

**Time-reversal-invariant scaling of light propagation in one-dimensional non-Hermitian systems**

Jose D. H. Rivero\* and Li Ge

*Department of Physics and Astronomy, College of Staten Island, CUNY, Staten Island, New York 10314, USA  
and The Graduate Center, CUNY, New York, New York 10016, USA*

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Light propagation through a normal medium is determined not only by the real part of the refractive index but also by its imaginary part, which represents optical gain and loss. Therefore, two media with different gain and loss landscapes can have very different transmission and reflection spectra, even when their real parts of the refractive index are identical. Here we show that while this observation is true for an arbitrary one-dimensional medium with refractive index  $n(x)$  and its time-reversed partner with refractive index  $n^*(x)$ , there exists a universal scaling that gives identical transmittance and reflectance in these corresponding systems. Interestingly, the scaled transmittance and reflectance reduce to their standard, unscaled forms in a time-reversal-invariant system, i.e., one without gain or loss.

DOI: [10.1103/PhysRevA.100.023819](https://doi.org/10.1103/PhysRevA.100.023819)**I. INTRODUCTION**

Recent advances in nanofabrication and integration of photonic devices have a profound technological impact on computation, communication, and sensing [1–4]. They rely on the transport of information through optical structures, which has made the study of wave transport, localization, and resonances essential to engineer their properties on demand. The scattering matrix is one of the central objects in this field of study [5–11]. Besides its wide range of applicability in optics and photonics, the scattering matrix has been used oftentimes to understand resonances in nuclear and particle physics [12,13] and transport in condensed matter [14] and, in general, to probe states of open quantum systems [15].

The scattering matrix connects the incoming channels to the outgoing channels in a system, and in one-dimensional (1D) systems it consists of the transmission and reflection coefficients from both sides of the system. These quantities in a normal medium are determined not only by the real part of the refractive index but also by its imaginary part, which represents optical gain and loss. Such non-Hermitian systems [16] have attracted enormous interest in optics because of unique emerging phenomena such as spontaneous symmetry breaking [17], coherent perfect absorption [18–21], anisotropic transmission resonances [22–24], self-sustained radiation [25], and asymmetric power oscillations [26,27], among others.

In general, scattering information contained in the reflection and transmission coefficients at a given frequency is different for waves in a gain or a loss system. Therefore, two media with different gain and loss landscapes can have very different transmission and reflection spectra, even when the real parts of their refractive indices are identical. For example, consider a single slab of uniform refractive index. The transmission and reflection can become more than one

order of magnitude larger when we add gain to overcompensate the intrinsic material loss. Furthermore, the frequency dependence of the transmission and reflection spectra can also become very different once the loss or gain landscape is changed, especially in more complicated systems. As a result, the notions of invariant transmittance and reflectance in different non-Hermitian systems have not been discussed since the general consensus is that they do not exist.

Here we show that such invariant transmittance and reflectance do exist for two 1D non-Hermitian media that are time-reversal partners, i.e., one with refractive index  $n(x)$  and the other with its complex conjugation  $n^*(x)$ . This observation does not depend on the complexity of the refractive index, e.g., whether it is homogeneous or random or whether it has only gain or only loss or a mixture of them. Interestingly, the scaled reflectance and transmittance acquire a physical significance under photon flux conservation in time-reversal-invariant systems, where no gain or loss is present.

Our treatment of the scattering matrix is useful to establish generalized conservation laws and diverse identities for a wide range of optical systems and would serve to study reciprocity and reflectance ratios between different channels and to design gain profiles in lasers with suitable properties for the aforementioned applications. For example, we find that the flux conservation relation  $T + R = 1$  in time-reversal-invariant systems can be preserved in 1D non-Hermitian systems, with the physical transmittance  $T$  and reflectance  $R$  evolved continuously to what we refer to as pseudotransmittance and pseudoreflectance. We also show that the diverging behaviors of the transmittance and reflectance in a normal 1D laser are the same to the leading order using time-reversal-invariant scaling.

This paper is organized as follows: in Sec. II we review the scattering matrix formalism and establish the universal scaling of transmission and reflection that becomes identical for time-reversed partners. We also provide some important identities about the scaled transmittance and reflectance. In Sec. III we exemplify these results and show their connection

\*jhernandezrivero@gradcenter.cuny.edu

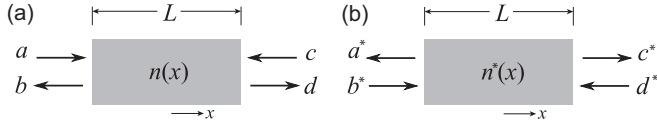


FIG. 1. Two-channel scattering of light from (a) a 1D optical system and (b) its time-reversed partner.

to the physical reflectance and transmittance, through some insightful discussions that range from the Hermitian limit to parity-time ( $\mathcal{PT}$ )-symmetric systems. We also relate our analysis to a special pair of time-reversed partners, i.e., a laser and a coherent perfect absorber (CPA). Finally, we provide some concluding remarks in Sec. IV, including its extension to higher dimensions.

## II. THE SCATTERING MATRIX

Consider the 1D optical system depicted in Fig. 1(a). We assume that electromagnetic waves propagate freely in space until they scatter off a cavity of finite length  $L$ . The refractive index landscape in the scattering region is described by the function  $n(x)$ . The transverse electric field  $E(x, t)$  in the scattering region satisfies the wave equation,

$$\left[ \partial_x^2 - \frac{n^2(x)}{c^2} \partial_t^2 \right] E(x, t) = 0, \quad (1)$$

where  $c$  is the speed of light in vacuum. Outside the scattering region the electric field takes the form

$$E(x, t) = \begin{cases} (ae^{ikx} + be^{-ikx})e^{-i\omega t}, & x < -L/2, \\ (ce^{-ikx} + de^{ikx})e^{-i\omega t}, & x > -L/2, \end{cases} \quad (2)$$

where  $\omega$  is the real-valued frequency and  $k = \omega/c$  is the wave vector in free space. The amplitudes  $a$ ,  $b$ ,  $c$ , and  $d$  in Eq. (2) are depicted in Fig. 1(a), and they are related through the scattering matrix  $S$  as

$$\begin{pmatrix} b \\ d \end{pmatrix} = \begin{bmatrix} r_L(\omega) & t(\omega) \\ t(\omega) & r_R(\omega) \end{bmatrix} \begin{pmatrix} a \\ c \end{pmatrix} \equiv S(\omega) \begin{pmatrix} a \\ c \end{pmatrix}. \quad (3)$$

Here  $r_L$  and  $r_R$  are the transmission coefficients from the left and right side, and  $t$  is the reciprocal transmission coefficient. As the wave equation, (1), is unchanged when the time  $t$  is replaced by  $-t$ , then

$$E(x, -t) = \begin{cases} (ae^{ikx} + be^{-ikx})e^{i\omega t}, & x < -L/2, \\ (ce^{-ikx} + de^{ikx})e^{i\omega t}, & x > -L/2, \end{cases} \quad (4)$$

is also a valid solution to the wave equation.

Next we take the complex conjugation of the wave equation, (1), which now describes the scattering of light from a medium with refractive index  $n^*(x)$ , with the loss and gain regions exchanged from the original non-Hermitian system. Therefore, these two systems are time-reversed partners, and the electric field given by

$$E^*(x, -t) = \begin{cases} (b^*e^{ikx} + a^*e^{-ikx})e^{-i\omega t}, & x < -L/2, \\ (d^*e^{-ikx} + c^*e^{ikx})e^{-i\omega t}, & x > -L/2, \end{cases} \quad (5)$$

indicates that the incoming amplitudes are now given by  $b^*$  and  $d^*$  and that they are scattered into outgoing amplitudes  $a^*$

and  $c^*$  [see Fig. 1(b)]. We denote the corresponding scattering matrix  $\tilde{S}$ ,

$$\begin{pmatrix} a^* \\ c^* \end{pmatrix} = \tilde{S}(\omega) \begin{pmatrix} b^* \\ d^* \end{pmatrix} = \begin{bmatrix} \tilde{r}_L(\omega) & \tilde{t}(\omega) \\ \tilde{t}(\omega) & \tilde{r}_R(\omega) \end{bmatrix} \begin{pmatrix} b^* \\ d^* \end{pmatrix} \quad (6)$$

or

$$\begin{pmatrix} a \\ c \end{pmatrix} = \tilde{S}^*(\omega) \begin{pmatrix} b \\ d \end{pmatrix}. \quad (7)$$

By multiplying  $\tilde{S}^*(\omega)$  to both sides of Eq. (3) from the left and simplifying the result using Eq. (7), we obtain

$$\tilde{S}^*(\omega)S(\omega) = \mathbf{1}, \quad (8)$$

which relates the scattering matrix  $S(\omega)$  of the original non-Hermitian system and the scattering matrix  $\tilde{S}(\omega)$  of its time-reversed partner. Here  $\mathbf{1}$  is the identity matrix.

When the determinant of  $S(\omega)$  is nonzero, i.e., away from a zero of the  $S$  matrix, we employ

$$S^{-1} = \frac{1}{r_L r_R - t^2} \begin{pmatrix} r_R & -t \\ -t & r_L \end{pmatrix} = \tilde{S}^* \quad (9)$$

to derive

$$\tilde{r}_{L,R}^* = \frac{r_{R,L}}{r_L r_R - t^2}, \quad \tilde{t}^* = \frac{-t}{r_L r_R - t^2}. \quad (10)$$

Furthermore, using the property that

$$\det(\tilde{S}^* S) = \det \tilde{S}^* \det S = 1 \quad (11)$$

or, more explicitly,

$$(r_L r_R - t^2)(\tilde{r}_L^* \tilde{r}_R^* - (\tilde{t}^*)^2) = 1, \quad (12)$$

we can rewrite Eq. (10) as

$$\frac{R_{R,L}}{|r_L r_R - t^2|} = \frac{\tilde{R}_{L,R}}{|\tilde{r}_L \tilde{r}_R - \tilde{t}^2|}, \quad (13)$$

$$\frac{T}{|r_L r_R - t^2|} = \frac{\tilde{T}}{|\tilde{r}_L \tilde{r}_R - \tilde{t}^2|}, \quad (14)$$

where  $T = |t|^2$ ,  $R_{L,R} = |r_{L,R}|^2$  and  $\tilde{T} = |\tilde{t}|^2$ ,  $\tilde{R}_{L,R} = |\tilde{r}_{L,R}|^2$  are the transmittance and reflectances in the two systems. Note that the subindices of the reflectances are switched in Eq. (13), e.g.,  $R_R$  is related to  $\tilde{R}_L$ . These two relations show that there exists a universal scaling of transmittance and reflectance that is invariant after time reversal, which utilizes the determinant of the respective scattering matrix.

This observation holds for 1D systems regardless of their symmetries and other properties of the refractive index. One representative example is shown in Fig. 2 for a heterostructure with randomly chosen refractive indices and its time-reversed partner. Not only does their transmittance (as well as reflectances) differ by as much as four orders of magnitude, but also the frequency dependence of these spectra shows distinct features. Nevertheless, once scaled by the universal factor given in Eqs. (13) and (14), their transmittances and corresponding reflectances become identical, respectively.

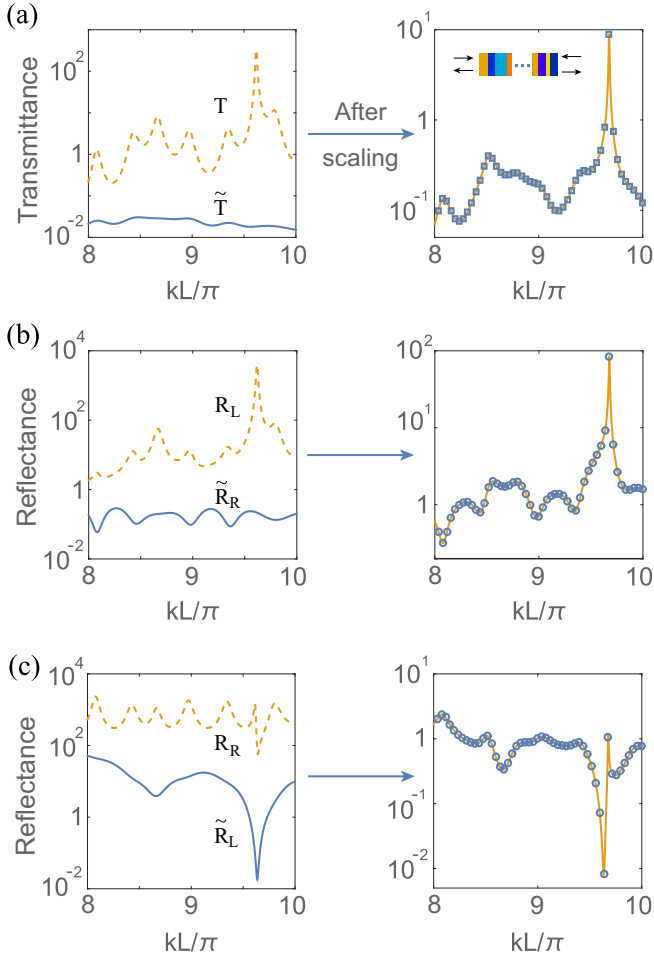


FIG. 2. Scattering from a 1D heterostructure and its time-reversed partner. There are 20 layers in each structure, and the refractive index is randomly set using  $\text{Re}[n] \in [1.5, 2.5]$  and  $\text{Im}[n] \in [-0.05, 0.05]$ . The layer thickness is also chosen randomly, with the widest twice as wide as the thinnest. (a) Transmittance  $T$  in this system and  $\tilde{T}$  in its time-reversed partner as a function of the wave number before (left) and after (right) the time-reversal-invariant scaling. The line and squares represent the scaled  $T$  and  $\tilde{T}$ , respectively. Inset: Schematic of the heterostructure. (b, c) The same for the reflectances. Lines and circles represent the scaled  $R_{L,R}$  and  $\tilde{R}_{L,R}$ , respectively.

The time-reversal-invariant scaling can also be derived using the  $R$ -matrix construction of the  $S$  matrix [19,28],

$$S = -e^{2in_0kx_0} [\mathbf{1} - in_0kR]^{-1} [\mathbf{1} + in_0kR], \quad (15)$$

where  $x_0 > L/2$  is an arbitrary boundary and  $n_0$  is the real-valued refractive index in free space. To obtain  $\tilde{S}$  for the time-reversed partner system, we use the same expression and utilize the fact that the  $R$  matrix becomes its complex conjugate when  $n(x)$  is replaced by  $n^*(x)$ . As a result, we find

$$\tilde{S}^* = -e^{-2in_0kx_0} [\mathbf{1} + in_0kR]^{-1} [\mathbf{1} - in_0kR], \quad (16)$$

which leads to Eq. (8) and in turn Eqs. (13) and (14).

As we work through some enlightening examples we unravel the physical significance of these and other scaled reflectances and transmittances in systems obeying particular symmetries, as described in the following sections.

### III. EXAMPLES

#### A. Systems with time-reversal symmetry

We start our exemplification by considering a dielectric slab with an arbitrary index profile  $n(x)$ , where the only constraint is that  $n(x)$  is real, i.e., the slab is dissipationless and gainless. Then the slab is its own time-reversed partner, and in this case it is interesting to note that the scaled transmittance and reflectance in Eqs. (13) and (14) reduce to their standard and unscaled forms. This is because now  $\tilde{S}$  is the same as  $S$ , leading to  $|\det S| = |r_L r_R - t^2| = 1$  in Eq. (12), i.e., the time-reversal-invariant scaling trivially becomes 1 in this case.

However, we should note that

$$\frac{T}{|r_L r_R - t^2|} + \frac{R_{R,L}}{|r_L r_R - t^2|} \neq 1 \quad (17)$$

in general, and hence one may wonder what the general “conservation law” is that reduces to  $T + R = 1$  in the time-reversal-invariant case, where  $R_L = R_R \equiv R$ . It turns out that such a conservation law is simply given by

$$\mathfrak{T} + \mathfrak{R} = 1, \quad (18)$$

where we have defined

$$\mathfrak{R} \equiv \frac{r_L r_R}{r_L r_R - t^2} \quad \text{and} \quad \mathfrak{T} \equiv \frac{-t^2}{r_L r_R - t^2} \quad (19)$$

as the pseudorefectance and pseudotransmittance.

$\mathfrak{R}$  and  $\mathfrak{T}$  are complex in general, but they also become the physical reflectance and transmittance in the time-reversal-symmetric case. To show explicitly that  $\mathfrak{R}$  and  $\mathfrak{T}$  are real in this case, we note that Eq. (10) now becomes

$$r_{L,R}^* = \frac{r_{R,L}}{r_L r_R - t^2}, \quad t^* = \frac{-t}{r_L r_R - t^2}. \quad (20)$$

Therefore, we find

$$|r_L|^2 = |r_R|^2 = \frac{r_L r_R}{r_L r_R - t^2} = \mathfrak{R}, \quad (21)$$

$$|t|^2 = \frac{-t^2}{r_L r_R - t^2} = \mathfrak{T}, \quad (22)$$

and the trivial conservation law given by Eq. (18) now becomes the actual flux conservation relation. The latter, of course, can also be obtained from the unitarity property of the scattering matrix, i.e.,  $S^\dagger S = 1$ , which can be derived using Eq. (8) with the assumed reciprocity ( $S^T = S$ ) in this case.

#### B. A dielectric slab with gain or loss

One of the simplest systems that exhibit nontrivial scattering features is a 1D slab which is capable of amplifying or absorbing radiation. The former is a typical model used to study solid-state laser cavities, and the consideration of its time-reversed partners has led to the discovery of coherent perfect absorbers [18–20].

However, such a consideration has only been explored for the extremes of the scattering matrix, namely, its poles and zeros. A pole (zero) of a scattering matrix is defined as the usually complex frequency where one or more eigenvalues of the scattering matrix become infinite (zero). Therefore, at a pole of a scattering matrix an infinitesimal input (e.g., noises as input  $a$  and  $c$  in Fig. 1) can lead to a finite scattering

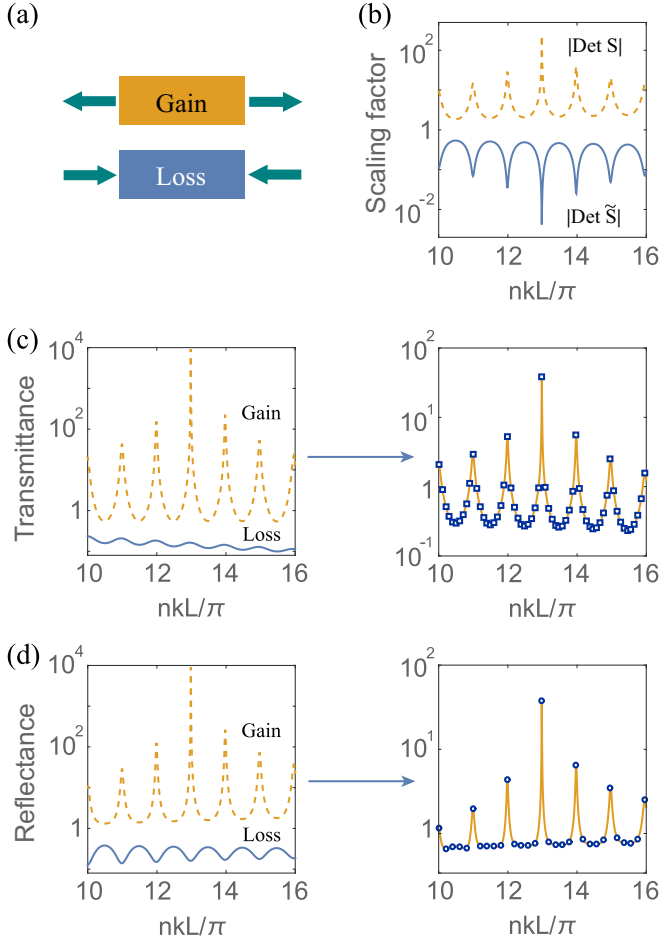


FIG. 3. Scattering from a pair of uniform dielectric slabs with gain and loss, respectively.  $n = 3 \pm 0.05i$ . (a) Schematics showing their respective lasing and CPA state. (b) Scaling factors as a function of the wave number. (c) Transmittance before (left) and after (right) the time-reversal-invariant scaling. (d) The same for the reflectance. Symbols are the same as in Fig. 2.

or output light intensity, which is one mathematical model used to describe a laser with a gain medium when the pole occurs on the real frequency axis. Clearly, a “time-reversed laser” then corresponds to a zero of the scattering matrix, where the incoming light satisfying certain coherent phase and amplitude configuration is absorbed perfectly by the time-reversed system with loss, after which the CPA is named.

Such a correspondence is a special case of Eq. (11): a pole (zero) implies that the determinant of the scattering matrix is infinite (zero). And the time-reversal relation between a laser and a CPA is described asymptotically by Eq. (11):  $\det S \rightarrow \infty$  in a laser cavity with gain, while  $\det \tilde{S} \rightarrow 0$  in the corresponding CPA with loss [see Fig. 3(b)].

It is well known that  $\det S = r_L r_R - t^2 \rightarrow \infty$  implies that  $r_L$ ,  $r_R$ , and  $t$  all diverge at the frequency of a lasing mode. However, one usually cannot verify analytically whether these quantities approach infinity at the same speed without knowing their explicit expressions. Together with Eq. (12), the scaling relation, (10), offers us a unique opportunity to overcome this difficulty. Specifically, they lead to  $\tilde{\mathfrak{R}}^* = \mathfrak{R}$  for the pseudorefectance in a pair of time-reversed partner

systems or, more explicitly,

$$\left[ \frac{\tilde{r}_L \tilde{r}_R}{\tilde{r}_L \tilde{r}_R - \tilde{t}^2} \right]^* = \frac{r_L r_R}{r_L r_R - t^2}. \quad (23)$$

As  $\det S = r_L r_R - t^2$  approaches infinity in a laser cavity,  $\det \tilde{S} = \tilde{r}_L \tilde{r}_R - \tilde{t}^2$  vanishes in the corresponding CPA as we have mentioned, and  $\tilde{r}_L \tilde{r}_R$  remains finite in this process [see Fig. 3(d), for example]. As a result, the left-hand side of Eq. (23) diverges, and so does its right-hand side. Because the denominator of the latter (i.e.,  $\det S$ ) also diverges, it indicates that  $r_L r_R$  approaches infinity faster than  $r_L r_R - t^2$ . In other words, the leading-order asymptotics of  $r_L r_R$  and  $t^2$  are the same. In a system with reflection symmetry such as the one shown in Fig. 3,  $r_L = r_R$  and hence they approach infinity at the same speed as  $t$  [see the highest peak around  $10^4$  at  $nkL/\pi \approx 13$  in Figs. 3(c) and 3(d)].

Similarly to what is shown in Fig. 2, here the physical reflectance and transmittance differ significantly in the gain and loss slabs [see the left panels in Figs. 3(c) and 3(d)]. After applying the scaling factor specified in Eqs. (13) and (14) and shown in Fig. 3(b), we again verify the time-reversal-invariant scaling of the reflectance and transmittance [see the right panels in Figs. 3(c) and 3(d)].

### C. $\mathcal{PT}$ -symmetric systems

If we compare three systems—the original non-Hermitian one (“A”) with refractive index  $n(x)$ , its time-reversed partner (“B”) with index  $n^*(x)$ , and its  $\mathcal{PT}$ -symmetric partner (“C”) with index  $n^*(-x)$ —the latter two are just mirror images of each other. Therefore, we can easily rephrase the time-reversal-invariant scaling relations (13) and (14) as  $\mathcal{PT}$ -invariant scaling relations. For this purpose, we denote the scattering matrix of system C by  $\tilde{S}$ , which satisfies the following relation:

$$\sigma_x \tilde{S}^*(\omega) \sigma_x S(\omega) = \mathbf{1}. \quad (24)$$

Here  $\omega$  is again real-valued and  $\sigma_x$  is the first Pauli matrix. The counterpart to Eq. (10) is then

$$\tilde{r}_{L,R}^* = \frac{r_{L,R}}{r_L r_R - t^2}, \quad \tilde{t}^* = \frac{-t}{r_L r_R - t^2}. \quad (25)$$

In other words, the additional parity operator simply leaves the reciprocal transmission unchanged (i.e.,  $\tilde{t} = \tilde{t}$ ) and exchanges the left and right reflection coefficients (i.e.,  $\tilde{r}_{L,R} = \tilde{r}_{R,L}$ ) when we compare systems B and C. The resulting  $\mathcal{PT}$ -invariant scaling is then given by

$$\frac{R_{L,R}}{|r_L r_R - t^2|} = \frac{\tilde{R}_{L,R}}{|\tilde{r}_L \tilde{r}_R - \tilde{t}^2|}, \quad (26)$$

$$\frac{T}{|r_L r_R - t^2|} = \frac{\tilde{T}}{|\tilde{r}_L \tilde{r}_R - \tilde{t}^2|}. \quad (27)$$

If a non-Hermitian system is  $\mathcal{PT}$  symmetric, we then find  $|\det S(\omega)| = 1$  from Eq. (24) using  $\tilde{S}(\omega) = S(\omega)$  and  $(\det \sigma_x)^2 = 1$ . Therefore, the scaling factors in Eqs. (26) and (27) also vanish, similarly to the case in Sec. III A where the system is time reversal invariant. In addition, we note that the generalized but trivial conservation law  $\mathfrak{T} + \mathfrak{R} = 1$  for the pseudotransmittance  $\mathfrak{T}$  and pseudorefectance  $\mathfrak{R}$  derived in that section applies to any 1D non-Hermitian system.

Therefore, it holds in a  $\mathcal{PT}$ -symmetric system as well but differs from the nontrivial conservation law given in Ref. [22]. To derive the latter, we can write the physical transmittance and reflectances as

$$R_{L,R} = \frac{r_{L,R}^2}{r_L r_R - t^2} \in \mathbb{R}, \quad T = \frac{-t^2}{r_L r_R - t^2} \in \mathbb{R} \quad (28)$$

using Eq. (25), which then leads to

$$|1 - T| = \sqrt{R_L R_R}. \quad (29)$$

#### IV. DISCUSSION AND CONCLUSIONS

In summary, we have shown that there exists a time-reversal-invariant scaling for wave propagation in 1D non-Hermitian systems. It applies to both transmission and reflection, no matter how different the spectra of these quantities are in this system and its time-reversed partner.

Although we have restricted our discussion to 1D systems so far, some of our observations can be easily extended to higher dimensions. For example, for a quasi-1D waveguide with multiple transverse channels [29], relation (8) still applies as long as the incoming and outgoing channels with the same index are related by time reversal. In this case the scattering matrices can be written in their block forms, i.e.,

$$S(\omega) = \begin{pmatrix} r_L & t \\ t & r_R \end{pmatrix}, \quad \tilde{S}(\omega) = \begin{pmatrix} \tilde{r}_L & \tilde{t} \\ \tilde{t} & \tilde{r}_R \end{pmatrix}, \quad (30)$$

where  $r_{L,R}$ ,  $t$  and  $\tilde{r}_{L,R}$ ,  $\tilde{t}$  become matrices themselves. We then find the identity

$$\underline{T} + \underline{R} = N, \quad (31)$$

where  $\underline{T} \equiv \tilde{t}^* t$ ,  $\underline{R} \equiv \text{Tr}(\tilde{r}_L^* r_L) = \text{Tr}(\tilde{r}_R^* r_R)$ , and  $N$  is the number of incoming (and outgoing) channels.

Similarly, the identity, (8), also holds for the scattering of cylindrical waves in two dimensions. Following the convention used in Ref. [11], we define the  $m$ th incoming and outgoing channels by

$$\Psi_m^-(r, \theta) = \frac{H_m^-(kr)}{H_m^-(kR)} e^{im\theta}, \quad \Psi_m^+(r, \theta) = \frac{H_m^+(kr)}{H_m^+(kR)} e^{im\theta}.$$

Here  $m \in \mathbb{Z}$  is the angular momentum number, and  $m > 0$  ( $m < 0$ ) describes counterclockwise (clockwise) waves.  $r$  and  $\theta$  are the radial position and the azimuthal angle,  $R$  is the radius of the scattering region, and  $H_m^\pm$  are the Bessel functions of the first and second kind. In the absence of scattering, the scattering matrix becomes an antidiagonal matrix, similarly to the 1D case defined in Eq. (3).

Although the time-invariant scaling of the scattering coefficients does not exist in higher dimensions in general, it reemerges in some special cases. Take the scattering of the cylindrical waves, for example. If the index modulation itself has an angular momentum  $M = 2$  (e.g., a quadruple cavity with a uniform refractive index [30,31]), then a cylindrical wave of angular momentum  $m$  will be scattered strongly into the  $m \pm 2$  channels in general [32]. More specifically, the  $m = \pm 1$  channels are scattered strongly into each other and the  $m' = \pm 3$  channels. If the index modulation is weak, then the scattering intensities into the  $m' = \pm 3$  channels are

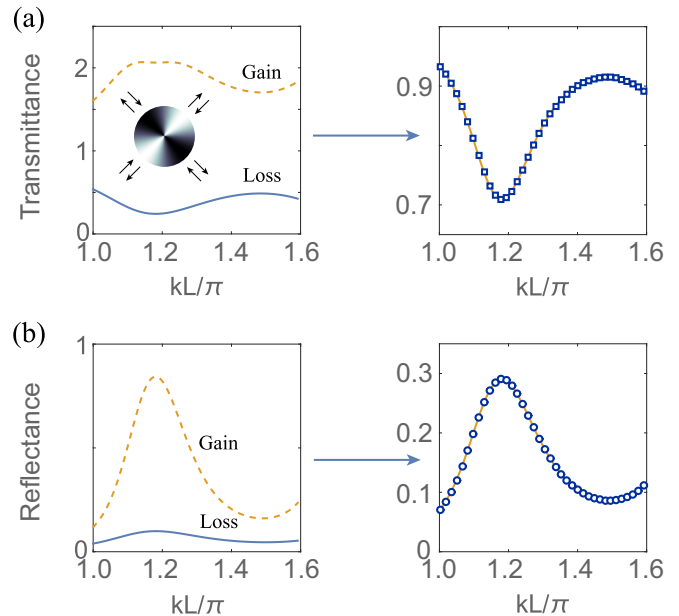


FIG. 4. Time-reversal-invariant scaling of a reduced scattering matrix in 2D disk geometry. (a) Transmittance as a function of the wave number before (left) and after (right) the time-reversal-invariant scaling. Inset: Schematic of the disk scatterer. (b) The same for the reflectance. Symbols are the same as in Fig. 2.

negligible due to a low spectral overlap factor [33]. Consequently, the  $m = \pm 1$  channels form a largely closed subspace of the scattering matrix, and the reduced-dimension scattering matrix  $S$  in these channels behave in the same way as in the 1D case.

Assuming reciprocity, we denote the scattering amplitudes between the  $m = \pm 1$  channels the “transmission coefficient”  $t$  and those back into the outgoing channels of the same indices the reflection coefficients  $r_{L,R}$ . The same time-reversal-invariant scaling shown in Eqs. (13) and (14) still holds, and we exemplify this result using two disks of radius  $R$  and refractive index  $n(r, \theta) = (1.5 + 0.1 \sin 2\theta) \pm 0.05i$ . To verify that the reduction of the scattering matrix into the  $2 \times 2$  form is a good approximation, we first mention that the ratio of the scattering intensities into the  $m' = 3$  and  $m' = 1$  ( $-1$ ) channels from the  $m = 1$  channel is  $1.3 \times 10^{-2}$  ( $6.1 \times 10^{-4}$ ) at  $kR = 3$ , which is typical for the range of wave numbers shown in Fig. 4. The reflectances  $R_{R,L}$  in this case are identical, because the  $m = \pm 1$  channels are exchanged when we simply change our perspective from the top view to the bottom view of the two-dimensional plane. In other words, the chirality of the channels is flipped when  $\theta \rightarrow -\theta$ . The transmittances  $T$  and  $\tilde{T}$  shown in Fig. 4 do not exhibit similarities in particular, but they become identical after applying the time-reversal-invariant scaling specified in Eq. (14).

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