Tightest conditions for violating the Bell inequality when measurement independence is relaxed

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It has been believed that statistical inequalities such as the Bell inequality should be modified once measurement independence (MI), the assumption that observers can freely choose measurement settings without changing the probability distribution of hidden variables, is relaxed. However, we show that there exists the possibility that the Bell inequality is still valid even if MI is relaxed. MI is only a sufficient condition to derive the Bell inequality when both determinism and setting independence, usually called local realism, are satisfied. We thus propose a condition necessary and sufficient for deriving the Bell inequality, called concealed measurement dependence.

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I. INTRODUCTION

One of the most remarkable achievements of modern physics is the discovery that quantum mechanics violates certain statistical inequalities such as the Bell inequality [1–4]. The Bell inequality is derived based upon several physical postulates, namely, determinism, setting independence, and measurement independence (MI). Determinism is the property that an outcome of any physical observable has a definite value all the time. Setting independence implies that the probability of observing an event associated with one setting is independent of the other setting so that it prohibits any information from transmitting faster than light, which is called no signaling.

MI is the assumption that measurement settings can be chosen independently of any underlying variables describing a system. Compared with determinism and setting independence, MI had not been seriously considered in literatures since it is believed that experimenters can freely choose an experimental setup. In this regard MI is often associated with the so-called "free will" of experimenters [5–7]. As clearly discussed in Ref. [8], however, there are *no* free or random events in a hidden-variable theory based upon truly classical mechanics so that MI cannot be naturally ensured.

MI is also related to practical aspects of quantum information processing tasks such as device-independent quantum key distribution [9–13], random-number generation, and randomness expansion [14–17]. Furthermore, there have been recent experiments partially closing the MI assumption using an astronomical source [18] and using real human choices instead of a random number generator [19,20].

So far considerable focus has been laid at constructing singlet correlation, maximally violating the Bell inequality by relaxing MI. Brans first showed that singlet correlation can be reproduced by completely relaxing MI [8]. More quantitative studies to obtain singlet correlation by partially relaxing MI has been performed by introducing various measures [21–23] or by using models [5,24–26]. In fact, singlet correlation has also been acquired by relaxing no signaling [27,28], determinism [29–32], or both MI and outcome independence [33–36]. The Bell-like inequalities specifically suited for a measurement-dependent scenario have been derived [37].

In this paper, instead of constructing singlet correlation by relaxing MI, we focus on the question whether MI is sufficient, necessary, or both a necessary and sufficient condition to fulfill the Bell inequality when all the other conditions such as determinism, setting independence, and others are assumed. We show that the Bell inequality can be still valid even if MI is relaxed. MI is only a sufficient condition to derive the Bell inequality when determinism and setting independence are satisfied. We thus propose a new condition necessary and sufficient for deriving the Bell inequality, called concealed measurement dependence (CMD). We also find that our CMD hidden-variable model may violate no signaling even if locality is assumed.

In Sec. II we briefly summarize three assumptions to derive the Bell inequality. In Sec. III we introduce a concept called as generalized correlation to define CMD. In Secs. IV and V we discuss the relation between CMD and no signaling and present its mathematical analyses in detail, respectively. In Sec. VI we also present a simple example using what we believe to be an original Bell inequality. Finally, a summary is presented in Sec. VII.

II. BELL INEQUALITY AND MEASUREMENT INDEPENDENCE

A quantum nonlocality test is given by a bound on the possible values of a combination of correlations of local

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observables in spacelike separated places. When Alice and Bob with such a distance measure their spin-1/2 particles along the X direction and along Y, respectively, the correlation of the outcomes is given by

$$E(X,Y)=\sum_{\alpha,\beta=\pm1}\alpha\beta P_{XY}(\alpha,\beta), \tag{1}$$

where $P_{XY}(\alpha,\beta)$ is the joint probability of obtaining the outcomes α and β. Here we assume that each observable is dichotomic, i.e., α and β are ±1. A quantum nonlocality test like the CHSH inequality is given by

$$\sum_{i,j=1,2} w_{ij}E(A_i,B_j)\leqslant C, \tag{2}$$

where C is a classical upper bound and $w_{ij}\in\mathbb{R}$ are weight coefficients. When the model is a deterministic hidden-variable model, the Bell inequalities (2) become the CHSH inequality for $C=2$ and $w_{11}=w_{12}=w_{21}=-w_{22}=\pm1$ [38].

Hidden variable theories assume that the joint probability distribution is given by

$$P_{XY}(\alpha,\beta)=\sum_{\lambda}P_{XY}(\alpha,\beta|\lambda)P_{XY}(\lambda), \tag{3}$$

where λ denotes the hidden variables [39]. However, it fails to correctly describe the subtle assumptions of MI, setting independence, and outcome independence. MI implies the hidden-variable distribution has nothing to do with any experimental setting. The setting independence implies the probability distribution of an outcome of a physical variable related to a local setting, for example, X, has nothing to do with the other local setting, for example, Y. The outcome independence implies the probability distribution of an outcome, for example, α, has nothing to do with that of another outcome, for example, β. Here we accept both the setting and outcome independences as usual [5] while MI is relaxed. To precisely express all of them one may explicitly introduce dependence on experimental settings X and Y as conditions:

$$P(\alpha,\beta|X,Y)=\sum_{\lambda}P(\alpha,\beta|\lambda,X,Y)P(\lambda|X,Y). \tag{4}$$

Locality assumes that physical properties cannot be influenced by spacelike-separated events in a superluminal way. This implies the joint probabilities conditioned by hidden variable is factorizable, which results in

$$P(\alpha,\beta|\lambda,X,Y)=P(\alpha|\lambda,X)P(\beta|\lambda,Y). \tag{5}$$

Here both the setting and outcome independences are incorporated. Based on a Bayesian rule, $P(\alpha,\beta|\lambda,X,Y)=P(\alpha|\beta,\lambda,X,Y)P(\beta|\lambda,X,Y)$, the setting independence forbids instantaneous change of a local setting from affecting the probability distribution in the other local setting separated (possibly) in a spacelike manner: $P(\beta|\lambda,X,Y)=P(\beta|\lambda,X',Y)=P(\beta|\lambda,Y)$ and similarly, $P(\alpha|\beta,\lambda,X,Y)=P(\alpha|\beta,\lambda,X,Y')=P(\alpha|\beta,\lambda,X)$. Furthermore, the outcome independence forbids the fact that one obtains a particular outcome β from affecting the probability distribution of another outcome α: $P(\alpha|\beta,\lambda,X)=P(\alpha|\lambda,X)$.

MI is the assumption that what measurement setting is chosen does not influence the probability distribution of hidden variables, so that Alice and Bob can freely choose their measurement settings with the hidden-variable distribution $P(\lambda|X,Y)$ unchanged:

$$P(\lambda|X,Y)=P(\lambda). \tag{6}$$

MI differs from the setting independence in that the former deals with the dependence of the hidden variable on the sets of measurement setting, namely, $\{X,Y\}$, while the latter is related to the interdependence among the local measurement settings, namely, X and Y, for the probability distribution of a physical variable.

If one of the assumptions is relaxed, the Bell inequality is modified such that the bound increases. It thus becomes more difficult to violate it in quantum mechanics. However, we emphasize that the bound does not always increase even if MI is relaxed. Interestingly there exists a certain condition that the bound remains unchanged so that the Bell inequality still holds with the same bound. We call such conditions *concealed measurement dependence* (CMD), the main theme of this paper.

III. GENERALIZED CORRELATION AND CONCEALED MEASUREMENT DEPENDENCE

MI is characterized by the conditional probability distribution $P(\lambda|X,Y)$. Once we relax MI, we immediately have more than one hidden-variable distribution depending on the settings labeled by X and Y. For example, we have four distributions, in CHSH inequality case, namely, $P(\lambda|X,Y)$ with $X\in\{A_1,A_2\}$ and $Y\in\{B_1,B_2\}$. Even when we obtain the correlation of A_1 and B_1, we need consider not only $P(\lambda|A_1,B_1)$ but also the other three distributions, $P(\lambda|A_1,B_2)$, $P(\lambda|A_2,B_1)$, $P(\lambda|A_2,B_2)$. In order to investigate such models, we introduce a modified correlation which we call generalized correlation:

$$E_{X'Y'}(X,Y)=\sum_{\lambda}\alpha(\lambda,X)\beta(\lambda,Y)P(\lambda|X',Y').$$

$E_{X'Y'}(X,Y)$ is the correlation between X and Y calculated by using the hidden-variable distribution conditioned on a measurement setting X' and Y' incorporating measurement dependence. For one measurement setting, namely, $X=A_1$ and $Y=B_1$, we have the hidden-variable distribution $P(\lambda|A_1,B_1)$ from which we obtain the correlation between X and Y, i.e., $E_{A_1B_1}(X,Y)$. For another setting, namely, A_1 and B_2, however, we now have $P(\lambda|A_1,B_2)$ from which we also obtain the correlation between X and Y, $E_{A_1B_2}(X,Y)$. In general, one finds $E_{A_1B_1}(X,Y)\neq E_{A_1B_2}(X,Y)$ due to $P(\lambda|A_1,B_1)\neq P(\lambda|A_1,B_2)$ once the MI assumption is relaxed.

We say a local deterministic hidden-variable model satisfies CMD if

$$E(X,Y)=E_{XY}(X,Y)=E_{A_1B_1}(X,Y) \tag{7}$$

is satisfied for all the possible settings $X\in\{A_1,A_2\}$ and $Y\in\{B_1,B_2\}$ with respect to reference observables A_1 and B_1 [40]. Even though the probability distribution of the CMD model, $P(\lambda|X,Y)$, is different from that of the hidden-variable model satisfying MI, $P(\lambda)$, the CMD condition guarantees that one still finds an appropriate $P(\lambda|X,Y)$ exhibiting at least the equivalent expectation value obtained from the corresponding

$P(\lambda)$. Hence, the CMD model, the relaxing MI condition, can still have the bound of a MI model:

$$\sum_{i,j=1,2} w_{ij} E_{A_iB_j}(A_i, B_j) = \sum_{i,j=1,2} w_{ij} E_{A_1B_1}(A_i, B_j) \leqslant C, \tag{8}$$

where the classical upper bound is given by the MI deterministic hidden-variable model. The CMD condition is weaker than MI since it restricts only correlations rather than the distributions of hidden variables.

It is worth noting that the existence of CMD does not contradict Fine's proof [41], which requires the correlations of the CMD model to be simulated by a certain MI deterministic hidden-variable model. Again, this is the case since $E_{A_1B_1}(X, Y)$ are averaged over a *single* hidden-variable distribution $P(\lambda|A_1, B_1)$ like the usual MI deterministic hidden-variable models.

IV. RELATION BETWEEN CONCEALED MEASUREMENT DEPENDENCE AND NO SIGNALING

Even though locality is assumed, CMD models may violate no-signaling constraints [42,43] expressed as

$$P(\alpha|X, Y) = P(\alpha|X, Y') \text{ and } P(\beta|X, Y) = P(\beta|X', Y), \tag{9}$$

where $P(\alpha|X, Y) = \sum_\beta P(\alpha, \beta|X, Y)$ and similarly for $P(\beta|X, Y)$. Using Eq. (4), the conditional probabilities of Eq. (9) are rewritten as

$$P(\alpha|X, Y) = \sum_\beta \sum_\lambda P(\alpha, \beta|\lambda, X, Y)P(\lambda|X, Y). \tag{10}$$

Here locality, setting independence, and outcome independence are all associated with $P(\alpha, \beta|\lambda, X, Y)$ but have nothing to do with $P(\lambda|X, Y)$. Thus, $P(\alpha|X, Y)$ may still depend on X or Y once the MI defined as Eq. (6) is relaxed; this implies that a signal can be instantaneously transmitted by altering measurement settings violating the no-signaling condition. We can also explain it in a slight different way. In CMD models MI is relaxed in a very specific form [Eq. (7)]; CMD includes a restriction not on local expectations $E(X)$ and $E(Y)$ but only on the correlations $E(X, Y)$. CMD models do not guarantee $E_{XY}(X) = E_{XY'}(X)$ and $E_{XY}(Y) = E_{X'Y}(Y)$, where $E_{XY}(Z) = \sum_\lambda \alpha(\lambda, Z)P(\lambda|X, Y)$. Therefore, no signaling (9) may be violated since probabilities are directly related to their expectations, i.e., $P(\alpha|X, Y) = [1 + \alpha E_{XY}(X)]/2$ and $P(\beta|X, Y) = [1 + \beta E_{XY}(Y)]/2$ [44]. It is remarkable that *satisfying the Bell inequalities of MI deterministic hidden-variable models is not sufficient for no signaling in CMD models.*

Now we find the relationships among the sets of hidden-variable models satisfying CMD, the Bell inequalities, and no signaling when MI is relaxed, while locality and determinism still hold. First, CMD implies Bell inequalities by definition. We will show below that the Bell inequalities also imply CMD, which implies that the set of CMD models is equivalent to that of the Bell inequalities. Second, as discussed above, CMD does not guarantee no signaling. However, the MI model should exist if both the Bell inequalities and no signaling are satisfied due to Fine's proof [41]. It implies that

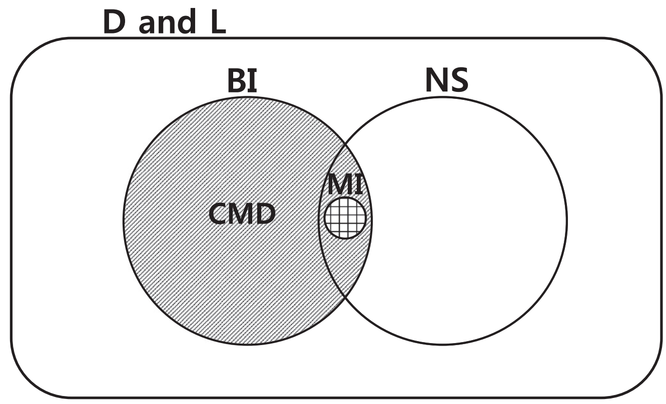

FIG. 1. Schematic diagram for the sets of hidden-variable distributions satisfying the Bell inequalities (BI), no signaling (NS), and CMD when determinism (D) and locality (L) are assumed but MI is relaxed. "MI" denotes the set representable with a MI deterministic hidden-variable model.

not only does there exist a nonzero intersection between the set of CMD models satisfying the Bell inequalities and that of the no-signaling models, but also the intersection should contain the set of MI models. The intersection is larger than the set of the MI deterministic hidden-variable model since the CMD model deals with the correlation of only two variables. The relations discussed here are schematically summarized in Fig. 1. We emphasize that *CMD models are not distinguishable from MI deterministic hidden-variable models by only testing the Bell inequalities.*

V. MATHEMATICAL ANALYSES

The measurement dependence of a hidden variable can be conveniently analyzed by introducing a variable $\xi_{XY\lambda}$ as

$$P(\lambda|X, Y) = P(\lambda|A_1, B_1) + \xi_{XY\lambda} \geqslant 0, \tag{11}$$

$$\sum_\lambda \xi_{XY\lambda} = 0, \tag{12}$$

where the second equation is the normalization condition. $\xi_{XY\lambda}$ expresses the difference of the probability distribution from the reference distribution, that is, $P(\lambda|A_1, B_1)$. Therefore, $\xi_{A_1B_1\lambda} = 0$ is trivially satisfied. With Eq. (11), the CMD conditions [Eq. (7)] can be written as

$$\sum_{(X,Y)\neq(A_1,B_1)} \sum_\lambda M^{X'Y'\eta}_{XY\lambda} \xi_{XY\lambda} = 0, \tag{13}$$

where $M^{X'Y'\eta=1}_{XY\lambda} = \alpha(\lambda, X)\beta(\lambda, Y)\delta_{XX'}\delta_{YY'}$ for the CMD conditions ($\eta = 1$) and $M^{X'Y'\eta=2}_{XY\lambda} = \delta_{XX'}\delta_{YY'}$ for the normalization conditions ($\eta = 2$). The CMD condition is compactly described as

$$\mathbf{M}\xi = \mathbf{0}. \tag{14}$$

Here ξ is a vector of 48(=3×16) dimensions, $\mathbf{0}$ a null vector, and M a 6×48 matrix. The 48 dimensions of ξ come from three measurement settings of XY ($XY \neq A_1B_1$) and 2^4 hidden variables of λ, and the six rows of $\mathbf{M}$ come from three measurement settings of $X'Y'$ and two conditions of

η. Equation (14) implies that ξ belongs to the kernel of $\mathbf{M}$ denoted as ker($\mathbf{M}$). All six row vectors of $\mathbf{M}$ are mutually orthogonal so that the dimension of ker($\mathbf{M}$), or the nullity of $\mathbf{M}$ is 42. Note that the MI corresponds to $\xi = 0$, which forms a zero-dimensional kernel. Even if MI is relaxed, i.e., $\xi \neq \mathbf{0}$, the Bell inequalities could remain intact as long as $\xi \in \ker(\mathbf{M})$ is satisfied (see Sec. VI for a simple example of CMD).

We show that if a measurement-dependent model satisfies the Bell's inequality with a bound C, it also satisfies the CMD condition. Using Eq. (11), the weighted combination of the generalized correlation is written as

$$\sum_{i,j=1,2} w_{ij} E_{A_iB_j}(A_i, B_j) = C + \gamma, \tag{15}$$

where

$$\gamma = \sum_{(i,j)\neq(1,1)} w_{ij} \sum_{(X,Y)\neq(A_1,B_1)} \sum_{\lambda} M^{A_iB_j\eta=1}_{XY\lambda}\xi_{XY\lambda} \equiv \mathbf{w}\cdot\tilde{\xi}. \tag{16}$$

Here $\mathbf{w} = (w_{12}, w_{21}, w_{22})^T$, and $\tilde{\xi} = \mathbf{M}^{\eta=1}\xi = \mathbf{M}^{\eta=1}\xi^{\perp}$ with $\xi^{\perp} = \xi - \xi^{\parallel}$ and $\xi^{\parallel} \in \ker(\mathbf{M})$ according to $\mathbf{M}^{\eta=1}\xi^{\parallel} = 0$. The maximum increase of the bound is given by

$$\gamma_M = \sup_{\mathbf{w}\in\mathbb{R}^3} \mathbf{w}\cdot\mathbf{M}^{\eta=1}\xi^{\perp}, \tag{17}$$

which is determined by $\xi^{\perp}$. It is worth noting that the classical bound defined as $\min\{2 + 3\mathcal{M}, 4\}$ with

$$\mathcal{M} \equiv \sup_{X,X',Y,Y'} \sum_{\lambda} |\xi_{XY\lambda} - \xi_{X'Y'\lambda}| \tag{18}$$

has been proposed in Ref. [22], but this differs from γ_M.

According to Eq. (16) the fact that the Bell inequalities are satisfied implies $\gamma = 0$, or equivalently

$$\mathbf{w}\cdot\mathbf{M}^{\eta=1}\xi = \mathbf{0}. \tag{19}$$

As far as every Bell inequality represented by all possible $\mathbf{w}$ are concerned, Eq. (19) is fulfilled if and only if $\mathbf{M}^{\eta=1}\xi = \mathbf{0}$. Together with the normalization condition $\mathbf{M}^{\eta=2}\xi = \mathbf{0}$, we reach $\mathbf{M}\xi = \mathbf{0}$. Therefore, we conclude that any measurement-dependent model satisfying the Bell inequalities satisfies the CMD condition, while the converse proposition is given by the definition of the CMD condition. Therefore, the sets of the hidden-variable models satisfying CMD is equal to the set satisfying the Bell inequalities.

In a similar way, we analyze no-signaling conditions. We set the measurement settings of Alice and Bob as (A_i, B_j) $(i, j = 1, 2)$. Using $P(\alpha|X, Y) = [1 + \alpha E_{XY}(X)]/2$ in the no-signaling condition [Eq. (9)] we obtain the following set of equations:

$$\mathbf{N}\xi \equiv \sum_{(X,Y)\neq(A_1,B_1)} \sum_{\lambda} N^{j\eta}_{XY\lambda}\xi_{XY\lambda} = \mathbf{0}, \tag{20}$$

where $N^{j\eta=1}_{XY\lambda} = \alpha(\lambda, X)\delta_{XA_j}(\delta_{YB_1} - \delta_{YB_2})$ and $N^{j\eta=2}_{XY\lambda} = \beta(\lambda, Y)(\delta_{XA_1} - \delta_{XA_2})\delta_{YB_j}$ for no signaling, and $N^{j\eta=3}_{XY\lambda} = \delta_{XA_j}\delta_{YB_1}$ and $N^{j\eta=4}_{XY\lambda} = \delta_{XA_j}\delta_{YB_2}$ for normalization. Taking $j\eta$ as the row index and $XY\lambda$ as the column index, $\mathbf{N}$ is represented by a 7×48-dimensional matrix. All the row vectors of $\mathbf{N}$ are mutually orthogonal, similarly in the CMD matrix $\mathbf{M}$, so that ker($\mathbf{N}$) is a 41-dimensional subspace.

Our results can be summarized by using the kernels of $\mathbf{M}$ and $\mathbf{N}$ as follows:

(S1) If and only if $\xi \in \ker(\mathbf{M})$ and $\xi \in \ker(\mathbf{N})$, there exists a MI deterministic hidden-variable model.

(S2) If $\xi \in \ker(\mathbf{M})$, there exists a CMD model. Furthermore, if $\xi \in \ker(\mathbf{M})$ but $\xi \notin \ker(\mathbf{N})$, no MI deterministic hidden-variable model exists.

(S3) If $\xi \notin \ker(\mathbf{M})$, there exists a measurement-dependent model satisfying the Bell inequalities with the increased classical bound determined by $\xi^{\perp}$.

(S4) If $\xi \notin \ker(\mathbf{M})$ and $\xi \in \ker(\mathbf{N})$, there exists a measurement-dependent and no-signaling model satisfying the Bell inequalities with the increased classical bound determined similarly in (S3).

VI. A SIMPLE EXAMPLE: CONCEALED MEASUREMENT DEPENDENCE IN THE BELL INEQUALITY

The original Bell inequality [1] is expressed as

$$|E(A, B) - E(A, C)| \leqslant 1 + E(B, C), \tag{21}$$

where $E(A, B)$ denotes the average of the spin correlation when Alice and Bob measure the spin of the correlated particles along the directions A and B, respectively. Alice and Bob are separated in space and the particles are perfectly anticorrelated to form the singlet state. $E(B, C)$ and $E(A, C)$ are similarly defined with the directions B and C and A and C, respectively.

CMD is defined as

$$E_{AB}(B, C) = E_{BC}(B, C), \quad E_{AB}(A, C) = E_{AC}(A, C). \tag{22}$$

We consider only a perfect anticorrelation case and assume determinism and locality. In this case, Table I can represent all possible hidden variable distributions.

The probabilities of the hidden variables are represented as P_i, P'_i, and P''_i for the settings $\{A, B\}$, $\{B, C\}$, and $\{A, C\}$, respectively. Without loss of generality, P'_i and P''_i can be expressed as $P_i + x_i$ and $P_i + y_i$ with appropriate x_i and y_i, respectively. x_i and y_i should satisfy the normalization

TABLE I. The probability distributions of the outcomes α and β in terms of the hidden variable λ for three different settings. The state is anticorrelated so that $\alpha\beta = -1$ for the same measurement direction.

	α(Alice)			β(Bob)					
λ	A	B	C	A	B	C	$P(\lambda_i\|A,B)$	$P(\lambda_i\|B,C)$	$P(\lambda_i\|A,C)$
λ_1	1	1	1	−1	−1	−1	P_1	P'_1	P''_1
λ_2	1	1	−1	−1	−1	1	P_2	P'_2	P''_2
λ_3	1	−1	1	−1	1	−1	P_3	P'_3	P''_3
λ_4	1	−1	−1	−1	1	1	P_4	P'_4	P''_4
λ_5	−1	1	1	1	−1	−1	P_5	P'_5	P''_5
λ_6	−1	1	−1	1	−1	1	P_6	P'_6	P''_6
λ_7	−1	−1	1	1	1	−1	P_7	P'_7	P''_7
λ_8	−1	−1	−1	1	1	1	P_8	P'_8	P''_8

$\sum_i x_i = \sum_i y_i = 0$. Together with the normalization conditions, the two CMD conditions are expressed as

$$\begin{pmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 \end{pmatrix} \vec{\xi} = \vec{0},$$

with $\vec{\xi} = (x_1, x_2, \dots, x_8, y_1, \dots, y_8)$, $\vec{0} = (0, 0, \dots, 0)$, and the 4×16 matrix, which is called $\mathbf{M}_B$ corresponds to $\mathbf{M}$ in the CHSH inequality. $\ker(\mathbf{M}_B)$ is 12-dimensional since all row vectors are orthogonal to each other. It means that the CMD is constructed in a 12-dimensional space of $\vec{\xi}$, represented as $\ker(\mathbf{M}_B)$. Note that the conventional MI forms a zero-dimensional space of $\vec{\xi}$ since it is achieved by $\vec{\xi} = \vec{0}$. Even if the MI is relaxed so that $\vec{\xi}$ no longer lies at $\vec{0}$, the Bell inequality is still valid if $\vec{\xi}$ lies on $\ker(\mathbf{M}_B)$.

VII. CONCLUSION

In conclusion, we have shown that the Bell inequalities still can be valid even if MI is relaxed. The necessary and sufficient condition for satisfying the Bell inequalities is given as CMD if both determinism and locality are assumed. The MI assumption is too strict for a local deterministic hidden variable model satisfying the Bell inequality. We also find that our CMD models may violate the no-signaling condition even if locality is assumed. In fact, the CMD model satisfying no signaling is not equivalent to the MI model. Furthermore, we characterized the CMD condition by introducing parameters describing measurement dependence of a model, and we obtained how the classical bound varies depending on the parameters.

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