Zero-error quantum hypothesis testing in finite time with quantum error correction

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Under the unitary evolutions it is always possible to distinguish different Hamiltonians with zero-error probability within a finite time. At the presence of noises, however, the error probability does not always go to zero within a finite time. In this article, we give a necessary and sufficient condition for Markovian dynamics to achieve zero-error quantum hypothesis testing in a finite time. We show that when the condition fails, the maximal fidelity of the output states under two different Hamiltonians, aided with arbitrary control operations, is always lower bounded by an exponential function, which remains positive at any finite time; zero-error quantum hypothesis testing thus cannot be achieved within any finite time. However, when the condition holds, quantum error corrections can be used to correct the noises and partially maintain the coherent evolutions, zero-error quantum hypothesis testing can then be achieved within a finite time.

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I. INTRODUCTION

Hypothesis testing, which studies the distinguishability between a set of models, plays a central role in science and technology. A common scenario of hypothesis testing is to differentiate between the null hypothesis, h_0, and the alternative hypothesis, h_1. In the classical setting, h_0 and h_1 are typically represented by two distributions, and one needs to determine the hypothesis by observing the data generated from the distribution. The error probability of the hypothesis testing typically decreases with the number of data. In particular, when the data are generated independently, the error probability decreases exponentially with the number of data. This is captured by the Chernoff bound, the Stein bound, the Hoeffding bound, and various other bounds in classical hypothesis testing [1–3] and is widely used in classical estimation and communications. Specifically with a finite number of data, zero-error hypothesis testing is only possible between orthogonal distributions.

This has been extended to the distinguishing of quantum states, pioneered by Helstrom, Holevo, and Yuen *et al.* [4–6]. The quantum versions of the Stein, Chernoff, and Hoeffding bounds on the error probability of hypothesis testing of the quantum states have been obtained [7–13]. Specifically with n identical copies of the states, $\rho_1^{\otimes n}$ and $\rho_2^{\otimes n}$, the error probability of distinguishing between ρ_1 and ρ_2 decreases exponentially with n in the asymptotical limit. Similarly for a finite n, zero-error hypothesis testing is only possible when ρ_1 and ρ_2 are orthogonal.

A much more complicated case of the hypothesis testing is to distinguish quantum dynamics. This includes the distinguishability of quantum states as a special case when one fixes the initial state and the evolution time. However, many different strategies, such as quantum comb and continuous measurement [14,15], can be employed to improve the successful probability of the hypothesis testing. In general, arbitrary operations are allowed during the evolution to improve the successful probability. These additional degrees of freedom lead to many different features in the hypothesis testing of quantum dynamics. For example, it is possible to perfectly distinguish between two nonorthogonal quantum channels with a finite number of queries [16–26].

The hypothesis testing of quantum dynamics can be divided according to discrete and continuous evolutions. For the distinguishability of discrete quantum channels, it is known that unitary channels can always be perfectly distinguished with a finite number of queries [16–19]. The condition for perfect distinguishability of two general quantum channels with a finite number of queries has also been obtained [21]. The optimal strategy, however, is computationally hard. Chernoff, Stein, and Hoeffding types of bounds for quantum dynamics that cannot be perfectly distinguished within a finite number of queries remain largely unknown. By using identical probe states repeatedly, one can obtain some upper bounds on the error probability of distinguishing quantum channels, which has an exponential form in the asymptotical limit. A general approach would be trying to get lower bounds, which also decrease exponentially in the asymptotical limit [27], and then reducing the gap between the exponents in the upper and lower bounds to get a tight bound.

In many practical applications we are interested in the distinguishability of the Hamiltonians that govern the continuous evolutions. A typical situation is to distinguish between two Hamiltonians, H_0 and H_1, which represent the null hypothesis and the alternative hypothesis. This is a fundamental problem and has many applications, for example, the detection of a field can be modeled as the hypothesis testing between H_0 and H_1, where H_0 is the Hamiltonian without the field and H_1 is the Hamiltonian with the field. Under unitary evolution, the optimal strategy has been obtained and zero-error hypothesis testing can always be achieved within a finite time [28,29]. In the presence of noises, however, little is known. In this article, we study the distinguishability of the Hamiltonians in the presence of Markovian noises. We first review previous results on the distinguishability of the Hamiltonians under unitary

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dynamics, and then we provide a necessary and sufficient condition on the Markovian noisy dynamics under which zero-error quantum hypothesis testing can be achieved within a finite time. Detailed proofs are then presented.

II. QUANTUM HYPOTHESIS UNDER CONTINUOUS UNITARY DYNAMICS

When there are no noises, the dynamics governed by H_0 and H_1 can be described by $\frac{d\rho}{dt} = -i[H_p, \rho]$, $p = 0$ and 1 (where we have taken $\hbar = 1$). Besides the freedom of choosing the initial probe state, the evolution time, and the measurement, one can also add controls during the evolution such as

$$\frac{d\rho}{dt} = -i[H_p + H_c, \rho], \quad p = 0 \text{ and } 1, \tag{1}$$

where H_c is the added control Hamiltonian. Ancilla systems and measurements during the evolution can also be used. Under the unitary dynamics, perfect distinguishability of the Hamiltonians can always be achieved within a finite time [28,29]. Specifically the optimal strategy is to choose the control as $H_c = -H_0$ (or $H_c = -H_1$) and prepare the initial probe state as $\frac{|\lambda_{\max}\rangle + |\lambda_{\min}\rangle}{\sqrt{2}}$, where $|\lambda_{\max/\min}\rangle$ is the eigenstate of $H_1 - H_0$ corresponding to the maximal/minimal eigenvalue. Under this optimal strategy, the fidelity between the states changes as $|\cos\frac{\lambda_{\max}-\lambda_{\min}}{2}t|$, which reaches zero at the time $t_{\min} = \frac{\pi}{\lambda_{\max}-\lambda_{\min}}$. Thus under the unitary dynamics, zero-error quantum hypothesis testing can always be achieved within a finite time.

III. CONDITION ON ZERO-ERROR QUANTUM HYPOTHESIS TESTING IN A FINITE TIME

In practice, noises are unavoidable. At the presence of noises, zero-error hypothesis testing of quantum dynamics is not always achievable within any finite time. For example, if the two dynamics are given by $\frac{d\rho}{dt} = -i[H_p, \rho] + \gamma(\sigma_z\rho\sigma_z - \rho)$, with $H_0 = \sigma_z$ and $H_1 = 2\sigma_z$, respectively, here σ_x, σ_y, and σ_z are Pauli matrices. Then with the control $H_c = -\sigma_z$ and the initial state $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$, the fidelity between the quantum states changes as $e^{-4\gamma t}\frac{2e^{4\gamma t}+\cos 2t-1}{2} \geqslant ce^{-4\gamma t}$, where $c = \min_{t\geqslant 0}\frac{2e^{4\gamma t}+\cos 2t-1}{2} > 0$. It is lower bounded by an exponential function which is always positive at any finite time; the states thus cannot become orthogonal within any finite time. This holds even if arbitrary operations are allowed during the evolution. Intuitively, one can understand this by taking the dephasing as the fluctuations of the fields with a white noise spectrum. Due to this fluctuation the fields can overlap with nonzero probabilities even if they have different mean values; this then makes the perfect distinguishability of the two fields impossible within any finite time.

We extend this intuition to the quantum hypothesis testing of the Hamiltonians under general Markovian dynamics described by the master equations as

$$\frac{d\rho}{dt} = -i[H_p, \rho] + \sum_{k=1}^{m}\left(L_k\rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}\right),$$
$$p = 0 \text{ and } 1. \tag{2}$$

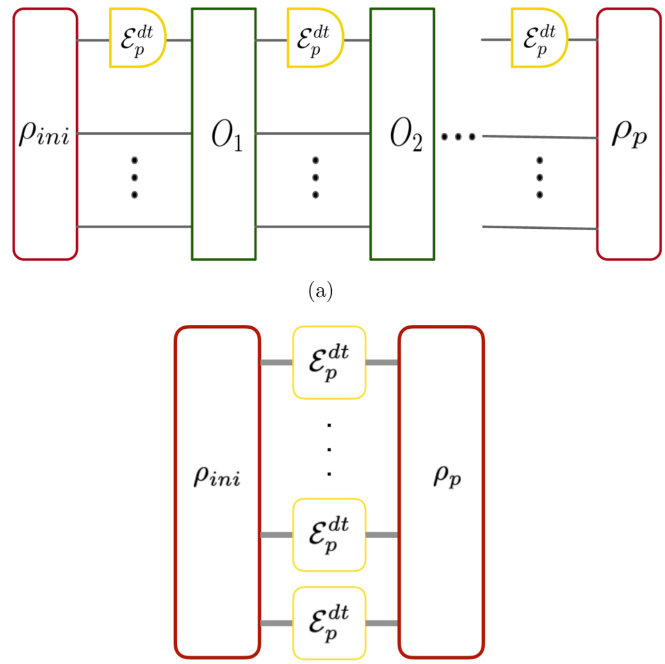

FIG. 1. (a) Sequential scheme. General scheme for quantum hypothesis testing with ancillary systems and arbitrary operations; here O_i is any physical operation. When O_i is taken as the SWAP gate between the system and the ith ancilla, the scheme reduces to the (b) parallel scheme where a large entangled state can be prepared with each part going through the evolution independently.

And arbitrary control operations can be added during the evolution. We provide a necessary and sufficient condition under which the zero-error quantum hypothesis testing can be achieved within a finite time. In particular, we show that when $H_1 - H_0 \in S = \text{span}\{I, L_k, L_k^\dagger, L_j^\dagger L_k | j, k = 1, \ldots, m\}$ the states under the two hypotheses cannot become orthogonal within any finite time. More specifically we show the maximal fidelity of the states is always lower bounded by an exponential function which remains positive at any finite time. This marks a step forward to form the quantum Chernoff type of bounds on continuous quantum evolutions. On the other hand if $H_1 - H_0 \notin S$, zero-error hypothesis testing can be achieved within a finite time, and it turns out that $H_1 - H_0 \notin S$ is exactly the condition for the quantum error correction [30–37]. A quantum error correction code can thus be constructed to eliminate the noise while (partially) maintaining the coherent evolutions. The evolutions are then essentially reduced to the unitary evolutions and the maximal fidelity of the states changes as a trigonometry function which can reach zero at a finite time.

A. Necessary

We first show that when $H_1 - H_0 \in S$ zero-error hypothesis testing cannot be achieved within any finite time. We divide the whole evolution period into small time intervals, each with a period of dt, and allow the use of ancillas and arbitrary operations between the intervals, as shown in Fig. 1. With $dt \to 0$, this can approximate any strategies to arbitrary precision. We then show that under the optimal strategy for the perfect discrimination of discrete quantum channels with arbitrary dt, the maximal fidelity of the states is always lower bounded by an exponential function which remains positive at any finite time. We note that a finite time is different from a finite number of queries. By letting $dt \to 0$, an infinite number of queries can be employed in a finite time. Without loss of generality the optimal strategy for quantum hypothesis testing in a finite time can always be assumed to contain an infinite number of queries, i.e., assume $dt \to 0$, as one can always choose to do nothing between some time intervals to simulate the strategies with only a finite number of queries.

We use the maximal fidelity of the quantum states to quantify the orthogonality between the states [21]. The maximal fidelity between two quantum states, ρ_0 and ρ_1, is defined as

$$\tilde{F}(\rho_0, \rho_1) = \max\{|\langle\psi|\phi\rangle|, |\psi\rangle \in \text{supp}(\rho_0), |\phi\rangle \in \text{supp}(\rho_1)\}, \tag{3}$$

where supp(ρ) denotes the support of ρ, which is the subspace spanned by the eigenvectors of ρ with nonzero eigenvalues. A few important properties of the maximal fidelity will be used. First, $\tilde{F}(\rho_0, \rho_1) \geqslant F(\rho_0, \rho_1)$ where $F(\rho_0, \rho_1) = Tr\sqrt{\rho_0^{\frac{1}{2}}\rho_1\rho_0^{\frac{1}{2}}}$ is the fidelity, and if ρ_0 and ρ_1 are pure states then $\tilde{F}(\rho_0, \rho_1) = F(\rho_0, \rho_1)$. Second, $F(\rho_0, \rho_1) = 0$ if and only if $\tilde{F}(\rho_0, \rho_1) = 0$; i.e., ρ_0 and ρ_1 are orthogonal if and only if $\tilde{F}(\rho_0, \rho_1) = 0$. Third, a necessary and sufficient condition for the existence of a physical operation that can deterministically transfer ρ_0 to a pure state $|\psi_0\rangle$ and ρ_1 to another pure state $|\psi_1\rangle$ is $\tilde{F}(\rho_0, \rho_1) \leqslant |\langle\psi_0|\psi_1\rangle|$ [21,38].

To achieve the zero-error hypothesis testing, the maximal fidelity needs to reach zero. The optimal procedure is then to decrease the maximal fidelity as fast as possible [21]. Suppose after the kth time interval, the two output states under the optimal strategy are $\rho_0(kdt)$ and $\rho_1(kdt)$. We denote the maximal fidelity of the two states as $q(kdt) = \tilde{F}[\rho_0(kdt), \rho_1(kdt)]$. The optimal strategy is to transform $\rho_0(kdt)$ and $\rho_1(kdt)$ to the two states $|\psi\rangle$ and $|\phi\rangle$ with $\langle\psi|\phi\rangle = q(kdt)$ and proceed with the evolution for the next dt [21]. These two states should be chosen optimally so the maximal fidelity at $(k+1)dt$ is minimized, i.e.,

$$q[(k+1)dt] = \min_{\langle\psi|\phi\rangle = q(kdt)} \tilde{F}\left[\mathcal{E}_0^{dt}(|\psi\rangle\langle\psi|), \mathcal{E}_1^{dt}(|\phi\rangle\langle\phi|)\right], \tag{4}$$

where $\mathcal{E}_i^{dt}$ is the dynamics with the ith Hamiltonian which evolved for the period of dt,

$$\mathcal{E}_0^{dt}(|\psi\rangle\langle\psi|) = \sum_{k=0}^{m} E_{0k}|\psi\rangle\langle\psi|E_{0k}^{\dagger},$$
$$\mathcal{E}_1^{dt}(|\phi\rangle\langle\phi|) = \sum_{k=0}^{m} E_{1k}|\phi\rangle\langle\phi|E_{1k}^{\dagger}. \tag{5}$$

Here $E_{00} = I - iH_0dt - \frac{1}{2}\sum_{k=1}^{m} L_k^{\dagger}L_k dt$, $E_{10} = I - iH_1dt - \frac{1}{2}\sum_{k=1}^{m} L_k^{\dagger}L_k dt$, and $E_{0k} = E_{1k} = L_k\sqrt{dt}$ $(k = 1, \ldots, m)$. We note that additional ancillary systems are allowed, in which case $|\psi\rangle$ and $|\phi\rangle$ can be entangled states of the system + ancilla, and the evolution $\mathcal{E}_i$ is understood as $\mathcal{E}_i \otimes I$ with $\mathcal{E}_i$ acting on the system and the identity operator acting on the ancilla. The optimal strategy typically requires the use of the ancilla, but the derivation here works for both cases.

In principle one can follow the procedure and obtain $q(t)$. However, in general it is not possible to solve $q(t)$ analytically. And due to the numerical precision, numerical simulations typically cannot be used to decide whether $q(t)$ has reached zero (numerical simulation also can only simulate the evolution for a finite time). Instead we derive an analytical lower bound of $q(t)$, which is our main contribution. In particular, we show that when $H_1 - H_0 \in S$, $q(t)$ is lower bounded by an exponential function which is always positive at any finite time.

We first note that given a pure state $|\psi\rangle$, under a quantum channel $\mathcal{E}(|\psi\rangle\langle\psi|) = \sum_k E_k|\psi\rangle\langle\psi|E_k^{\dagger}$, any pure state in the support of $\mathcal{E}(|\psi\rangle\langle\psi|)$ can be written as $\sum_k \alpha_k E_k|\psi\rangle$; i.e., the support of $\mathcal{E}(|\psi\rangle\langle\psi|)$ is the space spanned by $\{E_k|\psi\rangle\}$. This can be seen by choosing an equivalent Kraus operator so that $E_k|\psi\rangle$ and $E_j|\psi\rangle$ are orthogonal to each other when $k \neq j$. If they are not orthogonal initially, we can find an equivalent set of the Kraus operators by diagonalizing the Hermitian matrix A whose kjth entry is $A_{kj} = \langle\psi|E_k^{\dagger}E_j|\psi\rangle$. Let $\Lambda = U^{\dagger}AU$, with Λ being a diagonal matrix and U being a unitary matrix, and define the equivalent Kraus operators as $\tilde{E}_k = \sum_i u_{ik}E_i$. Then $\langle\psi|\tilde{E}_k^{\dagger}\tilde{E}_j|\psi\rangle = \sum_{i,p} u_{ik}^* A_{ip} u_{pj} = \Lambda_{kk}\delta_{k,j}$, where $\delta_{k,j} = 1$ when $k = j$ and $\delta_{k,j} = 0$ when $k \neq j$. Thus, $\mathcal{E}(|\psi\rangle\langle\psi|) = \sum_k \tilde{E}_k|\psi\rangle\langle\psi|\tilde{E}_k^{\dagger}$ are decompositions with orthogonal vectors $\tilde{E}_k|\psi\rangle$, which can be viewed as eigenvalue decompositions with the nonzero $\tilde{E}_k|\psi\rangle$ as the eigenvectors (some $\tilde{E}_k|\psi\rangle$ can be zero vectors which then do not contribute to the decomposition). Any pure state in the support can then be written as $\sum_k \tilde{\alpha}_k\tilde{E}_k|\psi\rangle = \sum_{k,i}\tilde{\alpha}_k u_{ik}E_i|\psi\rangle = \sum_i(\sum_k \tilde{\alpha}_k u_{ik})E_i|\psi\rangle = \sum_i \alpha_i E_i|\psi\rangle$, where $\alpha_i = \sum_k \tilde{\alpha}_k u_{ik}$. The support of $\mathcal{E}(|\psi\rangle\langle\psi|)$ is thus spanned by $\{E_k|\psi\rangle\}$ (with ancillas this is understood as $\{E_k \otimes I|\psi\rangle\}$, with $|\psi\rangle$ being a state of system + ancilla).

A pure state in the support of $\mathcal{E}_0^{dt}(|\psi\rangle\langle\psi|)$ can then be written as

$$|\psi(dt)\rangle = \frac{1}{N_0}\sum_{k=0}^{m} a_k E_{0k}|\psi\rangle, \tag{6}$$

where N_0 is the normalization factor. Similarly we can write any pure state in the support of $\mathcal{E}_1^{dt}(|\phi\rangle\langle\phi|)$ as

$$|\phi(dt)\rangle = \frac{1}{N_1}\sum_{k=0}^{m} b_k E_{1k}|\phi\rangle. \tag{7}$$

Assume $|\psi\rangle$ and $|\phi\rangle$ have been chosen optimally under the constraint $\langle\psi|\phi\rangle = q(t)$ to achieve the $\min \tilde{F}[\mathcal{E}_1^{dt}(|\psi\rangle\langle\psi|), \mathcal{E}_2^{dt}(|\phi\rangle\langle\phi|)]$, then up to the order of dt we have

$$\begin{aligned} q(t+dt) &= \tilde{F}\left[\mathcal{E}_1^{dt}(|\psi\rangle\langle\psi|), \mathcal{E}_2^{dt}(|\phi\rangle\langle\phi|)\right] \\ &= \max_{\{|\psi(dt)\rangle, |\phi(dt)\rangle\}} |\langle\psi(dt)|\phi(dt)\rangle| \\ &\geqslant \frac{1}{N_0N_1}|\langle\psi|a_0^*b_0\left[I - i(H_1 - H_0)dt - \sum_{k=1}^{m} L_k^{\dagger}L_k dt\right] \end{aligned}$$

$$+ \sum_{k=1}^{m} (a_k^* b_0 \sqrt{dt} L_k^\dagger + a_0^* b_k \sqrt{dt} L_k)$$
$$+ \sum_{k,j=1}^{m} a_k^* b_j L_k^\dagger L_j dt |\phi\rangle| + O(dt^{\frac{3}{2}}). \quad (8)$$

We can find useful lower bounds for $q(t+dt)$ by choosing $\{a_k\}$ and $\{b_k\}$ properly. Intuitively, if $H_1 - H_0 \in S$, the contribution of $H_1 - H_0$ can be canceled by the noisy terms, which then limits the speed of decay for $q(t)$. We refer the interested readers to Appendix A for the detailed procedure of how this is achieved; here we just state the result. If $H_1 - H_0 \in S$, we can write $i(H_1 - H_0) + \sum_{k=1}^{m} L_k^\dagger L_k = \sum_{k,j=0}^{m} B_{kj} L_k^\dagger L_j$, where we denote $L_0 = I$ and B_{kj} are the coefficients of the expansion. Then there exists a constant Q, which is determined by $\{B_{kj}\}$ and the Lindblad operators $\{L_k\}$, such that $q(t+dt) \geqslant q(t)(1 - Qdt)$. Specifically, we have $Q = \max\{(m+1)^2|B_{00}|, (m+1)^2|B_{l0}|\|L_l\|, (m+1)^2 |B_{0l}|\|L_l\|, \frac{(m+1)^2|B_{lp}|(\|L_l^\dagger L_l\| + \|L_p^\dagger L_p\| + 4\|L_l\|\|L_p\|)}{2} | l, \quad p = 1, \ldots, m\}$, here $\|\cdot\|$ denote the operator norm which equals the maximal singular value (see Appendix A for detailed derivation). This then gives a lower bound on $q(t)$ as $q(t) \geqslant e^{-Qt} q(0) = e^{-Qt}$. Thus, at any finite time, $q(t)$ is always lower bounded by a positive value of $\epsilon = e^{-Qt}$. Zero-error quantum hypothesis testing is then not possible within any finite time.

B. Sufficiency

We then show that when $H_1 - H_0 \notin S$, quantum error correction can be employed to render the dynamics to unitary evolution; zero-error hypothesis testing can thus be achieved at a finite time. The condition $H_1 - H_0 \in S$ can thus be used to define the "nonorthogonal" evolutions.

We first consider the condition for the quantum error correction that can correct the noises described by

$$\frac{d\rho}{dt} = \sum_{k=1}^{m} \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right). \quad (9)$$

We assume the quantum error correction can be executed much faster than the evolution. The noisy effect within a small time period dt can be described by the Kraus operators as

$$\rho(t+dt) = \sum_{k=0}^{m} E_0 \rho(t) E_0^\dagger, \quad (10)$$

where $E_0 = I - \frac{1}{2}\sum_{j=1}^{m} L_j^\dagger L_j dt$ and $E_j = L_j \sqrt{dt}$ $(j = 1, \ldots, m)$. If P_C is the projection onto the error-correcting code space, it needs to satisfy the condition of the quantum error correction as $P_C E_k^\dagger E_j P_C = \alpha_{kj} P_C, \forall k, j \in \{0, 1, \ldots, m\}$, where α_{kj} are some constants [30–35]. Up to the order of dt, this condition can be expressed in terms of the Lindblad operators as

$$P_C L_k^\dagger L_j P_C = \beta_{kj} P_C, \ \forall k, j \in \{0, 1, \ldots, m\}, \quad (11)$$

where $L_0 = I$. Under the quantum error correction, the effective Hamiltonian on the code subspace is $P_C H_p P_C$ ($p = 0$ and 1). In order to be able to distinguish H_0 and H_1 in the code space, their effective Hamiltonians should be different on the code space, which requires $P_C(H_1 - H_0)P_C \not\propto P_C$; i.e., the effective evolution on the code space cannot differ just by a global phase. This requires $H_1 - H_0 \notin S$, and when $H_1 - H_0 \notin S$ an error correction code can also be explicitly constructed. For completeness, we provide the construction of the quantum error-correcting code in Appendix B, which follows Ref. [36].

IV. EXAMPLES

With the obtained condition one can immediately tell that zero-error quantum hypothesis testing is not possible if the noises are full rank, i.e., when S is the whole space. For example, in the presence of depolarizing noise where the evolutions are given by $\frac{d\rho}{dt} = -i[H_p, \rho] + \sum_{k=x,y,z} \gamma(\sigma_k \rho \sigma_k^\dagger - \rho)$, with $p = 0$ and 1. In this case, $S = \text{span}\{I, \sigma_x, \sigma_y, \sigma_z\}$ contains all 2×2 matrices, perfect distinguishing of H_0 and H_1 within any finite time is thus not possible. This includes the cases with ancillary systems, as $S \otimes I$ contains $H_1 \otimes I - H_0 \otimes I$. It also includes the parallel strategy where N qubits can be prepared in an entangled state and each qubit evolves according to the master equation, as the parallel strategy can be taken as a suboptimal strategy which can be realized by taking SWAP operations between the system qubit and $N - 1$ ancillary qubits, as shown in Fig. 1. Perfect distinguishability thus cannot be achieved within any finite time under the parallel strategy. This can also be checked directly by taking the space spanned by the Lindblad operators for the independent evolution of N qubits under the depolarizing noise; it is easy to check that S contains the differences of the Hamiltonian that acts on the N qubits independently, perfect distinguishability is thus not possible. When $H_1 - H_0 \notin S$, quantum error correction can be employed to achieve zero-error quantum hypothesis testing. For example, under the dephasing noise, the evolutions are $\frac{d\rho}{dt} = -i[H_p, \rho] + \gamma(\sigma_z \rho \sigma_z^\dagger - \rho)$. If $H_0 = \sigma_z$ and $H_1 = \sigma_x$, we can use an ancillary qubit to construct the error-correcting code as $\{|++\rangle, |--\rangle\}$, where $|\pm\rangle$ are the eigenvectors of σ_x. In this case, $P_C = |++\rangle\langle++| + |--\rangle\langle--|$, we then have $P_C \sigma_z P_C = 0$ and $P_C \sigma_x P_C = |++\rangle\langle++| - |--\rangle\langle--|$. If we prepare the initial state as $\frac{|++\rangle + |--\rangle}{\sqrt{2}}$, then the output states under the evolutions with the quantum error correction are $|\psi_0(t)\rangle = \frac{|++\rangle + |--\rangle}{\sqrt{2}}$ and $|\psi_1(t)\rangle = \frac{e^{it}|++\rangle + e^{-it}|--\rangle}{\sqrt{2}}$, which are orthogonal to each other at $t = \frac{\pi}{2}$. However, if $H_0 = \sigma_z + \sigma_x$ and $H_1 = \sigma_x$, then under the dephasing noise it is not possible to achieve perfect distinguishability within a finite time. This implies that under the parallel strategy where N qubits can be prepared in any entangled state and each qubit evolves under the evolution, the zero-error quantum hypothesis testing is not possible within a finite time, as the parallel strategy can be taken as a suboptimal strategy, as shown in Fig. 1. Without the derived condition this will be hard to see.

V. DISCUSSION

Quantum error correction has been widely used in many applications. The closest application to quantum hypothesis testing is quantum metrology [36,37,39–46], which can be regarded as a special case of quantum hypothesis testing when H_0 and H_1 are taken as two neighboring Hamiltonians as $H(x)$ and $H(x+dx)$, with x as the parameter to be estimated and dx as a small shift of the parameter. If

$\partial_x H(x) = \lim_{dx\to 0} \frac{H(x+dx)-H(x)}{dx} \notin S$, quantum error correction can be applied to render the dynamics to unitary evolutions, and the Heisenberg scaling can be achieved where the quantum Fisher information (QFI) scales quadratically with time as ct^2 [36,37].

The Hamiltonians in quantum hypothesis testing can be more general and do not need to be of neighboring Hamiltonians. Even for neighboring Hamiltonians, the scaling of the QFI has no distinguishing power on zero-error hypothesis testing. The QFI, $J(t)$, is related to the output states of the two evolutions as $2 - 2F[\rho_x(t), \rho_{x+dx}(t)] \approx \frac{1}{4}J(t)dx^2$ [47], from which we have $F[\rho_x(t), \rho_{x+dx}(t)] \approx 1 - \frac{1}{8}J(t)dx^2$. Different from quantum metrology where dx can be varied and taken as arbitrarily small, in quantum hypothesis testing we have two fixed Hamiltonians; dx is thus also fixed. Even the QFI can only scale linearly as ct; this does not exclude the possibility that the two states become orthogonal when $t \geqslant \frac{8}{cdx^2}$. In fact, as long as $J(t)$ is unbounded, the no go of zero-error quantum hypothesis testing can not be obtained from the QFI.

VI. SUMMARY

We showed that, for the hypothesis testing of the Hamiltonians under the same Markovian noises, zero-error hypothesis testing can be achieved within a finite time if and only if $H_1 - H_0 \notin S = \text{span}\{I, L_k^\dagger, L_k, L_j^\dagger L_k | k, j = 1, \cdots m\}$, which coincides with the condition for quantum error correction. When $H_1 - H_0 \in S$, the evolutions are in some sense nonorthogonal, which resembles the hypothesis testing of nonorthogonal distributions and nonorthogonal quantum states. In particular when $H_1 - H_0 \in S$ the maximal fidelity of the output states is lower bounded by an exponential function which remains positive at any finite time. The condition can thus be used to characterize the "orthogonality" of quantum evolutions, which also builds a bridge between the quantum error correction and the quantum hypothesis testing. As the first exponential lower bound on the quantum hypothesis under the continuous evolution, this initiates the efforts on the identification of the exact exponents for the nonorthogonal continuous evolutions, for both the maximal fidelity and other possible measures of distinguishability.

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APPENDIX A: DERIVATION OF THE LOWER BOUND

We show that if $H_1 - H_0 \in \text{span}\{L_k, L_k^\dagger, L_k^\dagger L_j, \mathcal{I}\}$, then at any finite time T there exists $\epsilon > 0$ such that the maximal fidelity of the output states of the two channels, aided with the optimal operations, is at least ϵ. We prove this by deriving a lower bound on the evolution of the maximal fidelity $q(t) = \tilde{F}[\rho_0(t), \rho_1(t)]$, where $\rho_i(t)$ is the state that evolves under the ith evolution, which can be aided with ancillary systems and controls.

The whole evolution can be divided into small intervals, each with a duration dt, and optimal operations can be added in between. With the optimal procedure described in the main text, we have

$$q(t+dt) = \min_{\langle\psi|\phi\rangle = q(t)} \tilde{F}\left[\mathcal{E}_0^{dt}(|\psi\rangle\langle\psi|), \mathcal{E}_1^{dt}(|\phi\rangle\langle\phi|)\right], \quad \text{(A1)}$$

where $\mathcal{E}_i^{dt}$ is the ith dynamics evolved for the time dt:

$$\mathcal{E}_0^{dt}(|\psi\rangle\langle\psi|) = \sum_{k=0}^{m} E_{0k}|\psi\rangle\langle\psi|E_{0k}^\dagger,$$
$$\mathcal{E}_1^{dt}(|\phi\rangle\langle\phi|) = \sum_{k=0}^{m} E_{1k}|\phi\rangle\langle\phi|E_{1k}^\dagger, \quad \text{(A2)}$$

where $E_{00} = I - iH_0 dt - \frac{1}{2}\sum_{k=1}^m L_k^\dagger L_k dt$, $E_{10} = I - iH_1 dt - \frac{1}{2}\sum_{k=1}^m L_k^\dagger L_k dt$, and $E_{0k} = E_{1k} = L_k\sqrt{dt}$ ($k = 1, \ldots, m$). As the support of $\mathcal{E}_0^{dt}(|\psi\rangle\langle\psi|)$ is the span of $\{E_{0i}|\psi\rangle | i = 0, \cdot, m\}$, any pure state in the support of $\mathcal{E}_0^{dt}(|\psi\rangle)$ can thus be written as

$$|\psi(dt)\rangle = \frac{1}{N_0}\sum_{k=0}^{m} a_k E_{0k}|\psi\rangle, \quad \text{(A3)}$$

where N_1 is a normalization factor. Similarly we can write any pure state in the support of $\mathcal{E}_1^{dt}(|\phi\rangle)$ as

$$|\phi(dt)\rangle = \frac{1}{N_1}\sum_{k=0}^{m} b_k E_{1k}|\phi\rangle. \quad \text{(A4)}$$

Then, up to the order of dt, we have

$$\begin{aligned} q(t+dt) &= \tilde{F}\left[\mathcal{E}_0^{dt}(|\psi\rangle), \mathcal{E}_1^{dt}(|\phi\rangle)\right] \\ &= \max_{\{|\psi(dt)\rangle, |\phi(dt)\rangle\}} |\langle\psi(dt)|\phi(dt)\rangle| \\ &\geqslant \frac{1}{N_0 N_1}\Bigg|\langle\psi|a_0^* b_0\left[I - i(H_1 - H_0)dt - \sum_{k=1}^m L_k^\dagger L_k dt\right] \\ &\quad + \sum_{k=1}^m (a_k^* b_0\sqrt{dt}L_k^\dagger + a_0^* b_k\sqrt{dt}L_k) \\ &\quad + \sum_{k,j=1}^m a_k^* b_j L_k^\dagger L_j dt|\phi\rangle\Bigg|, \end{aligned} \quad \text{(A5)}$$

where we assume $|\psi\rangle$ and $|\phi\rangle$ are optimal states under the constraint $\langle\psi|\phi\rangle = q(t)$.

If $H_1 - H_0 \in \text{span}\{L_k^\dagger L_j | k, j = 0, 1, \ldots, m\}$, we have $i(H_1 - H_0) + \sum_{k=1}^m L_k^\dagger L_k = \sum_{k,j=0} B_{kj} L_k^\dagger L_j$, where $L_0 = \mathcal{I}$. Then,

$$\begin{aligned} &\left|\langle\psi|i(H_1 - H_0) + \sum_{k=1}^m L_k^\dagger L_k|\phi\rangle\right| \\ &= \left|\sum_{k,j=0}^m B_{kj}\langle\psi|L_k^\dagger L_j|\phi\rangle\right| \\ &\leqslant \sum_{k,j=0}^m |B_{kj}||\langle\psi|L_k^\dagger L_j|\phi\rangle| \\ &\leqslant (m+1)^2 \max_{k,j}\{|B_{kj}||\langle\psi|L_k^\dagger L_j|\phi\rangle|\}. \end{aligned} \quad \text{(A6)}$$

We discuss different cases.

First with the optimal choice of $|\psi\rangle$ and $|\phi\rangle$, if we have $|\langle\psi|i(H_1-H_0)+\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle|=0$, we can choose $a_0=b_0=1$ and $a_k=b_k=0$ for $k=1,\dots,m$. Up to the order of dt, we have

$$q(t+dt)\geqslant\frac{1}{N_0N_1}q(t). \tag{A7}$$

In this case, $N_0^2=N_1^2=|1-\langle\psi|\sum_{k=1}^m L_k^\dagger L_k|\psi\rangle|dt|\leqslant 1$; thus

$$q(t+dt)\geqslant q(t), \tag{A8}$$

$$\frac{dq}{dt}\geqslant 0=-Q_1q(t), \tag{A9}$$

where we take $Q_1=0$.

If $|\langle\psi|i(H_1-H_0)+\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle|\neq 0$, we denote $|B_{lp}||\langle\psi|L_l^\dagger L_p|\phi\rangle|=\max_{k,j}\{|B_{kj}||\langle\psi|L_k^\dagger L_j|\phi\rangle|\}$. Since $|\langle\psi|i(H_1-H_0)+\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle|\neq 0$, $|\langle\psi|L_l^\dagger L_p|\phi\rangle|$ also cannot be zero. From Eq. (A6) we then have

$$\frac{|\langle\psi|i(H_1-H_0)+\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle|}{|\langle\psi|L_l^\dagger L_p|\phi\rangle|}\leqslant(m+1)^2|B_{lp}|. \tag{A10}$$

We discuss different cases according to different l and p.

1. $l=p=0$

In this case we can choose $a_0=1+\alpha dt$, $b_0=1+\beta dt$, and $a_k=b_k=0$ for $k=1,\dots,m$. From Eq. (A5), up to the order of dt we get

$$q(t+dt)\geqslant\frac{1}{N_0N_1}\bigg|q(t)+(\alpha^*+\beta)\langle\psi|\phi\rangle dt - \langle\psi|i(H_1-H_0)+\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle dt\bigg|. \tag{A11}$$

Let $\alpha^*=\beta=\frac{1}{2}\frac{\langle\psi|i(H_1-H_0)+\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle}{\langle\psi|\phi\rangle}$, then

$$|\alpha|=|\beta|=\frac{1}{2}\frac{|\langle\psi|i(H_1-H_0)+\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle|}{|\langle\psi|\phi\rangle|}\leqslant\frac{1}{2}(m+1)^2|B_{00}| \tag{A12}$$

and

$$q(t+dt)\geqslant\frac{1}{N_0N_1}q(t). \tag{A13}$$

In this case,

$$\begin{aligned}N_0^2&=|\langle\psi(dt)|\psi(dt)\rangle|\\&=\bigg|1-\langle\psi|\sum_{k=1}^m L_k^\dagger L_k|\psi\rangle dt+(\alpha^*+\alpha)dt\bigg|\\&\leqslant\bigg|1-\langle\psi|\sum_{k=1}^m L_k^\dagger L_k|\psi\rangle dt\bigg|+(m+1)^2|B_{00}|dt\\&\leqslant 1+(m+1)^2|B_{00}|dt=1+Q_2dt,\end{aligned} \tag{A14}$$

where $Q_2=(m+1)^2|B_{00}|$. Similarly we can get $N_1^2\leqslant 1+Q_2dt$. Thus

$$q(t+dt)\geqslant\frac{1}{N_0N_1}q(t)\quad\geqslant\frac{q(t)}{1+Q_2dt}\quad\approx q(t)(1-Q_2dt) \tag{A15}$$

and

$$\frac{dq}{dt}\geqslant -Q_2q(t). \tag{A16}$$

2. $l\neq 0, p=0$

In this case, we choose $a_0=b_0=1$, $a_l=\alpha\sqrt{dt}$, $a_k=0$ for $k\neq l$, and $b_j=0$ for $j=1,\dots,m$. Then from Eq. (A5), up to the order of dt, we get

$$q(t+dt)\geqslant\frac{1}{N_0N_1}\bigg|q(t)+\alpha^*\langle\psi|L_l^\dagger|\phi\rangle dt-\langle\psi|i(H_1-H_0)+\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle dt\bigg|. \tag{A17}$$

We choose $\alpha^*=\frac{\langle\psi|i(H_1-H_0)+\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle}{\langle\psi|L_l^\dagger|\phi\rangle}$. From Eq. (A10), we get

$$|\alpha|=\frac{|\langle\psi|i(H_1-H_0)+\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle|}{|\langle\psi|\phi\rangle|}\leqslant(m+1)^2|B_{l0}|. \tag{A18}$$

With this choice we have

$$q(t+dt)\geqslant\frac{1}{N_0N_1}q(t), \tag{A19}$$

where

$$\begin{aligned}N_0^2&=\bigg|1-\langle\psi|\sum_{k=1}^m L_k^\dagger L_k|\psi\rangle dt+\langle\psi|\alpha^*L_l^\dagger+\alpha L_l|\psi\rangle dt\bigg|\\&\leqslant\bigg|1-\langle\psi|\sum_{k=1}^m L_k^\dagger L_k|\psi\rangle dt\bigg|+2|\alpha|\|L_l\|dt\\&\leqslant 1+2|\alpha|\|L_l\|dt\\&\leqslant 1+2(m+1)^2|B_{l0}|\|L_l\|dt\end{aligned} \tag{A20}$$

and

$$\begin{aligned}N_1^2&=\bigg|1-\langle\phi|\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle dt\bigg|\\&\leqslant 1.\end{aligned} \tag{A21}$$

Thus,

$$\begin{aligned}q(t+dt)&\geqslant\frac{1}{N_0N_1}q(t)\\&\geqslant q(t)(1-Q_3dt),\end{aligned} \tag{A22}$$

where $Q_3=(m+1)^2|B_{l0}|\|L_l\|$. We then have $\frac{dq}{dt}\geqslant -Q_3q$.

3. $l=0, p\neq 0$

This is symmetric to case 2 and we can similarly obtain $\frac{dq}{dt}\geqslant -Q_4q$, where $Q_4=(m+1)^2|B_{0p}|\|L_p\|$.

4. $l \neq 0, p \neq 0$

In this case, we choose $a_0 = b_0 = 1$, $a_k = 0$ when $k \neq l$, and $b_j = 0$ when $j \neq p$, where $k, j \in \{1, 2, \dots, m\}$. Then from Eq. (A5) we get

$$q(t+dt) \geqslant \frac{1}{N_0 N_1}\left| q(t) + a_l^* b_p \langle\psi|L_l^\dagger L_p|\phi\rangle dt - \langle\psi|i(H_1 - H_0) + \sum_{k=1}^m L_k^\dagger L_k|\phi\rangle dt + \big(a_l^*\langle\psi|L_l^\dagger|\phi\rangle + b_p\langle\psi|L_p|\phi\rangle\big)\sqrt{dt}\right|. \quad \text{(A23)}$$

Denote $\frac{\langle\psi|i(H_1-H_0)+\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle}{\langle\psi|L_l^\dagger L_p|\phi\rangle} = r_0 e^{i\theta_0}$, $\langle\psi|L_l^\dagger|\phi\rangle = r_1 e^{i\theta_1}$, and $\langle\psi|L_p|\phi\rangle = r_2 e^{i\theta_2}$, and we get $r_0 \leqslant (m+1)^2|B_{lp}|$, $r_1 \leqslant \|L_l\|$, and $r_2 \leqslant \|L_p\|$.

Let $a_l = \sqrt{r_0} e^{i(\theta_1+\gamma}$ and $b_p = \sqrt{r_0} e^{i\theta_0+\theta_1+\gamma}$, and we get

$$\begin{aligned} q(t+dt) &\geqslant \frac{1}{N_0 N_1}\big|q(t) + \big(a_l^* r_1 e^{i\theta_1} + b_p r_2 e^{i\theta_2}\big)\sqrt{dt}\big| \\ &= \frac{1}{N_0 N_1}\big|q(t) + \sqrt{r_0}\big(r_1 e^{-i\gamma} + r_2 e^{i(\theta_0+\theta_1+\theta_2+\gamma)}\big)\sqrt{dt}\big| \\ &= \frac{1}{N_0 N_1}\big|q(t) + \sqrt{r_0}\{r_1\cos\gamma + r_2\cos(\theta_0+\theta_1+\theta_2+\gamma) + i[-r_1\sin\gamma + r_2\sin(\theta_0+\theta_1+\theta_2+\gamma)]\}\sqrt{dt}\big| \\ &= \frac{1}{N_0 N_1}\big|q(t) + \sqrt{r_0}\{[r_1 + r_2\cos(\theta_0+\theta_1+\theta_2)]\cos\gamma - r_2\sin(\theta_0+\theta_1+\theta_2)\sin\gamma\} \\ &\quad + i(-r_1\sin\gamma + r_2\sin(\theta_0+\theta_1+\theta_2+\gamma)]\sqrt{dt}\big|. \end{aligned} \quad \text{(A24)}$$

If $r_2\sin(\theta_0+\theta_1+\theta_2) = 0$, we choose $\gamma = \frac{\pi}{2}$, otherwise we let $\tan\gamma = \frac{r_1 + r_2\cos(\theta_0+\theta_1+\theta_2)}{r_2\sin(\theta_0+\theta_1+\theta_2)}$. With this choice, we have $[r_1 + r_2\cos(\theta_0+\theta_1+\theta_2)]\cos\gamma - r_2\sin(\theta_0+\theta_1+\theta_2)\sin\gamma = 0$, then

$$\begin{aligned} q(t+dt) &= \frac{1}{N_0 N_1}\big|q(t) + i\sqrt{r_0}[-r_1\sin\gamma + r_2\sin(\theta_0+\theta_1+\theta_2+\gamma)]\sqrt{dt}\big| \\ &= \frac{1}{N_0 N_1}\sqrt{q^2(t) + r_0[-r_1\sin\gamma + r_2\sin(\theta_0+\theta_1+\theta_2+\gamma)]^2 dt} \\ &\geqslant \frac{1}{N_0 N_1} q(t). \end{aligned} \quad \text{(A25)}$$

Here

$$\begin{aligned} N_0 &= \sqrt{1 - \langle\psi|\sum_{k=1}^m L_k^\dagger L_k|\psi\rangle dt + \langle\psi|a_l^* a_l L_l^\dagger L_l|\psi\rangle dt + \langle\psi|a_l^* L_l^\dagger + a_l L_l|\psi\rangle\sqrt{dt}}, \\ N_1 &= \sqrt{1 - \langle\phi|\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle dt + \langle\phi|b_p^* b_p L_p^\dagger L_p|\phi\rangle dt + \langle\phi|b_p^* L_p^\dagger + b_p L_p|\phi\rangle\sqrt{dt}}, \end{aligned} \quad \text{(A26)}$$

$$N_0 N_1 = \sqrt{1 + C\sqrt{dt} + D dt}, \quad \text{(A27)}$$

where $C = \langle\psi|a_l^* L_l^\dagger + a_l L_l|\psi\rangle + \langle\phi|b_p^* L_p^\dagger + b_p L_p|\phi\rangle$ and

$$\begin{aligned} D &= \langle\psi|a_l^* a_l L_l^\dagger L_l|\psi\rangle - \langle\psi|\sum_{k=1}^m L_k^\dagger L_k|\psi\rangle + \langle\phi|b_p^* b_p L_p^\dagger L_p|\phi\rangle - \langle\phi|\sum_{k=1}^m L_k^\dagger L_k|\phi\rangle + \langle\psi|a_l^* L_l^\dagger + a_l L_l|\psi\rangle\langle\phi|b_p^* L_p^\dagger + b_p L_p|\phi\rangle \\ &\leqslant |a_l|^2\|L_l^\dagger L_l\| + |b_p|^2\|L_p^\dagger L_p\| + 4|a_l b_p|\|L_l\|\|L_p\| = r_0(\|L_l^\dagger L_l\| + \|L_p^\dagger L_p\| + 4\|L_l\|\|L_p\|) \\ &\leqslant (m+1)^2|B_{lp}|(\|L_l^\dagger L_l\| + \|L_p^\dagger L_p\| + 4\|L_l\|\|L_p\|). \end{aligned} \quad \text{(A28)}$$

Note that we can always choose a_l and b_p to make $C \leqslant 0$. As if $C > 0$, we can just change γ to $\tilde{\gamma} = \gamma + \pi$, then $\tilde{a}_l = -a_l$, $\tilde{b}_p = -b_p$, which flips the sign of C. This also flip the sign of the imaginary part in Eq. (A24) but it does not change the result of Eq. (A24). Thus we can assume $C \leqslant 0$, then up to the order of dt, $N_0 N_1 \leqslant 1 + Q_5 dt$, where $Q_5 = \frac{(m+1)^2|B_{lp}|(\|L_l^\dagger L_l\| + \|L_p^\dagger L_p\| + 4\|L_l\|\|L_p\|)}{2}$. We then have $q(t+dt) \geqslant q(t)(1 - Q_5 dt)$, $\frac{dq}{dt} \geqslant -Q_5 q$.

Let $Q = \max\{(m+1)^2|B_{00}|, (m+1)^2|B_{l0}|\|L_l\|, (m+1)^2|B_{0l}|\|L_l\|, \frac{(m+1)^2|B_{lp}|(\|L_l^\dagger L_l\| + \|L_p^\dagger L_p\| + 4\|L_l\|\|L_p\|)}{2}, \; l, p = 1, \dots, m\}$, we then have $\frac{dq}{dt} \geqslant -Qq$. Thus $q(T) \geqslant e^{-QT} q(0) = e^{-QT} > 0$; i.e., at any finite time T, $q(T)$ is always lower bounded by a positive value of $\epsilon = e^{-QT}$.

APPENDIX B: CONSTRUCTION OF THE QUANTUM ERROR-CORRECTION CODE

For completeness, we provide the construction of the quantum error-correcting code when $H_1 - H_0 \notin S$, which follows Ref. [36].

When $H_1 - H_0 \notin S$, $H_1 - H_0$ can be decomposed as $H_1 - H_0 = H_{||} + H_{\perp}$, where $H_{||} \in S$ and $\mathrm{Tr}(H_{\perp}O) = 0\,\forall O \in S$. Since $I \in S$ thus $\mathrm{Tr}(H_{\perp}) = 0$, $H_{\perp}$ can then be written as $H_{\perp} = \frac{\mathrm{Tr}|H_{\perp}|}{2}(\rho_1 - \rho_0)$, where ρ_1 and ρ_0 are two density matrices, which are positive semidefinite with trace equal to 1. Let $|C_0\rangle$ and $|C_1\rangle$ be the purified states of ρ_0 and ρ_1, respectively; then, with an additional qubit, we can construct an error-correcting code as $\{|C_0\rangle|0\rangle, |C_1\rangle|1\rangle\}$. It is straightforward to check that this code satisfies the conditions of quantum error correction as $\langle C_0|\langle 0|L_k^{\dagger}L_j|C_1\rangle|1\rangle = 0$ and $\langle C_0|\langle 0|L_k^{\dagger}L_j|C_0\rangle|0\rangle = \langle C_1|\langle 1|L_k^{\dagger}L_j|C_1\rangle|1\rangle$. Also $\langle C_0|\langle 0|H_1 - H_0|C_0\rangle|0\rangle - \langle C_1|\langle 1|H_1 - H_0|C_1\rangle|1\rangle = \frac{2\mathrm{Tr}(H_{\perp}^2)}{\mathrm{Tr}|H_{\perp}|} > 0$; $H_1 - H_0$ thus acts nontrivially on the code space. With this error-correcting code, if the initial state is taken as $\frac{|C_0\rangle|0\rangle + |C_1\rangle|1\rangle}{\sqrt{2}}$, then the zero-error hypothesis testing can be achieved at $t = \frac{\mathrm{Tr}|H_{\perp}|\pi}{2\mathrm{Tr}(H_{\perp}^2)}$.

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