Rapid Communications

Relativistic quantum-mechanical description of twisted paraxial electron and photon beams

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The analysis of twisted (vortex) paraxial photons and electrons is fulfilled in the framework of relativistic quantum mechanics. The use of the Foldy-Wouthuysen representation radically simplifies the description of relativistic electrons and clarifies the fundamental properties of twisted particles. It is demonstrated that the twisted and other structured photons are luminal. Their subluminality apparently takes place because the photon energy is also contributed by a hidden motion. This motion is vanished by averaging and disappears in the semiclassical description based on expectation values of the momentum and velocity operators. It is proven that semiclassical quanta of structured light are subluminal and massive. The quantum-mechanical and semiclassical descriptions of twisted and other structured electrons lead to similar results. The effect of a quantization of the velocity and the effective mass of the structured photon and electron is predicted. This effect is observable for the photon. The twisted and untwisted semiclassical photons and electrons modeled by the centroids are considered in the accelerated and rotating noninertial frame. The coincidence of their inertial masses with kinematic ones is shown. The orbital magnetic moment of the Laguerre-Gauss electron does not depend on the radial quantum number.

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The prediction and discovery of twisted (vortex) states of photons [1,2] and electrons [3,4] has opened new horizons in contemporary physics. In these states, photons and electrons have orbital angular momenta (OAMs). At present, twisted photon and electron beams are objects of intensive studies and have many practical applications (see Refs. [5–20] and references therein). The most important kind of such beams is a paraxial (Laguerre-Gauss) wave beam [1,18,21] satisfying the paraxial approximation ($p_\perp \ll p$). This beam is unlocalized in the longitudinal direction z and transversely two-dimensionally (2D) localized. It is described by the paraxial equation

$$\left(\nabla_\perp^2+2ik\frac{\partial}{\partial z}\right)\Psi=0,\quad \nabla_\perp^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}. \tag{1}$$

The known solutions in cylindrical coordinates are the Laguerre-Gauss beams [1,22,23],

$$\Psi=\mathcal{A}\exp(i\Phi),$$
$$\mathcal{A}=\frac{C_{nl}}{w(z)}\left(\frac{\sqrt{2}r}{w(z)}\right)^{|l|}L_n^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(-\frac{r^2}{w^2(z)}\right),$$
$$\Phi=l\phi+\frac{kr^2}{2R(z)}-(2n+|l|+1)\varphi(z),$$
$$C_{nl}=\sqrt{\frac{2n!}{\pi(n+|l|)!}},\quad w(z)=w_0\sqrt{1+\frac{4z^2}{k^2w_0^4}},$$
$$R(z)=z+\frac{k^2w_0^4}{4z},\quad \varphi(z)=\arctan\left(\frac{2z}{kw_0^2}\right), \tag{2}$$
$$\int\Psi^\dagger\Psi r dr d\phi=1, \tag{3}$$

where the real functions $\mathcal{A}$ and Φ define the amplitude and phase of the beam, k is the beam wave number, w_0 is the minimum beamwidth, $L_n^{|l|}$ is the generalized Laguerre polynomial, and $n=0,1,2,\dots$ is the radial quantum number. A frequently encountered inexactness [18–20] is the superfluous factor $\exp(ikz)$. Other quantum-mechanical solutions are 3D-localized particle wave packets [18,24–29]. Quantum numbers of twisted photons have been determined in Ref. [30].

We assume that $\hbar=1$, $c=1$ but include $\hbar$ and c into some formulas when this inclusion clarifies the problem.

One of the most mysterious physical phenomena is the recently discovered subluminality of free twisted photons for Bessel [31] (see also Ref. [32]) and Laguerre-Gauss [33,34] beams. Special relativity asserts that massless particles in vacuum should be luminal. Therefore, a definite solution of this puzzle should be based on a description of single photons. In the present Rapid Communication, we make this description in the framework of relativistic quantum mechanics (QM). We investigate the properties of twisted particles and untwisted but structured ones, changing the usual perception of such particles.

The possibility to use a quantum-mechanical approach for a description of light quanta is nontrivial and should be

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substantiated. For the photon in optics, Ψ is not a wave function in the same sense as for the electron and is simply a function that determines the relative amplitude of the electric field [1,21,35]. The full description of an electromagnetic field including its interaction with matter is based on the quantum field theory (see Refs. [36,37]). However, the propagation of light in a free space can be adequately described with the Riemann-Silberstein vector

$$\boldsymbol{F} = \frac{1}{\sqrt{2}}(\boldsymbol{E} + i\boldsymbol{B}).$$

It allows one to reduce the Maxwell equations and to present them in the form [30,38]

$$i\hbar\partial_t \boldsymbol{F} = c(\boldsymbol{\tau}\cdot\boldsymbol{p})\boldsymbol{F}, \tag{4}$$

where $\boldsymbol{\tau}$ is a vector in which the components are conventional spin-1 matrices acting on three components of $\boldsymbol{F}$. This equation is similar to the Weyl equation for a massless spin-1/2 neutrino [38]. When the six-component wave function is defined by [39]

$$\psi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi\\ \chi\end{pmatrix}, \quad \phi = \begin{pmatrix}E_x\\ E_y\\ E_z\end{pmatrix}, \quad \chi = \begin{pmatrix}iB_x\\ iB_y\\ iB_z\end{pmatrix}, \tag{5}$$

a Dirac-like equation for the free electromagnetic field can be obtained,

$$i\hbar\frac{\partial\psi}{\partial t} = \boldsymbol{\alpha}\cdot\boldsymbol{p}\,\psi, \quad \boldsymbol{\alpha} = \begin{pmatrix}0 & \boldsymbol{\tau}\\ \boldsymbol{\tau} & 0\end{pmatrix}. \tag{6}$$

In this connection, we can mention the existence of bosonic symmetries of the standard Dirac equation [40–45]. For twisted paraxial photons and electrons, we use the Foldy-Wouthuysen (FW) representation [46] in relativistic QM obtained by appropriate unitary transformations of initial Hamiltonians and wave functions. Excellent advantages of this representation are restoring the Schrödinger form of relativistic QM and unifying relativistic QM for particles with different spins [46–49]. In the FW representation, the Hamiltonian and all fundamental operators are block-diagonal (diagonal in two spinors or spinorlike parts of wave functions). The passage to the classical limit usually reduces to a replacement of the operators in quantum-mechanical Hamiltonians and equations of motion with the corresponding classical quantities [50]. The FW wave function being a generalization of the Schrödinger wave function on the relativistic case permits the probabilistic interpretation [51]. Owing to these properties, the FW representation provides the best possibility to obtain a meaningful classical limit of relativistic QM not only for the stationary case [46,50,52,53] but also for the nonstationary one [54].

We use the results obtained in Ref. [39]. The FW transformation of the Dirac-like Hamiltonian $\boldsymbol{\alpha}\cdot\boldsymbol{p}$ is straightforward and the FW Hamiltonian for a free photon is defined by [39]

$$\mathcal{H}_{\rm FW}\Psi_{\rm FW} = \beta|\boldsymbol{p}|\Psi_{\rm FW}, \quad \beta = {\rm diag}(1, 1, 1, -1, -1, -1). \tag{7}$$

While the wave functions Ψ and $\Psi_{\rm FW}$ have different definitions, a connection between $\boldsymbol{E}$ and $\boldsymbol{B}$ provides for their similarity. Nevertheless, $\Psi_{\rm FW}$ is proportional to a field amplitude. For a plane electromagnetic wave, $\boldsymbol{B} = \boldsymbol{p}\times\boldsymbol{E}/p$. Importantly, the quantum-mechanical approach allows one to introduce operators and to calculate their expectation values.

The corresponding FW Hamiltonian for a free electron [46] is similar to that for the free photon,

$$\mathcal{H}_{\rm FW} = \beta\sqrt{m^2 + \boldsymbol{p}^2}, \quad \beta = {\rm diag}(1, 1, -1, -1). \tag{8}$$

FW Hamiltonians describing free massive spin-1 particles [55,56] and massive and massless scalar ones [47] are also similar. The number of components of the corresponding wave functions depends on the spin and is equal to $2(2s+1)$.

Despite the similarity of the FW Hamiltonians, the wave functions for the photon and electron substantially differ from each other. The photon wave function $\Psi_{\rm FW}$ characterizes the relative amplitude of the electromagnetic field [39] and cannot be regarded as the probability amplitude of the spatial localization of the photon (see Ref. [57], p. 12). On the other hand, the corresponding wave function for the electron enables the probabilistic interpretation [51]. However, the photon wave function defines eigenvalues or expectation values of all operators. Its squared magnitude $|\Psi_{\rm FW}|^2$ is proportional to the light energy density. The physical reality of the wave functions of twisted photons has been confirmed in Ref. [58].

Lower spinors or spinorlike parts of FW wave functions vanish [59]. Hereinafter, they will be eliminated and β matrices will be removed. The simple form of Eqs. (7) and (8) clearly shows preferences for the approach based on the FW transformation.

The standard quantum-mechanical approach based on the Proca equations brings a result which is in accordance with Eq. (7). For massive and massless free spin-1 particles, these equations lead to the following second-order equation [see Ref. [57], Eq. (14.4)],

$$(p_0^2 - \boldsymbol{p}^2 - m^2)\psi_\mu = 0, \quad p_0 \equiv i\frac{\partial}{\partial t}, \tag{9}$$

where ψ_μ ($\mu = 0, 1, 2, 3$) has three *independent* components. For the photon, $m = 0$ and Eqs. (7) and (9) agree.

Optical and quantum-mechanical approaches significantly differ. Optics studies the light field and determines its *local* velocities. Certainly, the phase and group velocities are different. A local phase velocity (LPV) is defined by the phase front $\Phi(\boldsymbol{r})$, $v_p = \omega/|\nabla\Phi(\boldsymbol{r})|$, where $\omega = ck$ is the angular frequency [33,60]. Another frequently used formula for the LPV has been obtained in Ref. [61] (see also Ref. [62]),

$$v_p = c\left[1 + \frac{\nabla^2\mathcal{A}(\boldsymbol{r})}{k^2\mathcal{A}(\boldsymbol{r})}\right]^{-1/2}.$$

The local group velocity is given by $v_g = |\partial_\omega\nabla\Phi(\boldsymbol{r})|^{-1}$ [33,60] (see also Ref. [63] for details). The analysis shows [33,62–72] that both velocities can be subluminal and superluminal depending on the region. Certainly, both the local phase and group velocities characterize the important properties of twisted light beams. For example, the LPV defines an electron acceleration in a laser beam [65,67,73]. The distribution of the LPV has been measured in Ref. [74].

However, any free photon at any time extends over the whole 3D space and thus the optical approach based on the *local* phase and group velocities may fail to determine its fundamental properties (quantum numbers [30] and eigenvalues and expectation values of operators). On the other hand, the quantum-mechanical approach providing for a *single-particle* description perfectly solves this problem.

For stationary states ($\mathcal{H}_{\rm FW}\Psi_{\rm FW} = E\Psi_{\rm FW}$), squaring Eq. (8) for the upper spinor and applying the paraxial approximation for $p_z > 0$ results in (cf. Ref. [39])

$$p = \sqrt{p_\perp^2 + p_z^2} \approx p_z + \frac{p_\perp^2}{2p}, \quad p = \hbar k = \sqrt{E^2 - m^2}. \quad (10)$$

The operator form of Eq. (10) reads

$$\left(\nabla_\perp^2 + 2ik\frac{\partial}{\partial z}\right)\Psi_{\rm FW} = -2k^2\Psi_{\rm FW}. \quad (11)$$

The substitution $\Psi_{\rm FW} = \exp(ikz)\Psi$ brings the paraxial equation (1). Within the paraxial approximation, it *exactly* describes photons and electrons of arbitrary energies. Therefore, the FW transformation radically simplifies a description of relativistic electrons (cf. Ref. [75]). We underline the difference between $\Psi_{\rm FW}$ and Ψ.

The subluminality of twisted (and untwisted) light finds a straightforward explanation and description in relativistic QM which is a part of quantum optics. All beam parameters are defined by expectation values or eigenvalues of related operators. QM shows that the twisted photon is luminal and its subluminality is *apparent*. The group velocity operator, $v \equiv \sqrt{v_r^2 + v_\phi^2 + v_z^2}$, depends on a hidden motion in the horizontal plane [76]. As follows from Eq. (8),

$$\boldsymbol{v} = \frac{\partial\mathcal{H}_{\rm FW}}{\partial\boldsymbol{p}} = \frac{c\boldsymbol{p}}{p}, \quad v = c. \quad (12)$$

We use the term "hidden motion" for a motion which does not contribute to the expectation values of operators defining some components of the velocity and momentum but affects both expectation values of squares of these operators and eigenvalues of the energy operator. In the considered case, the expectation values of two Cartesian velocity components are zero ($\langle v_i\rangle = 0,\ i = x, y$). However, $\langle v_x^2 + v_y^2\rangle = \langle v_r^2 + v_\phi^2\rangle \neq 0$. Importantly, just the expectation values of the main operators define *measurable* beam parameters. For the electron, the velocity operator reads

$$\boldsymbol{v} = \frac{c\boldsymbol{p}}{\sqrt{m^2c^2 + \boldsymbol{p}^2}}. \quad (13)$$

Certainly, only the z component of the group velocity $\boldsymbol{v}$ can be *directly* measured. For the photon, it is less than c. This fact creates the impression that the twisted photon is subluminal.

QM is a foundation of contemporary physics and measurable quantities are expectation values of the velocity and momentum operators. Therefore, the *classical* model of light quanta (Einstein quanta) which velocity, energy, and momentum are defined by expectation values or eigenvalues of the corresponding operators remains very important. For twisted and any other structured light, the result is nontrivial.

The calculation of expectation values of v_z is straightforward. It follows from Eqs. (1), (10), and (12) that

$$\frac{v_z}{c} = \sqrt{1 - \frac{p_\perp^2}{p^2}} = \sqrt{1 - \frac{2i}{k}\frac{\partial}{\partial z}} \approx 1 - \frac{i}{k}\frac{\partial}{\partial z}. \quad (14)$$

As $\mathcal{A}^\dagger = \mathcal{A}$, $\mathcal{A}^2 = |\Psi|^2$,

$$\int \Psi^\dagger\frac{\partial\Psi}{\partial z}rdrd\phi = \int \mathcal{A}\frac{\partial\mathcal{A}}{\partial z}rdrd\phi + i\int|\Psi|^2\frac{\partial\Phi}{\partial z}rdrd\phi. \quad (15)$$

The first integral in the right-hand side vanishes,

$$\int \mathcal{A}\frac{\partial\mathcal{A}}{\partial z}rdrd\phi = \frac{1}{2}\frac{d}{dz}\int|\Psi|^2 rdrd\phi = 0.$$

The second integral can be calculated exactly. Since

$$\frac{\partial\Phi}{\partial z} = \frac{2}{kw^2(z)}\left\{\frac{r^2}{w_0^2}\left[1 - \frac{8z^2}{k^2w_0^2w^2(z)}\right] - \zeta\right\},$$
$$\zeta = 2n + |l| + 1, \quad (16)$$

averaging (see Ref. [77]) results in

$$\langle r^2\rangle = \frac{\zeta w^2(z)}{2}, \quad \left\langle\frac{\partial\Phi}{\partial z}\right\rangle = -\frac{\zeta}{kw_0^2}, \quad \langle p_\perp^2\rangle = \frac{2\zeta}{w_0^2}, \quad (17)$$

$$\langle v_z\rangle = c\left(1 + \frac{1}{k}\left\langle\frac{\partial\Phi}{\partial z}\right\rangle\right) = c\left(1 - \frac{2n + |l| + 1}{k^2w_0^2}\right). \quad (18)$$

A comparison of Eqs. (16) and (18) shows that the contributions from regions with small and large values of r to v_z are subluminal and superluminal, respectively.

Equation (18) has been previously derived in Ref. [78]. However, the correct interpretation of this equation can be based only on relativistic QM. Our approach connects the result (18) with initial quantum-mechanical equations (7) and (9) and, therefore, attributes it to a *single* photon. All twisted and untwisted Laguerre-Gauss modes, including the fundamental mode $n = l = 0$, are subluminal. Our results do not support the formula obtained in Ref. [79] by averaging the local field velocity which does not characterize the single photon. For the electron, Eq. (17) remains unchanged and the longitudinal velocity is given by

$$\langle v_z\rangle = \frac{ck}{\sqrt{k^2 + K^2}}\left(1 - \frac{2n + |l| + 1}{k^2w_0^2}\right), \quad K = \frac{mc}{\hbar}. \quad (19)$$

We predict another property of twisted particles consisting in a quantization of the group velocity and following from Eqs. (18) and (19). We suppose that this quantization can be observed because the modes n and l are measurable [80]. Experimental data [33] obtained for mixtures of modes with different n agree with our prediction but cannot prove it.

Some properties of twisted particles characterize a local field while other properties are attributed to the photon or electron extending over the whole spacetime. In particular, $\langle r^2\rangle$ depends on z and depicts local field properties. On the other hand, $\langle p_\perp^2\rangle$ and $\langle v_z\rangle$ are independent of z and define the general quantum-mechanical parameters of the twisted photon and electron.

Since the wave properties of twisted particles are defined by $\langle p_z\rangle$ and $\langle v_z\rangle$ and a detailed analysis of the hidden transversal motion can often be avoided, it is convenient to consider such particles as extended objects (the so-called centroids [16,18]) moving in the z direction. This model remains applicable for twisted particles in external fields [16,18,81–84]. The transition to the semiclassical approximation allows us to determine the mechanical properties of the centroids. In this case, angular brackets can be omitted and we can consider a twisted photon such as a centroid with the constant laboratory frame energy $E = \sqrt{p_z^2 + p_\perp^2}$. The velocity of the centroid is defined by $v_x = v_y = 0$, $v_z = p_z/E$, and the origin of an internal motion defined by $p_\perp^2$ can be disregarded. Certainly, such a quasiparticle satisfies the requirements of special relativity only if it possesses the mass $M = \sqrt{E^2 - p_z^2}$.

The validity of the introduction of the light mass was previously studied only for *groups of photons*. It is known [85] that two photons with equal frequencies and with the angle 2θ between the directions of their wave vectors acquire the Lorentz-invariant mass $m = (2\hbar\omega/c^2)\sin\theta$. In Refs. [86,87], this property has been applied to groups of *nonidentical and noncollinear* photons containing Gaussian pulses. It has been underlined [86,87] that such an approach is inapplicable for single photons or groups of identical photons being the objects of our study. We can add that the Gaussian pulses describe neither twisted states nor untwisted states with a nonzero radial quantum number. In particular, the average velocity obtained in Refs. [86–88] reads $\langle v_z\rangle = c[1 - (2k^2w_0^2)^{-1}]$ [cf. Eq. (18)].

To verify a possibility to model the Laguerre-Gauss photon by a massive centroid, we need to pass to an arbitrary inertial frame. Let us make the Lorentz boost to the centroid rest frame ($v_z^{(0)} = 0$). In this frame, $E^{(0)} = \sqrt{p_\perp^2}$, $p_x^{(0)} = p_x = 0$, $p_y^{(0)} = p_y = 0$, $p_z^{(0)} = 0$. We can now consider the second boost to the frame denoted by primes and moving with the arbitrary velocity $-\boldsymbol{V}$ relative to the centroid rest frame. If we change the coordinates and direct the X axis along the vector $\boldsymbol{V}$, the Lorentz transformation results in

$$p'_X = \frac{VE'}{c^2}, \quad p'_Y = p'_Z = 0, \quad E' = \frac{E^{(0)}}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{20}$$

It is easy to check that arbitrary Lorentz transformations for the centroid are equivalent to those for a massive particle with the mass $M = E^{(0)}/c^2$. When $\hbar, c$ are included, the effective mass of the twisted photon (centroid mass) reads

$$M = \frac{\sqrt{2(2n + |l| + 1)}\hbar}{cw_0}. \tag{21}$$

Its relation to the centroid velocity is defined by

$$M = \frac{\hbar k}{c}\sqrt{2\left(1 - \frac{\langle v_z\rangle}{c}\right)} = \frac{\hbar k}{c}\sqrt{1 - \frac{\langle v_z\rangle^2}{c^2}}. \tag{22}$$

This result shows a nontrivial possibility of conversion of the Lorentz-noninvariant hidden momentum into the Lorentz-invariant mass. The mass-energy ratio is given by

$$\frac{Mc^2}{E} = \frac{\sqrt{2(2n + |l| + 1)}\lambda}{2\pi w_0}, \quad \lambda = \frac{2\pi}{k}. \tag{23}$$

The second boost, unlike the first one, changes the OAM [81].

We can easily extend our analysis to the other forms of structured light. Equations (7) and (9) remain valid in the general case. Our derivation covers Gaussian beams because the presence or absence of the OAM is not important in this case. The other forms of structured light are also characterized by the hidden motion. For the 3D-localized particle wave packets (light bullets) [25,27,89], wave functions are 3D normalized ($\int \Psi^\dagger\Psi d^3x = 1$) and this motion takes place in three directions. The Lorentz boost from the wave-packet rest frame to the laboratory frame also satisfies Eq. (20) for any chosen direction X. In this case, $E^{(0)} = \sqrt{(p_x^{(0)})^2 + (p_y^{(0)})^2 + (p_z^{(0)})^2}$. Thus, arbitrary Lorentz transformations for the light wave packet are equivalent to those for a massive particle with mass $M = E^{(0)}/c^2$. Relation (22) also remains unchanged. Equations (7) and (9) demonstrate that the velocity operator is equal to c for any form of light. For wave packets, one can also determine the parameters of semiclassical light quanta (Einstein quanta) by averaging the momentum and velocity operators. Evidently, such semiclassical quanta are subluminal and massive.

To complete the analysis, we need only to consider Laguerre-Gauss and other structured particles in noninertial frames. This consideration allows us to determine an *inertial mass* which is important in the processes of beam acceleration and rotation. Light beam acceleration and rotation are largely investigated (see Refs. [90,91] and references therein). The problem is rather nontrivial. In particular, the kinematic ("Lorentz-invariant" [86–88]) mass of the group of noncollinear photons may not manifest itself in inertial and gravitational interactions [85–88]. The practical importance of the related problem of Laguerre-Gauss photons in gravitational fields is poor.

For spinning and spinless single particles in noninertial frames, relativistic FW Hamiltonians and equations of motion as well as their classical counterparts have been derived in Refs. [49,92–95]. We may disregard spin effects because the corresponding terms in the Hamiltonians are relatively small. In the semiclassical approximation, the Hamiltonian of a particle in a noninertial frame accelerated with the acceleration $\boldsymbol{a}$ and rotating with the angular velocity $\boldsymbol{\omega}$ has the form [93]

$$\mathcal{H} = (1 + \boldsymbol{a}\cdot\boldsymbol{r})\sqrt{m^2 + \boldsymbol{p}^2} - \boldsymbol{\omega}\cdot\boldsymbol{l}. \tag{24}$$

Here, $\boldsymbol{a}$ and $\boldsymbol{\omega}$ are independent of the spatial coordinates but may arbitrarily depend on time [93] and $\boldsymbol{l}$ is the total angular momentum. The particle motion is affected by the accelerator, Coriolis, and centrifugal forces. If the sizes of the light beam are negligible as compared with those of the beam trajectory in the inertial field, $\boldsymbol{l} = \boldsymbol{r}\times\boldsymbol{p} + \boldsymbol{L}$, where $\boldsymbol{L}$ is the intrinsic OAM. For the Laguerre-Gauss light beam ($m = 0$) formed by identical photons, the semiclassical approximation consists in $\boldsymbol{p}^2 \to p_z^2 + p_\perp^2$, $\boldsymbol{l} \to (\boldsymbol{r}\times\boldsymbol{e}_z)p_z + \boldsymbol{L}$. The z axis is longitudinal. Evidently, the paraxial photon should be modeled by the *massive* centroid with the *inertial* mass M defined by Eq. (21). Twisted and untwisted photons with the same energy have different momenta, velocities, and Lorentz factors, and can be distinguished. These conclusions remain valid for other structured photons (in particular, for light wave packets).

The nonzero mass as well as the subluminal velocity are extraordinary properties of the Laguerre-Gauss photon. The longitudinal beam shape depends on z/z_R, where $z_R = kw_0^2/2$ is the Rayleigh diffraction length. The last quantity, in particular, does not satisfy the Lorentz transformations for a segment length. Therefore, the independence of centroid parameters from z is necessary to use the model of the centroid.

Despite the paraxial approximation ($\langle p_\perp^2\rangle \ll p^2$), the Laguerre-Gauss photon mass is not very small. Under the experimental conditions used in Ref. [33], $Mc^2/E \approx 0.02$ when $\zeta = 100$ and $E = 1.56$ eV.

The presented consideration remains valid for the twisted electron. When the hidden transversal motion is taken into account, the electron velocity v satisfies Eq. (13). However, the twisted electron can also be regarded as a centroid with velocity v_z given by Eq. (19) and with mass equal to

$$M = \sqrt{m^2 + \langle p_\perp^2\rangle} = \sqrt{m^2 + \frac{2(2n+|l|+1)}{w_0^2}}. \tag{25}$$

Amazingly, the relation between the mass and velocity of the Laguerre-Gauss electron almost coincides with Eq. (22),

$$M = \frac{E}{c^2}\sqrt{1 - \frac{\langle v_z\rangle^2}{c^2}}. \tag{26}$$

The centroid momentum is equal to

$$\langle p_z\rangle = \sqrt{E^2 - m^2}\left[1 - \frac{2n+|l|+1}{(E^2-m^2)w_0^2}\right].$$

For the paraxial electron in noninertial frames, the only difference from the paraxial photon is the nonzero mass m. The *inertial* mass of the corresponding centroid is defined by $M = \sqrt{m^2 + p_\perp^2}$ and coincides with the kinematic mass (25).

A similar effect of an increase of the kinematic (Lorentz-invariant) mass of *3D-localized wave packets* of free twisted electrons as compared with m has been found in Ref. [29].

Importantly, the effective masses of the twisted paraxial photon and electron (i.e., the corresponding centroid masses) are quantized. The twisted wave packets also possess this property. The quantization of the mass and the group velocity can be discovered simultaneously in view of Eqs. (22) and (26).

We underline that all Laguerre-Gauss beams, even the mode $n = l = 0$, and all twisted and untwisted wave packets move slower than the plane wave and have mass $M > m$ ($v_g < c$ and $M > 0$ for light).

Since the Laguerre-Gauss electron is charged, it possesses a magnetic moment. Due to the connection between the FW operators of the OAM and the orbital magnetic moment, $\boldsymbol{\mu}_L = e\boldsymbol{L}/(2E)$ [81,82,96–98], the latter is not influenced by the radial quantum number. The total magnetic moment contains also a spin part (see Refs. [99–101]).

In this Rapid Communication, we have performed a general description of twisted paraxial photons and electrons in the framework of relativistic QM. The use of the FW representation has allowed us to find and investigate their properties, changing the usual perception of such particles. In this representation, twisted paraxial photons and electrons of arbitrary energies are characterized by the well-known wave function (2) and, therefore, a description of relativistic electrons is radically simplified. Moreover, the quantum-mechanical approach clarifies the fundamental properties of single photons and electrons. We have checked that twisted and other structured photons are luminal. Their subluminality is *apparent* and appears because the photon energy is contributed by the hidden motion. For Laguerre-Gauss light beams, this motion is transversal and the average transversal momentum vanishes. For light wave packets, the hidden motion occurs in three directions. We have presented quantum-mechanical and semiclassical descriptions of the structured photon. In QM, such a photon is a massless particle moving with velocity c. The semiclassical description applies expectation values of the momentum and velocity operators and disregards the hidden motion. As a result, Einstein quanta of structured light are subluminal and massive. In the semiclassical case, one should use the model of the centroid *with a nonzero kinematic (Lorentz-invariant) mass*. The analysis fulfilled unambiguously shows that the semiclassical description is self-consistent as well as the quantum-mechanical one. For Laguerre-Gauss light beams, the applicability of the model of a massive centroid is a nontrivial consequence of the independence of centroid parameters from the longitudinal coordinate. The properties of the Laguerre-Gauss electron are very similar. We predict the effect of a quantization of the velocity and mass of the structured photon and electron. This effect is observable for the photon. We have considered the twisted and untwisted semiclassical photons and electrons (modeled by the centroids) in the accelerated and rotating noninertial frame and have determined their *inertial* masses. Amazingly, the kinematic and inertial masses of these particles coincide. The orbital magnetic moment of the Laguerre-Gauss electron does not depend on the radial quantum number.

A deep similarity between the fundamental properties of the structured photon and electron illustrates the validity of the statement that the results for the photon can be well applied to all paraxial beams [35].

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