unitary systems using observables in a non-Hermitian representation [21] the weight of a perturbation is *not* measured by its norm in $\mathcal{H}^{(F)}_{\text{(friendly)}}$. By explicit constructive calculations we will clarify why it must be measured by the norm in another, *physical*, unitarily nonequivalent Hilbert space of states with standard probabilistic interpretation (denoted, say, by symbol $\mathcal{H}^{(S)}_{\text{(standard)}}$ of Table 2 in [20]).

The difference between the two norms increases when we get closer to the EP boundary of the "admissible" (i.e., unitarity-compatible) domain of parameters. For this reason we found it maximally instructive to restrict the attention of our readers just to the systems living in a small vicinity of one of their EP singularities. This enabled us to make our message

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compact and persuasive. Indeed, whenever the system moves closer to its EP boundary, the inner-product-related anisotropy of geometry of the associated physical Hilbert space $\mathcal{H}^{(S)}_{(\mathrm{standard})}$ grows and approaches its non-Hermitian-degeneracy supremum (cf. [22]).

We will show that and how this induces a "hierarchization" of the weights of the influence of the separate components of the fluctuations of the separate matrix elements of the Hamiltonian. Indeed, even if we keep calling these fluctuations "perturbations," we must also take their anisotropy dependence fully into account. Due to our choice of not too complicated illustrative examples we will be able to simplify some technicalities significantly. The presentation of our results will start, in Sec. II, by a concise explanation of the situation in which the vicinity of the EP singularity can be connected, by a continuous change of the parameters, with the bulk parametric domain of a less anomalous dynamical regime of the system.

Conveniently, the admissible, unitarity-compatible parametric domain near an EP will be called "corridor." By definition, the energies inside the corridor will be required real. The concept of the corridor connecting a stable unitary dynamical regime with its limiting EP boundary is given a more concrete form in Sec. III. We recall and extend there a few constructive results of our preceding papers [23,24]. We also reconfirm there that, under the quite common [17] but not sufficiently restrictive assumption that the perturbations are uniformly bounded, the vicinity of generic EP-limiting H *does not contain any* "broad" corridors at all.

The apparent paradox is resolved in Sec. IV, where we introduce a concept of a "narrow" corridor for which the "sufficiently small" perturbations are defined via a certain *ad hoc* redefinition of the space of variability of the "admissible" matrix elements of perturbation V. Explicit formulas for the boundaries of the corridors are presented there at the first few matrix dimensions N. The subsequent more general and N-independent results will be then presented in Sec. V. In a way based on an extrapolation of the preceding N-dependent observations to all N we will formulate there our main result.

This will only explicitly reconfirm our *a priori* expectations that in the non-Hermitian closed-system theories the basic phenomenological concept of the "smallness" of the stability-compatible perturbations V must be specified in a far from trivial manner. In our last, less technical discussion in Sec. VI we will finally complement this conclusion by a few comments on its consequences and interpretation.

II. UNITARITY CORRIDORS

In a way reflecting the recent trends [25] we intend to perform a deeper analysis of the mathematical guarantees of the reality of the spectrum attributed, often, to the spontaneously unbroken $\mathcal{PT}$ symmetry of H [1], or to the existence of a similarity between H and a self-adjoint operator [6,7]. Such a project led us to the search for correspondence of the underlying mathematics with the parallel conceptual physical questions concerning, first of all, the protection of a quantum system against the loss of its observability under too strong a perturbation.

A. Boundaries of observability

In the literature devoted to the analyses of quantum stability one mostly finds just various entirely routine descriptions which mainly fall into two subcategories. In the more common approach one simply assumes that both the unperturbed and perturbed Hamiltonians are self-adjoint. This, in essence, makes the problem trivial. Indeed, the reality of the bound-state energies remains "robust." One also does not need to pay too much attention to the EP singular values $g^{(EP)}$ of parameters because they are, by definition, out of consideration, incompatible simply with the self-adjointness assumption [2].

In the conventional Hermitian theories the influence of small perturbations is described by the pseudospectrum and it remains fully under our control. In the open-system theories the study of pseudospectra clarifies a number of features of various realistic systems. *Pars pro toto* we may name the study of perturbations of the Bose-Hubbard N-by-N-matrix forms of Hamiltonians $H^{(N)}(g)$ [4]. In this case the non-Hermitian formalism of perturbation expansions helped to clarify even some aspects of the behavior of the Bose-Einstein condensates. Another particularly impressive result of this type was a quite unexpected discovery of the generic failure of adiabatic approximation in the open, nonunitary quantum dynamical systems when forced to encircle their EP singularity [26].

The problems are much more challenging in the case of the closed quantum systems, especially in the models in which Kato's EP singularity is of the Nth order with $N > 2$ (in this case we shall usually use the acronym EPN). Indeed, after an arbitrarily small perturbation the initially strictly nondiagonalizable EPN-related Hamiltonians $H^{(N)}(g^{(EPN)})$ cannot be assigned their canonical Jordan-block form anymore (i.e., they become diagonalizable). At the same time, the brute-force numerical diagonalization of these perturbed Hamiltonians

$$H^{(N)} = H^{(N)}(g^{(EPN)}) + \lambda\, H^{(N)}_{(\mathrm{int})} \qquad (1)$$

remains almost prohibitively ill conditioned [27]. In what follows a remedy will be sought in perturbation theory (cf. its outline in our preceding paper [24]). On this background we will separate perturbations $H^{(N)}_{(\mathrm{int})}$ into two subfamilies. For the subfamily of our present interest (in which the energy spectra will be real) the parameters will form a unitarity-compatible corridor.

Working, for the sake of definiteness, with multiparametric and real but, otherwise, entirely general N by N matrix Hamiltonians $H^{(N)}$ we shall restrict our study just to the models lying "not too far" from an EP singularity. In these models, secondly, the non-Hermitian EP degeneracy will be assumed "maximal," i.e., N-tuple, with $g^{EP} \equiv g^{EPN}$. We should emphasize that these restrictions of the scope of our paper were motivated by the needs of physics. In particular, we wanted to complement the open-system results of Ref. [4] or the closed-system results of Refs. [28] (exhibiting already all features of a quantum phase transition [29]) by another family of the less realistic and less numerical but more universal and more transparent N by N matrix model.

B. Exceptional points and Jordan blocks

In the EP limit itself (also known as non-Hermitian degeneracy [3]) the Hamiltonian, by definition, ceases to be diagonalizable. This means that it loses its standard physical interpretation [30]. At the same time, the study and understanding of the behavior of quantum systems in the vicinity of EPs is of paramount descriptive [4] as well as conceptual [31] and practical numerical [27] relevance and importance.

Special attention is to be paid to the scenarios in which we may ignore the role of the EP-unrelated part of the physical Hilbert space. This enables us to restrict attention to the N by N (sub)matrices $H = H^{(N)}(g)$ of the Hamiltonian, especially when the parameter is able to acquire a maximal, Nth-order exceptional-point value, $g \to g^{(EPN)}$ [28]. In similar cases the EPN limit of the truncated Hamiltonian is usually assigned its Jordan-block canonical form,

$$H^{(N)}(g^{(EPN)}) \sim J^{(N)}(E_0) = \begin{bmatrix} E_0 & 1 & 0 & \dots & 0 \\ 0 & E_0 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & E_0 & 1 \\ 0 & \dots & 0 & 0 & E_0 \end{bmatrix}.$$

A mutual map is defined, in terms of the so-called transition matrix $Q^{(N)}$, by relations

$$H^{(N)}(g^{(EPN)})\, Q^{(N)} = Q^{(N)}\, J^{(N)}(E_0). \tag{2}$$

In quantum physics, such a “generalized diagonalization” of the Hamiltonian offers an efficient tool for analysis in perturbation theory [24].

C. One-parametric corridor in an exactly solvable example

As an elementary illustrative example let us recall the following exactly solvable N-state quantum Hamiltonian of dimension $N = 8$,

$$H^{(8)}_{(ES)}(g) = \begin{bmatrix} 0 & -1+\delta & 0 & 0 & 0 & 0 & 0 & 0 \\ -1-\delta & 0 & -1+\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & -1-\gamma & 0 & -1+\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -1-\beta & 0 & -1+\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-\alpha & 0 & -1+\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & -1-\beta & 0 & -1+\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & -1-\gamma & 0 & -1+\delta \\ 0 & 0 & 0 & 0 & 0 & 0 & -1-\delta & 0 \end{bmatrix}. \tag{3}$$

This matrix is non-Hermitian but $\mathcal{PT}$ symmetric, where $\mathcal{P}$ is parity (i.e., a matrix with units along the second main diagonal) while the nonlinear operator of transposition $\mathcal{T}$ mimics the time reversal [9]. In a standard decomposition $H = T + V$ of this Hamiltonian the kinetic energy term T coincides with the conventional discrete Laplacean, while the four-parametric antisymmetric tridiagonal matrix V plays the role of a weakly nonlocal interaction.

The EP8 limit of the model is reached at $\alpha = \beta = \gamma = \delta = 1$. The resulting Hamiltonian matrix with the mere $N - 1$ nonvanishing elements $H_{j+1,j} = -2$, $j = 1, 2, \dots, N-1$ may be given its Jordan-block form via Eq. (2) in terms of an antidiagonal transition matrix with N nonvanishing elements $Q_{N-j+1,j} = (-2)^{j-1}$, $j = 1, 2, \dots, N$.

For the specific g dependence of parameters

$$\alpha = \sqrt{1 - a\,g}, \quad \beta = \sqrt{1 - b\,g}, \quad \gamma = \sqrt{1 - c\,g}, \quad \delta = \sqrt{1 - d\,g}, \tag{4}$$

with a quadruplet of positive constants a, b, c, and d, the spectrum is sampled, in Fig. 1, at $a = 2$, $b = 1.8$, $c = 1.6$, and $d = 1.4$. It is all real and discrete at any real $g > 0$. With $g^{(EP8)}_{(ES)} = 0$ and with the trivial degenerate energy $E_0 = E^{(EP8)}_{(ES)} = 0$, the g dependence of the energies can even be specified by the remarkable exact formula $E_n(g) = E_n(1)\sqrt{g}$, which immediately follows from the g dependence of the secular polynomial.

III. EXCEPTIONAL POINTS AND THEIR BOUNDED PERTURBATIONS

The stability of quantum systems with respect to perturbations is usually studied in the framework of conventional quantum mechanics in which the Hamiltonians (i.e., the generators of evolution) are self-adjoint [30]. From this perspective our present study of manifestly non-Hermitian perturbed Hamiltonians (1) living in a small vicinity of an EPN singularity represents a true methodical challenge.

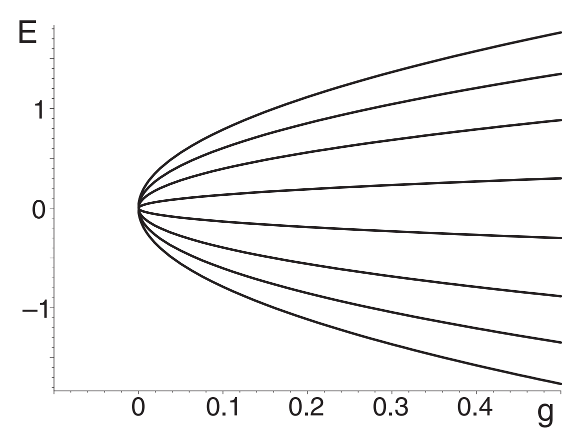

FIG. 1. Sample of degeneracy of the real spectrum of Hamiltonian $H^{(8)}_{(ES)}(g)$ in the EP8 limit of $g \to 0$ at $a = 2$, $b = 1.8$, $c = 1.6$, and $d = 1.4$ (both g and E are dimensionless here).

In the first step towards a disentanglement of the problems let us recall Eq. (2) and let us replace the unperturbed Hamiltonian $H^{(N)}(g^{(EPN)})$ of Eq. (1) by its canonical Jordan form. This yields the fairly general family of the EPN-related perturbed N by N real-matrix Hamiltonians of our interest,

$$H^{(N)} = J^{(N)}(0) + \lambda V. \quad (5)$$

In our analysis we shall initially assume that the matrix elements of the perturbation are uniformly bounded, $V_{i,j} = O(1)$. The smallness of the perturbation then becomes controlled by a single, "sufficiently small" positive parameter λ tractable as a coupling constant.

A. Exactly solvable $N = 2$ model

Jordan block with $N = 2$,

$$J^{(2)}(E_0) = \begin{bmatrix} E_0 & 1 \\ 0 & E_0 \end{bmatrix},$$

and with, say, $E_0 = 0$ can be perceived, in the light of Eq. (2), as a generic representative of an arbitrary $N = 2$ one-parametric Hamiltonian $H^{(2)}(g)$ in its EP2 limit. Thus, up to a trivial incorporation of transition matrix $Q^{(2)}$ via Eq. (2), we may replace any given unperturbed Hamiltonian $H^{(2)}(g^{(EP2)})$ in Eq. (1) by its canonical form $J^{(2)}(0)$. Even when adding an arbitrary (and, say, real) $N = 2$ perturbation matrix

$$V = \begin{bmatrix} \alpha_1 & \mu \\ \beta & \alpha_2 \end{bmatrix}$$

with bounded elements $V_{j,k} = O(1)$, the exhaustive construction of all of the bound states remains non-numerical. Its detailed presentation may be found in Sec. III A of [24]. For the present methodical purposes we only need to recall the elementary scaling rule

$$E_{\pm}^{(2)} = \pm\sqrt{\lambda\,\beta} + O(\lambda) \quad (6)$$

characterizing the order of magnitude of the complete perturbed bound-state energy spectrum. This rule immediately follows from secular equation

$$\det(H - E) = \begin{bmatrix} \lambda\,\alpha_1 - \epsilon\sqrt{\lambda} & 1 + \lambda\,\mu \\ \lambda\,\beta & \lambda\,\alpha_2 - \epsilon\sqrt{\lambda} \end{bmatrix} = 0,$$

$$\epsilon = E/\sqrt{\lambda} = O(1),$$

i.e., from the implicit definition of the spectrum

$$(\alpha_1\alpha_2 - \beta\,\mu)\lambda^2 + (-\alpha_1\epsilon - \epsilon\,\alpha_2)\lambda^{3/2} + (-\beta + \epsilon^2)\lambda = 0.$$

The conclusion is that in the leading-order approximation we get the two real energy roots $E_{\pm}$ of Eq. (6) if and only if $\beta \geqslant 0$. In such a broad "physical" parametric corridor the time evolution of our quantum system remains unitary in a nonempty interval of small $\lambda \in (0, \lambda_{\max})$. In contrast, the eigenvalues become purely imaginary whenever $\beta < 0$, $\epsilon \approx \epsilon_{\pm} = \pm i\sqrt{|\beta|}$. In other words, the vicinity of the EP2 singularity splits into the "admissible," unitarity-compatible corridor and its "unphysical," unitarity-incompatible complement. Thus the choice of $\beta > 0$ guarantees the existence of a nonempty corridor connecting the interior of the domain of the stable dynamical regime with its EP2-supporting boundary.

What remains to be discussed is the behavior of the $N = 2$ bound-state energies in the limit $\beta \to 0$. Incidentally, for the analysis the perturbation approximation approach is not needed. The eigenvalue formulas $E_{1,2} = \lambda\,\alpha_{1,2}$ become exact at $\beta = 0$. What is remarkable is only an enhancement of their order of smallness, $E_{1,2} = O(\lambda)$. We will see below that such a rescaling behavior will also reemerge at the larger matrix dimensions $N > 2$.

B. Nontrivial model with $N = 3$

The existence of transition matrices $Q^{(3)}$ and the routine solvability of the EPN-related Eq. (2) at $N = 3$ enable us to restrict attention, without any loss of generality, to the perturbed Jordan-block Hamiltonians

$$H^{(3)}(\lambda) = J^{(3)}(0) + \lambda V. \quad (7)$$

A partial analysis of consequences may already be found described in Sec. III C of the paper in [24]. Unfortunately, our conclusions in *loc. cit.* were negative. In the EP3 vicinity the quantum systems in question appeared nonunitary and unstable. In the real space of parameters of perturbation V we failed to localize a unitarity-compatible corridor which would provide a $\lambda \neq 0$ access to the EP3 singularity in the limit of $\lambda \to 0$.

In retrospect, the main reason for the failure may be traced back to the fact that we tried to follow the guidance provided by the simpler $N = 2$ model too closely. The use of the mere λ-independent real perturbation matrix with elements $V_{j,k} = O(1)$, i.e.,

$$V = \begin{bmatrix} \alpha_1 & \mu_1 & \nu \\ \beta_1 & \alpha_2 & \mu_2 \\ \gamma & \beta_2 & \alpha_3 \end{bmatrix}, \quad (8)$$

appeared insufficient. In fact, we only too heavily relied upon the existence of the specific "exact" representation of the $N = 3$ spectrum in terms of Cardano formulas. After all, this strategy led already to overcomplicated formulas and did not offer any insight. Thirdly, in a way guided by the results at $N = 2$ we "skipped ...the discussion of models with vanishing $\gamma = 0$" [24]. In other words, having restricted our attention to the mere search for a "broad" corridor with $\gamma \neq 0$ we missed the opportunity. We did not manage to find *any* reasonable construction of the corridor of stability at *any* nonvanishing $\lambda \neq 0$ in Eq. (7) (see the list of the related comments at the end of Sec. III in [24]).

The nonexistence of the corridor at $N = 3$ and $\gamma \neq 0$ may be given an elementary proof. In its outline let us return to ansatz (8). We may rescale the energies, in a way recommended in [24], whenever $\gamma \neq 0$, $E_n = \epsilon_n\sqrt[3]{\lambda}$. An implicit definition of the spectrum is then immediately provided by secular equation

$$\det\begin{bmatrix} \lambda\,\alpha_1 - \epsilon\sqrt[3]{\lambda} & 1 + \lambda\,\mu_1 & \lambda\,\nu \\ \lambda\,\beta_1 & \lambda\,\alpha_2 - \epsilon\sqrt[3]{\lambda} & 1 + \lambda\,\mu_2 \\ \lambda\,\gamma & \lambda\,\beta_2 & \lambda\,\alpha_3 - \epsilon\sqrt[3]{\lambda} \end{bmatrix} = 0.$$

Although the resulting secular polynomial is too long for print, its leading-order part is short and yields the final, explicit closed-form result

$$\epsilon \approx \epsilon_{1,2,3} = \sqrt[3]{\gamma}.$$

This reconfirms that the *whole* spectrum cannot be real (and the system compatible with unitarity) unless $\gamma = 0$.

IV. CONSTRUCTION OF THE CORRIDORS

In [24] we did not study the case of vanishing $\gamma = 0$ because we found it overcomplicated. Now we shall accept a different strategy, assuming that the limiting constraint $\gamma = 0$ is only valid in the leading-order approximation in λ. In other words, we will consider generalized, manifestly λ-dependent versions of perturbations.

A. Corridor and its boundaries at $N = 3$

Transition to manifestly λ-dependent real perturbation matrices

$$V = \begin{bmatrix} \alpha_1 & \mu_1 & \nu \\ \beta_1 & \alpha_2 & \mu_2 \\ \gamma & \beta_2 & \alpha_3 \end{bmatrix} + \sqrt{\lambda}\, V' + \cdots ,$$

$$V' = \begin{bmatrix} \alpha_1' & \mu_1' & \nu' \\ \beta_1' & \alpha_2' & \mu_2' \\ \gamma' & \beta_2' & \alpha_3' \end{bmatrix}, \dots \tag{9}$$

is an enrichment of the representation of dynamics at $N = 3$. It immediately leads us to a very natural resolution of the puzzle. Let us now outline its main technical ingredients. First, at $\gamma = 0$ we have to change the energy scaling: $E_n = \epsilon_n \sqrt{\lambda}$. From the resulting amended secular equation

$$\det \begin{bmatrix} \lambda\,\alpha_1 - \epsilon\sqrt{\lambda} & 1 + \lambda\,\mu_1 & \lambda\,\nu \\ \lambda\,\beta_1 & \lambda\,\alpha_2 - \epsilon\sqrt{\lambda} & 1 + \lambda\,\mu_2 \\ \lambda^{3/2}\gamma' & \lambda\,\beta_2 & \lambda\,\alpha_3 - \epsilon\sqrt{\lambda} \end{bmatrix} = 0$$

we are allowed to omit all of the higher-order corrections as irrelevant. Preserving merely the $O(\lambda^{3/2})$ leading-order part of secular equation

$$\gamma' + (\beta_1 + \beta_2)\,\epsilon - \epsilon^3 = 0,$$

we only need to reflect the role and influence of the parameter γ'. In a preparatory stage we may try to simplify the task and to fix, tentatively, $\gamma' = 0$. This would yield the two sample roots $\epsilon_\pm = \pm\sqrt{\beta_1 + \beta_2}$, which are both real if and only if $\beta_1 + \beta_2 \geqslant 0$. Thus relation $\beta_1 + \beta_2 = 0$ seems to offer the first nontrivial specification of the boundary of the corridor at $\gamma' = 0$. Unfortunately, the property of reality of the third energy root (which, in the leading-order approximation, vanishes) remains uncertain. Thus we have to return to the full-fledged analysis of the model at $\gamma' \neq 0$. Along these lines we abbreviated $\beta_1 + \beta_2 = 3\varrho^2$ and came to the following $N = 3$ result.

Lemma 1. For Hamiltonians (7) with small λ and arbitrary real perturbations (9) the energy spectra are real for parameters inside an EP3-attached corridor such that $\gamma = 0$ and $\gamma' \in (-\varrho^3, \varrho^3)$.

Proof. The graph of the curve $y(\epsilon) = \gamma' + (\beta_1 + \beta_2)\,\epsilon - \epsilon^3$ (with zeros equal to the energies) diverges to $\pm\infty$ at large and positive or negative ϵ, respectively. The γ'-independent derivative $y'(\epsilon) = \beta_1 + \beta_2 - 3\,\epsilon^2$ has zeros $\epsilon_\pm = \pm\varrho$ which determine the local minimum or maximum of $y(\epsilon)$. It must be negative or positive, respectively, but this is guaranteed by our constraint upon γ'. ■

B. Boundaries at $N = 4$

The perturbed Jordan-block Hamiltonians

$$H^{(4)}(\lambda) = J^{(4)}(0) + \lambda\, V \tag{10}$$

will be studied here with the following reduced, ten parametric real perturbation matrix

$$V = \begin{bmatrix} \mu_1 & 0 & 0 & 0 \\ \alpha_1 & \mu_2 & 0 & 0 \\ \beta_1 & \alpha_2 & \mu_3 & 0 \\ \gamma & \beta_2 & \alpha_3 & \mu_4 \end{bmatrix}. \tag{11}$$

Bound state energies $E_n = \epsilon_n \sqrt[4]{\lambda}$ may now be defined via roots of secular equation

$$\det(H - E) = \det \begin{bmatrix} \lambda\,\mu_1 - \epsilon\sqrt[4]{\lambda} & 1 & 0 & 0 \\ \lambda\,\alpha_1 & \lambda\,\mu_2 - \epsilon\sqrt[4]{\lambda} & 1 & 0 \\ \lambda\,\beta_1 & \lambda\,\alpha_2 & \lambda\,\mu_3 - \epsilon\sqrt[4]{\lambda} & 1 \\ \lambda\,\gamma & \lambda\,\beta_2 & \lambda\,\alpha_3 & \lambda\,\mu_4 - \epsilon\sqrt[4]{\lambda} \end{bmatrix} = 0.$$

In the leading-order approximation this yields the entirely elementary quadruplet of solutions

$$\epsilon_n \approx \gamma^{1/4}.$$

At both signs of nonvanishing real γ two of those roots are purely imaginary so that at arbitrarily small $\gamma \neq 0$ and $\lambda \neq 0$ the perturbed system becomes nonunitary. In other words, our quantum system with $\gamma \neq 0$ is unstable and it does not possess any suitable physical Hilbert space of states $\mathcal{H}^{(S)}{}_{(\text{standard})}$. The system must be interpreted as having performed a phase transition at $\lambda = 0$ [9,29]. At $\gamma \neq 0$ and $\lambda \neq 0$ its "energy" H is not an observable anymore.

The essence of the paradox was clarified in our preceding paper [24]. We emphasized there that in the quantum mechanics of closed systems it only makes sense to consider the "realizable" perturbations under which the perturbed Hamiltonian still operates in a suitable $\mathcal{H}^{(\text{physical})}$. One has to require that the "strength" of the perturbations is "measured" in $\mathcal{H}^{(\text{physical})}$ rather than in any of its unitarily nonequivalent, manifestly unphysical alternatives $\mathcal{H}^{(\text{auxiliary})}$ with, typically, a "friendlier" inner product [21].

In our present continuation of the EP4-related study of realizable perturbations let us reopen the search for a stable corridor in a restricted parametric domain where $\gamma \approx 0$. With this aim we replace the constant-perturbation ansatz (11) by a more sophisticated, λ-dependent $N = 4$ analog of Eq. (9). Recalling the strategy used at $N = 3$ we have to modify also the scaling of the bound-state energies and put $E_n = \epsilon_n \sqrt[3]{\lambda}$ in an amended secular equation

$$\det(H-E) = \det\begin{bmatrix} \lambda\,\mu_1 - \epsilon\,\sqrt[3]{\lambda} & 1 & 0 & 0\\ \lambda\,\alpha_1 & \lambda\,\mu_2 - \epsilon\,\sqrt[3]{\lambda} & 1 & 0\\ \lambda\,\beta_1 & \lambda\,\alpha_2 & \lambda\,\mu_3 - \epsilon\,\sqrt[3]{\lambda} & 1\\ \lambda^{4/3}\gamma' & \lambda\,\beta_2 & \lambda\,\alpha_3 & \lambda\,\mu_4 - \epsilon\,\sqrt[3]{\lambda}\end{bmatrix} = 0.$$

Its leading-order component of order $\lambda^{4/3}$ must vanish,

$$\epsilon^4 - \beta_1\epsilon - \beta_2\epsilon - \gamma' = 0. \tag{12}$$

Such an upgrade of secular polynomial has still strictly two or four complex roots. The evolution of the system remains nonunitary and unstable unless we set $\beta_1 + \beta_2 \to 0$ and $\gamma' \to 0$ making all of the roots of the leading-order secular equation (12) vanish as well.

The way out of the difficulty is found in the next-step lowering of the order of magnitude of all of the coefficients in approximate Eq. (12). In the language of physics this means that we have to introduce certain *ad hoc* higher-order perturbations. Thus, proceeding along the same lines as before, we weaken the dominant components of the perturbation, $\beta_1 \to \beta_1' \sqrt[2]{\lambda}$, $\beta_2 \to \beta_2' \sqrt[2]{\lambda}$, and $\gamma' \to \gamma''\,\lambda$. This induces the change in the scaling of the energies, $E_n = \epsilon_n \sqrt[2]{\lambda}$. The replacements lead to the following ultimate amendment of the Schrödinger operator:

$$H - E = \begin{bmatrix} \lambda\,\mu_1 - \epsilon\,\sqrt{\lambda} & 1 & 0 & 0\\ \lambda\,\alpha_1 & \lambda\,\mu_2 - \epsilon\,\sqrt{\lambda} & 1 & 0\\ \lambda^{3/2}\beta_1' & \lambda\,\alpha_2 & \lambda\,\mu_3 - \epsilon\,\sqrt{\lambda} & 1\\ \lambda^{2}\gamma'' & \lambda^{3/2}\beta_2' & \lambda\,\alpha_3 & \lambda\,\mu_4 - \epsilon\,\sqrt{\lambda}\end{bmatrix}. \tag{13}$$

Up to the higher-order $O(\lambda^{5/2})$ corrections the exact secular equation $\det(H-E) = 0$ degenerates to the vanishing of the effective secular polynomial,

$$z(\epsilon) = -\widetilde{\gamma} - \widetilde{\beta}\,\epsilon - \widetilde{\alpha}\,\epsilon^2 + \epsilon^4 = 0, \tag{14}$$

where

$$\widetilde{\gamma} = \gamma'' - \alpha_1\alpha_3, \quad \widetilde{\beta} = \beta_1' + \beta_2', \quad \widetilde{\alpha} = \alpha_1 + \alpha_2 + \alpha_3. \tag{15}$$

We arrive at our final answer.

Lemma 2. For a sufficiently small λ in the hierarchically perturbed four by four Hamiltonian of Eq. (13) the energy spectrum remains real inside a nonempty EP4-attached corridor of parameters.

Proof. The graph of the left-hand-side function $z(\epsilon)$ of Eq. (15) (with its four zeros equal to the energies) has its three real extremes localized at the zeros of its $\widetilde{\gamma}$-independent derivative $z'(\epsilon) = -\widetilde{\beta} - 2\,\widetilde{\alpha}\epsilon + 4\,\epsilon^3$. In the proof of Lemma 1 we saw that the latter triplet of zeros $\xi_{0,\pm}$ was real for $\widetilde{\alpha} = 3\varrho^2 > 0$ and $\widetilde{\beta} \in (-\varrho^3, \varrho^3)$. Thus, up to the parameters at the end points of these constraints, the zeros $\xi_{0,\pm}$ [i.e., the coordinates of the local extremes of $z(\epsilon)$] are real and nondegenerate. Thus the local maximum of $z(\epsilon)$ is sharply larger than both of the local minima, $z(\xi_0) > \max z(\xi_\pm)$. As a consequence, the interval of variability of our last free parameter $\widetilde{\gamma}$ guaranteeing that $z(\xi_0) > 0$ while $\max z(\xi_\pm) < 0$ is nonempty. ■

V. CORRIDORS AT ARBITRARY N

The form of H in Eq. (13) is instructive in revealing a general hierarchy of relevance of the individual matrix elements of V under the natural phenomenological requirement of the preservation of the unitarity of the evolution. The pattern can tentatively be extrapolated to the higher matrix dimensions N with, in particular, $E = \epsilon \sqrt[2]{\lambda}$ in the $N = 5$ Schrödinger operator

$$H - E = \begin{bmatrix} \lambda\,\nu_1 - \epsilon\,\sqrt{\lambda} & 1 & 0 & 0 & 0\\ \lambda\,\mu_1 & \lambda\,\nu_2 - \epsilon\,\sqrt{\lambda} & 1 & 0 & 0\\ \lambda^{3/2}\alpha_1' & \lambda\,\mu_2 & \lambda\,\nu_3 - \epsilon\,\sqrt{\lambda} & 1 & 0\\ \lambda^{2}\beta_1'' & \lambda^{3/2}\alpha_2' & \lambda\,\mu_3 & \lambda\,\nu_4 - \epsilon\,\sqrt{\lambda} & 1\\ \lambda^{5/2}\gamma''' & \lambda^{2}\beta_2'' & \lambda^{3/2}\alpha_3' & \lambda\,\mu_4 & \lambda\,\nu_5 - \epsilon\,\sqrt{\lambda}\end{bmatrix},$$

etc. (see also the illustrative explicit rederivation of such a form of the corridor-compatible matrix in Sec. V B below).

A. Extrapolation pattern

We saw that at $N = 2$, $N = 3$, and $N = 4$ the perturbation-expansion construction of the energy spectrum near the Jordan-block extreme $H^{(N)}(g^{(EPN)})$ was straightforward. The same technique can equally well be applied at any larger matrix dimension N. Our specific additional physical requirement of the reality of the spectrum (i.e., of the unitarity of the time evolution of the quantum systems in question) has been found to be satisfied inside a specific nonempty domain which we called corridor to EPN. We also saw that at $N = 2$, $N = 3$, and $N = 4$ the corridor can be defined by certain very specific choices of λ-dependent perturbations $\lambda\, V(\lambda)$ in which the matrix elements are of *unequal* orders of smallness. The pattern appeared amenable to a rigorous extrapolation beyond $N = 4$.

Theorem 3. At any $N = 2, 3, \dots$ and for all sufficiently small $\lambda > 0$ the reality of the bound-state spectrum of energies $E_n = \epsilon_n \sqrt[2]{\lambda}$ with $\epsilon_n = O(1)$ can be guaranteed by an appropriate choice of parameters $\mu_{jk} = O(1)$ in the real N by N matrix Hamiltonian $H = J^{(N)}(0) + \lambda\, V$ with

$$\lambda V = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ \lambda\,\mu_{21} & 0 & \dots & 0 & 0 & 0 \\ \lambda^{3/2}\,\mu_{31} & \lambda\,\mu_{32} & \ddots & \vdots & \vdots & 0 \\ \lambda^{2}\mu_{41} & \lambda^{3/2}\,\mu_{42} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \lambda\,\mu_{N-1N-2} & 0 & 0 \\ \lambda^{N/2}\mu_{N1} & \lambda^{(N-1)/2}\mu_{N2} & \dots & \lambda^{3/2}\,\mu_{NN-2} & \lambda\,\mu_{NN-1} & 0 \end{bmatrix}. \tag{16}$$

Proof. Once we guessed the appropriate λ dependence of the general Schrödinger operator it is entirely straightforward to deduce the general leading-order part of the secular determinant, and to recall the independence and the free variability of the coefficients in the secular polynomial. ∎

B. Boundaries of corridor at $N = 5$

Let us start from the naive ten-parametric constant-matrix perturbation

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu_1 & 0 & 0 & 0 & 0 \\ \alpha_1 & \mu_2 & 0 & 0 & 0 \\ \beta_1 & \alpha_2 & \mu_3 & 0 & 0 \\ \gamma & \beta_2 & \alpha_3 & \mu_4 & 0 \end{bmatrix}$$

and from the unperturbed Jordan-block matrix $H_0 = J^{(5)}(0)$. The Schrödinger operator with $E = \epsilon \sqrt[5]{\lambda}$ then reads

$$H - E = \begin{bmatrix} -\epsilon\sqrt[5]{\lambda} & 1 & 0 & 0 & 0 \\ \lambda\,\mu_1 & -\epsilon\sqrt[5]{\lambda} & 1 & 0 & 0 \\ \lambda\,\alpha_1 & \lambda\,\mu_2 & -\epsilon\sqrt[5]{\lambda} & 1 & 0 \\ \lambda\,\beta_1 & \lambda\,\alpha_2 & \lambda\,\mu_3 & -\epsilon\sqrt[5]{\lambda} & 1 \\ \lambda\,\gamma & \lambda\,\beta_2 & \lambda\,\alpha_3 & \lambda\,\mu_4 & -\epsilon\sqrt[5]{\lambda} \end{bmatrix}.$$

After we reduce the secular polynomial to its dominant part we get the five elementary energy roots $\epsilon \approx \gamma^{1/5}$. Such a spectrum cannot be all real unless $\gamma = 0$. This confirms the necessity of diminishing the matrix element of perturbation in its left lower corner, $\gamma \to \gamma' \sqrt[4]{\lambda}$ (we may and will drop the primes). This forces us to change, consistently, the scale of $E = \epsilon \sqrt[4]{\lambda}$. The resulting effective (i.e., leading-order) secular equation $(-\epsilon^5 + \beta_1\epsilon + \gamma + \epsilon\,\beta_2)\lambda^{5/4} = 0$ is now found to lead, in the nontrivial case, to at least two complex, nonreal energy roots. In the same corner of perturbation matrix as above we have to diminish, therefore, the relevant matrix elements again. Once we do so and once we drop the primes in $\beta_j \to \beta_l' \sqrt[3]{\lambda}$, $\gamma' \to \gamma'' \sqrt[3]{\lambda^2}$, and $E = \epsilon' \sqrt[3]{\lambda}$, we get the following tentative amendment of the Schrödinger operator:

$$H - E = \begin{bmatrix} -\epsilon\sqrt[3]{\lambda} & 1 & 0 & 0 & 0 \\ \lambda\,\mu_1 & -\epsilon\sqrt[3]{\lambda} & 1 & 0 & 0 \\ \lambda\,\alpha_1 & \lambda\,\mu_2 & -\epsilon\sqrt[3]{\lambda} & 1 & 0 \\ \lambda^{4/3}\beta_1 & \lambda\,\alpha_2 & \lambda\,\mu_3 & -\epsilon\sqrt[3]{\lambda} & 1 \\ \lambda^{5/3}\gamma & \lambda^{4/3}\beta_2 & \lambda\,\alpha_3 & \lambda\,\mu_4 & -\epsilon\sqrt[3]{\lambda} \end{bmatrix}.$$

Recycling the abbreviations of Eq. (15) the dominant part of the effective secular equation acquires the explicit three-parametric form

$$-\widehat{\gamma} - \widetilde{\beta}\,\epsilon - \widetilde{\alpha}\,\epsilon^2 + \epsilon^5 = 0, \quad \widehat{\gamma} = \gamma''. \tag{17}$$

Its roots still cannot be all real unless they vanish in the given order of precision. Making now the story short and iterating the procedure once more we arrive, at last, at the ultimate hierarchized and corridor-supporting perturbation matrix as given by Theorem 3,

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu_1 & 0 & 0 & 0 & 0 \\ \sqrt{\lambda}\,\alpha_1 & \mu_2 & 0 & 0 & 0 \\ \lambda\,\beta_1 & \sqrt{\lambda}\,\alpha_2 & \mu_3 & 0 & 0 \\ \lambda\sqrt{\lambda}\,\gamma & \lambda\,\beta_2 & \sqrt{\lambda}\,\alpha_3 & \mu_4 & 0 \end{bmatrix}.$$

The effective $O(\lambda^{5/2})$ part of the secular determinant $\det(H - E)$ leads now to the explicit form of the polynomial secular equation:

$$\begin{aligned} &-\epsilon^5 + (\mu_2 + \mu_1 + \mu_4 + \mu_3)\epsilon^3 + (\alpha_1 + \alpha_2 + \alpha_3)\epsilon^2 \\ &+ (\beta_1 - \mu_2\mu_4 - \mu_1\mu_3 + \beta_2 - \mu_1\mu_4)\epsilon - \alpha_1\mu_4 + \gamma - \mu_1\alpha_3 \\ &= 0. \end{aligned} \tag{18}$$

It has the ultimate four-parametric flexibility as required. The nonempty unitarity-preserving corridor to the $\lambda = 0$ EP5 vertex does exist, with the leading-order boundaries prescribed, in an implicit but still user-friendly manner, by Eq. (18).

VI. DISCUSSION

The recent successful localizations of the EP singularities in various experimental setups revealed a perceivable increase of their relevance in applied physics as well as in the quantum physics of resonant and unstable open systems [5]. In the quantum theory of stable systems the role of EP singularities used to be traditionally restricted to their purely mathematical role of an obstruction of convergence in perturbation theory [2]. Such a situation was only slowly improving with the emergence of the first realistic models in relativistic quantum mechanics where the EP marks an onset of instability [10]. An analogous phenomenological phase-transition interpretation was then also assigned to the EPs in many other unitary quantum systems [9,29,32].

In a conventional perspective these innovations seem to contradict the well-known Stone theorem [33]. Due to this theorem any unitary evolution (say, in $\mathcal{H}^{(S)}_{\text{(standard)}}$) must necessarily be generated by a Hamiltonian which is self-adjoint (naturally, in the same Hilbert space $\mathcal{H}^{(S)}_{\text{(standard)}}$). From this point of view the innovation of quantum theory of unitary systems may be presented and advocated in two ways. First, in an abstract manner, as a purely technical simplification of the physical inner product, i.e., as a reduction of our standard physical Hilbert space into its auxiliary partner, i.e., as a replacement $\mathcal{H}^{(S)}_{\text{(standard)}} \to \mathcal{H}^{(F)}_{\text{(friendly)}}$ leading to a friendlier mathematics. Secondly, alternatively, in an opposite direction and in a very concrete spirit, one picks up an auxiliary Hilbert space first of all. Then one replaces its unphysical but user-friendly inner product by a less friendly but correct physical amendment.

This is the most common formulation of the recipe. In practice, what is then required is just the Hamiltonian-dependent construction of the Hamiltonian-Hermitizing metric operator Θ. Naturally, the existence of such a metric requires the reality of the spectrum; there is no consistent (unitary) quantum theory without such a constraint [7]. Once the spectrum is shown real, we map $\mathcal{H}^{(F)}_{\text{(friendly)}} \to \mathcal{H}^{(S)}_{\text{(standard)}}$ and convert the initial, "friendly but false" Hilbert space with "natural" metric $\Theta^{\text{(false)}} = I$ into its model-dependent physical amendment with metric $\Theta^{\text{(standard)}} \neq I$.

In the present continuation of the related considerations in Ref. [24] we were able to explain that the conjecture of the nonexistence of an "admissible" access corridor to the EP3 limit only meant the nonexistence of a "broad" corridor (which we found to exist at $N = 2$ but not at $N = 3$). We came now with a corrigendum: the corridors of a stable access to the EPN extremes do exist. The only constraint is that they are "narrow" in the sense of Theorem 3.

The reason for the nonexistence of a "broad" corridor at $N \geqslant 3$ has been shown here to lie in the fact that at least some of the elements of the class of perturbations which are only required bounded in the auxiliary Hilbert space $\mathcal{H}^{(F)}_{\text{(friendly)}}$ may happen to be too large in $\mathcal{H}^{(S)}_{\text{(standard)}}$. Then, they can move the system out of a given (or, better, out of any eligible) physical Hilbert space of course. For this reason, the perturbations which are merely bounded in the auxiliary space $\mathcal{H}^{(F)}_{\text{(friendly)}}$ become a purely formal construct because they are only small with respect to a phenomenologically irrelevant metric $\Theta^{\text{(false)}} = I$. Even without an explicit reference to the metric we have shown that at any dimension N and at any, *arbitrarily small* but nonvanishing $V^{(N)}_{N,1} = O(1)$ and $\lambda > 0$ the perturbed Hamiltonians (5) *cannot* be assigned *any* physical meaning or experimental realization.

In the second, main step of our considerations we inverted the ordering of questions. In the light of our main interest in the system's stability we decided to search for a consistent, "admissible" subset of perturbations λV which would still keep the perturbed Hamiltonian compatible with the quantum theory of reviews [1,21]. We felt encouraged by a preparatory analysis of our one-parametric illustrative model (3) which appeared easily converted into its canonical Jordan-block form. Having used these blocks as certain strong-coupling EP-related unperturbed Hamiltonians we were then able to leave the elementary model and to extend the scope of our considerations to the entirely general N by N real-matrix class of perturbations $\lambda V^{(N)}$.

We may summarize that we managed to specify the structure of admissible, observability nonviolating perturbation matrices $V^{(N)} = V^{(N)}(\lambda)$ at all N. Besides the proof of existence we also described the method of an (implicit) determination of the leading-order boundaries of the unitarity-compatible corridors $\mathcal{S}$ in the Euclidean real space of the variable matrix elements of $V^{(N)}(\lambda)$. Inside these domains of "admissible" parameters the evolution remains unitary. We may conclude that, in a way contradicting the scepticism of conclusions based on the constructions of the pseudospectra [17,18] and/or of the "broad" corridors [24], the quantum systems in question remain stable and closed inside corridors $\mathcal{S}$, which may be called "narrow."

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