Interference in the time domain of a decaying particle with itself as the physical mechanism for the exponential-nonexponential transition in quantum decay

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By using an exact analytical non-Hermitian approach in terms of resonance states, we show that the exponential-nonexponential transition of decay at long times physically represents a process of interference in the time domain of the decaying particle with itself. We also show that actually the regeneration mechanism proposed by Fonda and Ghirardi [L. Fonda and G. C. Ghirardi, Nuovo Cimento A **7**, 180 (1972)] is not the relevant mechanism for nonexponential decay.

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I. INTRODUCTION

In the early days of quantum mechanics, Gamow imposed on physical grounds outgoing (radiative) boundary conditions to the solutions of the Schrödinger equation to describe α decay in radioactive nuclei [1–3]. This led to complex energy eigenvalues and to the derivation of the exponential decay law $\exp(-\Gamma t/\hbar)$ for the evolving probability density, where the decay rate Γ corresponds to the imaginary part of the complex energy eigenvalue. Around three decades later, Khalfin [4] pointed out that in decaying systems where the energy spectra are bounded by below, i.e., $E \in (0,\infty)$, which includes most physical systems of interest, it follows, due to a theorem by Paley and Wiener [5], that the exponential decay law cannot be valid at all times. Khalfin considered in his analysis the survival probability, which yields the probability that at time t the decaying particle remains in its initial state. He was able to show that this quantity exhibits, in addition to a purely decaying exponential behavior that follows by assuming a complex pole located on the energy plane, an integral contribution that behaves at long times as an inverse power of time. Most subsequent work on this subject [6–13] has been strongly influenced by the work by Khalfin.

One should note, however, that the result obtained by Khalfin is based on a mathematical argument and hence it does not provide a physical mechanism to understand the exponential-nonexponential transition. An approach to deal with this question was considered by Fonda and Ghirardi [14], who, following the work by Ersak [15], argued that the physical mechanism for the deviation from exponential decay law at long times is a partial regeneration process of the initial state caused by rescattering of the decayed states [14]. However, as discussed below, exact model calculations show that the proposed mechanism yields a negligible contribution to the exact nonexponential behavior at long times and more importantly it does not provide a description of the exponential-nonexponential transition.

It is worth mentioning that the failure to find deviations of the exponential decay law at long times in radioactive nuclei [16,17] contributed to the widespread view that nonexponential decay contributions were beyond experimental reach or even to the alternative explanation that the interaction of the decaying system with the environment would enforce exponential decay at all times [18,19]. However, the experimental verification in recent times of short-time deviations from exponential decay [20] and the quantum Zeno effect [21,22] together with the measurement of the deviations from exponential decay law at long times in organic molecules in solution, which exhibited distinct inverse power behaviors in time [23], have demonstrated that nonexponential decay is an observable quantum effect.

The present work rests on an exact analytical non-Hermitian formulation of quantum tunneling decay [24] which involves the complex poles of the propagator and the resonance (quasinormal) states to the problem to address the issue of the physical mechanism of the exponential-nonexponential transition at long times. We demonstrate that the decaying wave function may be written as the sum of exponentially and nonexponentially decaying wave functions. The latter involves the propagation of almost vanishing values of the wave number and as time evolves eventually interferes with the exponentially decaying terms which refer to wave components close to the resonance energies. This interference yields the exponential-nonexponential transition. We show that physically it corresponds to a phenomenon of interference in the time domain of the decaying particle with itself.

The formulation considered here refers to the full Hamiltonian to the problem and hence it differs from approaches where the Hamiltonian is separated into a part corresponding to a closed system and a part responsible for the decay which is usually treated to some degree in perturbation theory, as in the work by Weisskopf and Wigner on the exponential decay of an excited atom interacting with a quantized radiation field [25] or in studies concerning the deviation of exponential decay in these systems [26].

The paper is organized as follows. Section II provides a brief review of the formalism of resonance states. In Sec. III we discuss the interference in the time domain of the decaying

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function. Section IV considers the approach by Fonda and Ghirardi on the mechanism to produce the deviation from exponential decay. In Sec. V we discuss a model calculation to exemplify our findings. Section VI presents some concluding remarks.

II. RESONANCE-STATE FORMALISM

The eigenfunctions associated with complex energy eigenvalues, the resonance states, increase beyond the interaction region exponentially with distance, implying that the usual rules concerning normalization and completeness do not apply. For these reasons, the approach by Gamow has been considered a phenomenological and no fundamental approximation for the description of decaying systems. However, modern developments of the formalism of resonance states have solved, in a consistent fashion, the above issues [24]. It has been shown that this non-Hermitian approach yields exactly the same results for the time evolution of decay as a Hermitian approach based on continuum wave solutions for generic exactly solvable models [27–29].

Here we briefly recall the relevant aspects of the derivation of the decaying wave solution for a single particle confined initially within the internal region of a spherically symmetric real potential with the condition, imposed on physical grounds, that it vanishes beyond a distance, i.e., $V(r) = 0$ for $r > a$. We choose natural units $\hbar = 2m = 1$ and for simplicity of the discussion and without loss of generality we refer to s waves. The solution to the time-dependent Schrödinger equation may be written in terms of the retarded Green's function $g(r, r'; t)$ of the problem as [24]

$$\Psi(r,t) = \int_0^a g(r,r';t)\Psi(r',0)dr', \tag{1}$$

where $\Psi(r, 0)$ stands for an arbitrary initial state which is confined within the internal interaction region. The retarded time-dependent Green's function $g(r, r'; t)$ is the relevant quantity to study the time evolution of the initial state. It may be evaluated by a Laplace transformation into the complex wave number plane k aimed to exploit the analytical properties of the outgoing Green's function to the problem $G^+(r, r'; k)$ [24,30],

$$g(r,r';t) = \frac{1}{2\pi i}\int_{c_0} G^+(r,r';k)e^{-ik^2t}2k\,dk, \tag{2}$$

where c_0 refers to the Bromwich contour which corresponds to a hyperbolic contour along the first quadrant of the k plane. A consequence of the condition that the potential vanishes after a distance is that $G^+(r, r'; k)$ may be extended analytically to the whole complex k plane where it has an infinite number of complex poles distributed in a well known manner [31]. Resonance states and complex energy poles are intimately related. Resonance states are solutions to the radial Schrödinger equation $[E_n - H]u_n(r) = 0$ obeying outgoing (radiative) boundary conditions $[du_n(r)/dr]_{r=a} = i\kappa_n(a)$, where $E_n = \kappa_n^2 = \mathcal{E}_n - i\Gamma_n/2$. Here $\mathcal{E}_n$ stands for the resonance energy of the decaying particle and Γ_n for the corresponding resonance width. As is well known, the longest lifetime sets up the timescale of the decay process. Resonance states may be also obtained from the residues at the complex poles $\{\kappa_n\}$ of the outgoing Green's function which also provides its normalization condition [24,30], namely, $\int_0^a u_n^2(r)dr + iu_n^2(a)/2\kappa_n = 1$. It is worth mentioning that resonance states satisfy flux conservation [24]. The above considerations allow for the rigorous derivation of the resonance expansion of the outgoing Green's function [24]

$$G^+(r,r';k) = \sum_{n=-\infty}^{\infty}\frac{u_n(r)u_n(r')}{2\kappa_n(k-\kappa_n)}, \quad (r,r')^\dagger \leqslant a, \tag{3}$$

where the above sum includes the resonance states $u_{-n}(r)$ and poles κ_{-n} located on the third quadrant of the k plane which are related to those located on the fourth quadrant by symmetry relations that follow from time reversal invariance: $\kappa_{-n} = -\kappa_n^*$ and $u_{-n}(r) = u_n^*(r)$ [24,32]; the notation $(r, r')^\dagger \leqslant a$ means that the point $r = r' = a$ is excluded in the above expansion, since otherwise it diverges.

The representation of $G^+(r, r'; k)$ given by (3) satisfies the closure relation [24,27]

$$\mathrm{Re}\left\{\sum_{n=1}^{\infty}u_n(r)u_n(r')\right\} = \delta(r-r'), \quad (r,r')^\dagger \leqslant a, \tag{4}$$

and the sum rules [24]

$$\sum_{n=-\infty}^{\infty}\frac{u_n(r)u_n(r')}{\kappa_n} = 0, \quad (r,r')^\dagger \leqslant a \tag{5}$$

and

$$\sum_{n=-\infty}^{\infty}u_n(r)u_n(r')\kappa_n = 0, \quad (r,r')^\dagger \leqslant a. \tag{6}$$

One may also write the resonance expansion of the Green's function given by (3), using the identity $1/[2\kappa_n(k-\kappa_n)] \equiv (1/2k)[1/(k-\kappa_n) + 1/\kappa_n]$ and (5) as

$$G^+(r,r';k) = \frac{1}{2k}\sum_{n=-\infty}^{\infty}\frac{u_n(r)u_n(r')}{k-\kappa_n}, \quad (r,r')^\dagger \leqslant a \tag{7}$$

The evaluation of $g(r, r'; t)$ as a resonance-state expansion involving the poles of $G^+(r, r'; k)$ may be obtained by distinct deformations of the contour c_0. One of them leads to an integral extending along the full real k axis [24]. Then substitution of (7) into (1) allow us to write the time-dependent decaying wave function as [24,27]

$$\Psi(r,t) = \sum_{n=-\infty}^{\infty}\begin{cases}C_nu_n(r)M(y_n^0), & r \leqslant a\\ C_nu_n(a)M(y_n), & r \geqslant a,\end{cases} \tag{8}$$

where the sums run over the full set of poles, the coefficients C_n are defined by

$$C_n = \int_0^a \Psi(r,0)u_n(r)dr, \tag{9}$$

and the functions $M(y_n)$, the so-called Moshinsky functions, are defined as [24]

$$M(y_n) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik(r-a)}e^{-ik^2t}}{k-\kappa_n}dk = \frac{1}{2}e^{i(r-a)^2/4t}w(iy_n), \tag{10}$$

with $y_n = e^{-i\pi/4}(1/4t)^{1/2}[(r-a) - 2\kappa_n t]$, and the function $w(z) = \exp(-z^2)\text{erfc}(-iz)$ in (10) stands for the Faddeyeva-Terent'ev or complex error function [33] for which there exist efficient computational tools to calculate it [34]. The argument y_n^0 of the functions $M(y_n^0)$ in (8) is that of y_n with $r = a$, namely,

$$y_n^\circ = -e^{-i\pi/4}\kappa_n t^{1/2}. \tag{11}$$

Assuming that the initial state $\Psi(r,0)$ is normalized to unity, it follows from the closure relation (4) that

$$\text{Re}\sum_{n=1}^{\infty}\{C_n\bar{C}_n\} = 1, \tag{12}$$

where $\bar{C}_n$ follows by taking the conjugate of $\Psi(r,0)$ in (9). Equation (12) indicates that $\text{Re}\{C_n\bar{C}_n\}$ cannot be interpreted as a probability, since in general it is not a positive-definite quantity. Nevertheless, one may see that it represents the strength or weight of the initial state in the corresponding resonant state. One may see the coefficients $\text{Re}\{C_n\bar{C}_n\}$ as some sort of quasiprobabilities.

The solution $\Psi(r,t)$ for $r \leqslant a$, given by the first equation in (8), is the relevant ingredient to calculate the survival probability, as discussed in [24,27,28]. For $r \geqslant a$, the solution $\Psi(r,t)$, given by the second equation in (8), describes the propagation of a single decaying particle along the external region. This has been discussed in Refs. [24,27,29].

The exponential and nonexponential explicit behavior of $\Psi(r,t)$ for $r \leqslant a$ may be achieved by using the symmetry relations mentioned above among the poles located on the third and fourth quadrants on the k plane, namely, $\kappa_{-n} = -\kappa_n^*$, and correspondingly for the resonance states, $u_{-n}(r) = u_n^*(r)$. As a result, one may write $\Psi(r,t)$ for $r \leqslant a$ as

$$\Psi(r,t) = \sum_{n=1}^{\infty}\left[C_n u_n(r) M\left(y_n^0\right) + \bar{C}_n^* u_n^*(r) M\left(y_{-n}^0\right)\right]. \tag{13}$$

One then may utilize a property of the functions $M(y_n^0)$ that establishes that $M(y_n^0) = \exp(-i\kappa_n^2 t) - M(-y_n^0)$, provided $\pi/2 < \arg(y_n^0) < 3\pi/2$ [24,35]. This is in fact the case for resonance poles with $\alpha_n > \beta_n$, the so-called proper resonance poles. In such a case, the arguments of both $M(-y_n^0)$ and $M(y_{-n}^0)$ satisfy $-\pi/2 < \arg(y_n^0) < \pi/2$ and hence do not exhibit an exponential behavior. As a result, one may write (13) as

$$\Psi(r,t) = \Psi_e(r,t) + \Psi_{ne}(r,t), \quad r \leqslant a, \tag{14}$$

where $\Psi_e(r,t)$ corresponds to the sum of exponentially decaying terms

$$\Psi_e(r,t) = \sum_{n=1}^{\infty} C_n u_n(r) e^{-i\mathcal{E}_n t} e^{-\Gamma_n t/2} \tag{15}$$

and $\Psi_{ne}(r,t)$ stands for the nonexponential contribution

$$\Psi_{ne}(r,t) = -\sum_{n=1}^{\infty}\left[C_n u_n(r) M\left(-y_n^0\right) - \bar{C}_n^* u_n^*(r) M\left(y_{-n}^0\right)\right], \tag{16}$$

where y_{-n}^0 follows from (11) by replacing κ_n by $-\kappa_n^*$. The expressions for $\Psi_e(r,t)$ and $\Psi_{ne}(r,t)$, given by (15) and (16), satisfy the time-dependent Schrödinger equation. This is easily verified for (15), whereas for (16) it is required, in view of (10), to make use of $dw(z)/dz = -2zw(z) + 2i/\sqrt{\pi}$ [33] and also of the sum rule (6).

Here we analyze the exponential-nonexponential transition, without loss of generality, for the time evolution of the survival probability

$$S(t) = |A(t)|^2, \tag{17}$$

where $A(t)$ stands for the survival amplitude $A(t)$, which is defined as

$$A(t) = \int_0^a \Psi^*(r,0)\Psi(r,t)dr. \tag{18}$$

Using the first equation in (8), one may write (18) as

$$A(t) = \sum_{n=-\infty}^{\infty} C_n\bar{C}_n M(y_n^0). \tag{19}$$

Hence, using (15) and (16), one may write (19) as

$$A(t) = A_e(t) + A_{ne}(t), \tag{20}$$

where

$$A_e(t) = \sum_{n=1}^{\infty} C_n\bar{C}_n e^{-i\mathcal{E}_n t} e^{-\Gamma_n t/2} \tag{21}$$

and

$$A_{ne}(t) = -\sum_{n=1}^{\infty}\left[C_n\bar{C}_n M\left(-y_n^0\right) - (C_n\bar{C}_n)^* M\left(y_{-n}^0\right)\right]. \tag{22}$$

One sees immediately by inspection of (21) that the exponentially decaying behavior of the survival amplitude corresponds to the sum of distinct resonance energies weighted by expansion coefficients that fulfill (12). It turns out that for large values of the argument, the M functions in (22) exhibit an asymptotic expansion that goes as $M(z_q) \sim 1/z_q - 1/z_q^3 + \cdots$, with $z_q = -y_n^0$ or y_{-n}^0 [24,33]. The leading term in these expansions, using (11) and (5), vanishes exactly and hence

$$A_{ne}(t) \approx -i\eta\,\text{Im}\left[\sum_{n=1}^{\infty}\frac{C_n\bar{C}_n}{\kappa_n^3}\right]\frac{1}{t^{3/2}}, \tag{23}$$

with $\eta = 1/(4\pi i)^{1/2}$. Note that for $l > 0$ the nonexponential contributions go as $t^{-(l+3/2)}$ [36], which implies that the leading nonexponential contribution comes from s waves.

III. INTERFERENCE IN THE TIME DOMAIN FOR THE EXPONENTIAL-NONEXPONENTIAL TRANSITION

In order to establish which energies (wave numbers) contribute to the nonexponential expression given by (23), it is convenient to evaluate $g(r, r'; t)$ by closing the contour c_0 in (2) in a different fashion, namely, by considering a line 45° off the real axis that may be evaluated by deforming c_0 and using the theorem of residues to get explicitly the exponentially decaying contributions plus an integral term along that line [24,28]. The integral term may be evaluated by the steepest-descent method. The essential point is that the saddle point is at $k = 0$, and making a Taylor expansion of $G^+(r, r'; k)$

around that point, one may write [24,28]

$$g(r,r';t) \approx \sum_{n=1}^{\infty} u_n(r)u_n(r')e^{-i\mathcal{E}_n t}e^{-\Gamma_n t/2} - i\eta\left\{\frac{\partial}{\partial k}G^{+}(r,r';k)\right\}_{k=0}\frac{1}{t^{3/2}}, \quad (r,r')^{\dagger} \leqslant a. \tag{24}$$

Equation (24) shows beyond any doubt that the nonexponential contribution arises from almost vanishing values of the wave number k and hence of the energy. This behavior is known [12,37], but has not been used to investigate the physical mechanism for the exponential-nonexponential transition. One may use (3) to express the second term in (24) in terms of resonance states to obtain

$$-i\eta\,\mathrm{Im}\left\{\sum_{n=1}^{\infty}\frac{u_n(r)u_n(r')}{\kappa_n^3}\right\}\frac{1}{t^{3/2}}, \quad (r,r')^{\dagger} \leqslant a. \tag{25}$$

It follows then, using (1) and (20), that (25) leads exactly to the expression of the nonexponential survival amplitude at long times given by (23). In an analogous way, the first term in (24) corresponds exactly to (21).

Using (20), the survival probability may be written as

$$S(t) = |A_e(t)|^2 + |A_{ne}(t)|^2 + 2\,\mathrm{Re}[A_e^*(t)A_{ne}(t)], \tag{26}$$

where $A_e(t)$ and $A_{ne}(t)$ are given by (21) and (22). Depending on the parameters of the potential, there is a time t_0 where the first and second terms in (26) are necessarily of the same order of magnitude. For $t < t_0$, $|A_e(t)|^2 > |A_{ne}(t)|^2$, and for $t > t_0$ it is the other way around. Around t_0, the last term in (26) may be written using (23) as

$$2\,\mathrm{Re}[A_e^*(t)A_{ne}(t)] \approx -2\,\mathrm{Re}\left\{\left[\sum_{n=1}^{\infty} C_n\bar{C}_n e^{-i\mathcal{E}_n t}e^{-\Gamma_n t/2}\right]^* \times \left[i\eta\,\mathrm{Im}\sum_{n=1}^{\infty}\frac{C_n\bar{C}_n}{\kappa_n^3 t^{3/2}}\right]\right\}, \tag{27}$$

which describes analytically the interference in the time domain of the exponential and nonexponential contributions to the decay process in the transition region. The essential point is that both $A_e(t)$ and $A_{ne}(t)$ originate from the decaying wave function (14) and hence one may conclude that the exponential-nonexponential transition process corresponds to the interference in the time domain of the decaying particle with itself.

Since the coefficients $C_n\bar{C}_n$ satisfy the closure relation given by (12), in practice, depending on the initial state, a finite number of terms are needed to evaluate (27). It is straightforward to see that the above considerations hold also for the time evolution of the probability density $|\Psi(r,t)|^2$.

IV. FONDA-GHIRARDI MECHANISM FOR NONEXPONENTIAL DECAY

Let us now refer to the physical mechanism proposed by Fonda and Ghirardi to produce the deviations from the exponential decay law [14]. Following an approach considered by Ersak [15], these authors argue that during the decay process of an unstable system one may only observe whether the system remains undecayed or has decayed. As a consequence they write the Hamiltonian of the Hilbert space of the system as the sum of two orthogonal subspaces H_u and $H_\perp$, namely, $H = H_u + H_\perp$, and assume that the initial unstable state $|0\rangle$ is part of an orthonormal set of states which is complete in H_u. Then as time evolves one may write, for $t' \geqslant 0$ [14],

$$e^{-iHt'}|0\rangle = A(t')|0\rangle + |\phi_{t'}\rangle, \quad t' \geqslant 0, \tag{28}$$

where $A(t')$ is the survival amplitude defined by (18) and $|\phi_{t'}\rangle$ stands for the decayed state at time t'. From (28) it follows, provided the initial state is normalized to unity, i.e., $\langle 0|0\rangle = 1$, that

$$\langle 0|\phi_{t'}\rangle = 0. \tag{29}$$

The decayed state $|\phi_t'\rangle$ that arises after measurement determines that the system has decayed. Therefore $|\phi_{t'}\rangle$ represents presumably a collapsed state at time t'. By applying to both sides of (28) the operator $\exp(-iHt)$ with $t \geqslant 0$, it follows, after a simple mathematical manipulation, that [14,15]

$$A(t+t') = A(t)A(t') + I(t,t'), \tag{30}$$

where

$$I(t,t') = \langle 0|\exp(-iHt)|\phi_{t'}\rangle \tag{31}$$

is the expression that provides, according to Ref. [14], the physical mechanism for producing the deviation from the exponential decay law. This follows as a consequence of the fact that $I(t,t') = 0$ implies, using (30), that $A(t)$ decays exponentially and since this contradicts the Paley-Wiener theorem, $I(t,t')$ must be $\neq 0$. Following the above reasoning, they affirm that the decayed state reconstructs partially the initial state through a process of rescattering. These authors did not discuss or suggest a suitable analytical expression for $|\phi_{t'}\rangle$ satisfying the orthogonality condition (29) to evaluate (31). However, from (30) we may express exactly $I(t,t')$ as

$$I(t,t') = A(t+t') - A(t)A(t'), \tag{32}$$

which may be evaluated by using a solvable model for $A(t)$. Notice, however, that $I(t,t') \neq 0$ implies that $A(t)$ and $A(t')$ must contain intrinsic nonexponential contributions because otherwise $I(t,t')$ should vanish exactly as discussed above. Fonda and Ghirardi ignored this intrinsic nonexponential contribution and focused their discussion on the term $I(t,t')$. Here we will address a comparison between the intrinsic nonexponential contribution of $A(t)$ and the rescattering term $I(t,t')$. We find using a generic exact solvable model that, contrary to the claim by Fonda and Ghirardi, the reconstruction of the initial state by the decayed state is not the relevant mechanism to originate the nonexponential decay behavior.

It is of interest to mention that Fonda and Ghirardi [14] did not discuss the short-time behavior of the nonexponential term $I(t,t')$. As is well known, the survival probability $S(t)$, defined by (17), exhibits at short times a deviation from the exponential decay law that in general follows a quadratic behavior, namely,

$$S(t) \approx 1 - dt^2, \tag{33}$$

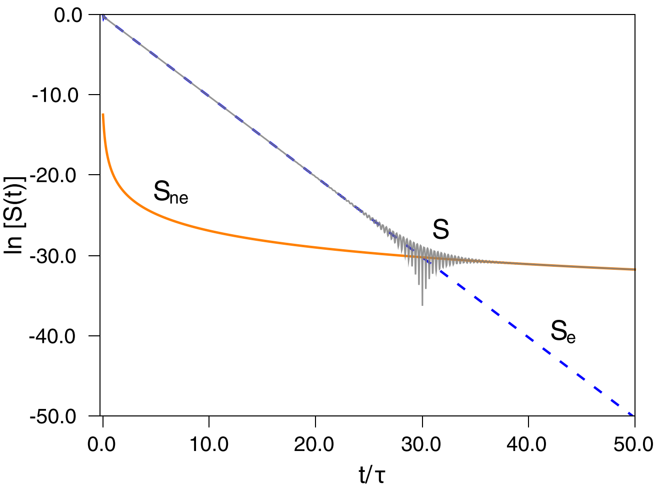

FIG. 1. Natural logarithm of the survival probability $S(t)$ (solid gray line) as a function of the time t/τ in lifetime units for the potential parameters given in the text. Also shown are the exponential $S_e(t) = |A_e(t)|^2$ and nonexponential $S_{ne}(t) = |A_{ne}(t)|^2$ contributions. Notice the interference region in the time domain for $t/\tau \approx 30.0$.

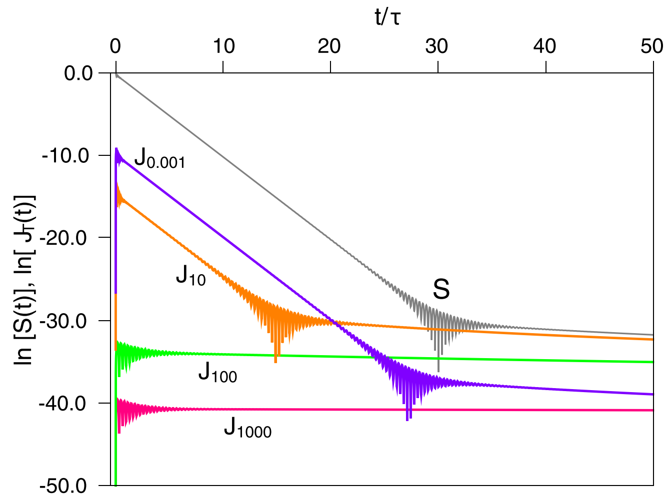

FIG. 2. Natural logarithm of the Ersak term $J_T(t) = |I(t, T|^2$ for several values of T as indicated in the figure and its comparison with $\ln S(t)$, which exhibits the intrinsic nonexponential behavior at long times. The parameters of the potential are the same as in Fig. 1. See the text for further details.

with d a constant larger than zero [38]. Using the resonance state formalism, the short-time behavior of $S(t)$ has been analyzed for distinct initial states in Ref. [39]. Since the initial state is normalized, it follows from (18) that $A(0) = 1$ and hence it is straightforward to convince oneself, using (32) for any $t' = T$, that

$$I(0, T) = 0. \tag{34}$$

The above result is in agreement with an inequality derived by Fleming [40], i.e., Eq. (3.2) therein, which in our notation reads

$$|I(t, t')|^2 \leqslant [1 - S(t)][1 - S(t')]. \tag{35}$$

For $t' = T$ and $t = 0$, Eq. (35) yields also $I(0, T) = 0$ as in (34). Hence, the above considerations imply that the rescattering mechanism proposed by Fonda and Ghirardi does not describe the deviation from exponential decay at short times. The above result is exemplified below in a model calculation.

V. MODEL

As an example, let us consider the barrier shell potential, which consists of an internal region of width w, followed by a rectangular potential barrier of width b, so $w + b = a$. The system parameters are the barrier height $V = 30.0$, well width $w = 1.0$, and barrier width $b = 0.3$. The resonance state solutions to the Schrödinger equation obeying outgoing boundary conditions with the usual continuity conditions at the distinct interfaces of the potential lead to the equation whose solution yields, following known procedures [41–43], the κ_n's to the problem. As the initial state we use the quantum-box state $\Psi(r, 0) = \sqrt{2/w}\,\sin(\pi r/w)$ placed at the center of the quantum well.

Figure 1 shows a typical survival probability graph along the exponential and long-time regimes (solid grey line), which exhibits in particular the exponential-nonexponential transition which is described analytically by (27). Figure 1 also displays the exponential $\ln S_e(t)$ and nonexponential $\ln S_{ne}(t)$ contributions calculated using (21) and (22). It is worth pointing out that (23) yields the same result as (22) except at very small times. In some cases the exponential-nonexponential transition time t_0 may be evaluated analytically [44]. In order to guarantee the t^{-3} asymptotic behavior of the survival probability, it is sufficient to consider 50 poles in the calculations.

Figure 2 exhibits the exact calculation of the Ersak term $\ln J_T(t)$ versus time in lifetime units, where $J_T(t) = |I(t, T)|^2$ follows using (32). Since Fonda and Ghirardi did not provide a criterion to choose the value of T, we evaluate $\ln J_T(t)$ for a range of values of T and compare it with $\ln S(t)$ with the same potential parameters as in Fig. 1. It is also worth noting in Fig. 2 that as the time t approaches zero, the contribution of $J_T(t)$ diminishes as T increases and that at $t = 0$ it goes to zero, i.e., $\ln J_T(0) = -\infty$, in agreement with (34) and (35). This implies, as discussed in the preceding section, that the Fonda and Ghirardi mechanism says nothing about the deviation of exponential decay at short times. Note that for $T = 0.001$ and 10, which are within the exponential regime of $S(t)$, $J_T(t)$ vs t exhibits both exponential and nonexponential regimes. We also consider values of T corresponding to the nonexponential regime of the survival probability. These are the values of $T = 100$ and 1000 of $\ln J_T(t)$ in Fig. 2. One may understand the distinct behaviors of $J_T(t)$ exhibited in Fig. 2 by writing down an analytic expression for $I(t, T)$ using Eqs. (20), (21), and (23). Since the expansion coefficient $\mathrm{Re}\, C_1^2 = 0.899$ is of the order of unity, in view of (12), we make the single-term approximation $n = 1$ in (21) and (23) to write the leading contributions of $I(t, T)$ as

$$I(t, T) \approx \left(C_1^2 - C_1^4\right) e^{-i\mathcal{E}_1(t+T)} e^{-\Gamma_1(t+T)/2} - \frac{D_1}{(t+T)^{3/2}} - \frac{D_1^2}{t^{3/2}T^{3/2}}, \tag{36}$$

where $D_1 = i\eta\, \mathrm{Im}(C_1^2/\kappa_1^3)$. Notice that the coefficient $(C_1^2 - C_1^4)$ does not vanish, which explains the exponentially decaying behavior of $J_T(t)$ observed in Fig. 2. Similarly, the second and third terms on the right-hand side of (36) describe the leading inverse power of time behavior of $J_T(t)$. These terms become the dominant contribution for very large values of T, because then the exponential contribution is negligible. We point out that the above approximate formula holds for any values of t and T, not too close to zero because then the exact expressions (21) and (22) must be used. Notice that all calculations yield values much smaller than those of the survival probability $S(t)$. This implies that the nonexponential contribution $A_{ne}(t)$, given by (23), is larger than the Ersak term $I(t,T)$, given by (32). The above results show that the so-called regeneration or rescattering mechanism proposed by Fonda and Ghirardi is not the relevant mechanism of the long-time deviation of exponential decay and says nothing of the short-time behavior.

VI. CONCLUSION

The main contribution of this work is to show that the physical mechanism for the exponential-nonexponential transition exhibited by the survival probability is the interference in the time domain of the decaying particle with itself. We also found that the regeneration term proposed several decades ago by Fonda and Ghirardi is unable to provide the physical mechanism for the origin of nonexponential decay and as a consequence it does not describe the exponential-nonexponential transition. It is worth emphasizing that the calculation of the survival probability using the non-Hermitian approach discussed here yields exactly the same results as numerical calculations using the corresponding Hermitian approach in terms of continuum wave functions [27–29], which however do not provide any clue to the underlying physical mechanism that yields the non-Hermitian approach. In our view, our finding exhibits a unknown feature of nonstationary solutions to the Schrödinger equation and therefore suggests to explore its implications on foundational issues of quantum mechanics. Our approach follows a line of inquiry aimed to extend the standard formalism of quantum mechanics to incorporate in a fundamental fashion the present non-Hermitian treatment of the Hamiltonian to describe dynamical aspects of quantum systems [27,45].

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