Criteria to detect genuine multipartite entanglement using spin measurements

R. Y. Teh and M. D. Reid
Centre for Quantum and Optical Science Swinburne, University of Technology, Melbourne, Australia

(Received 24 April 2019; published 27 August 2019)

We derive conditions in the form of inequalities to detect the genuine N-partite entanglement of N systems. The inequalities are expressed in terms of variances of spin operators and can be tested by local spin measurements performed on the individual systems. Violation of the inequalities is sufficient (but not necessary) to certify the multipartite entanglement and occurs when a type of spin squeezing is created. The inequalities are similar to those derived for continuous-variable systems, but instead are based on the Heisenberg spin-uncertainty relation $\Delta J_x \Delta J_y \geqslant |\langle J_z \rangle|/2$. We also extend previous work to derive spin-variance inequalities that certify the full tripartite inseparability or genuine multipartite entanglement among systems with fixed spin J, as in Greenberger–Horne–Zeilinger (GHZ) states and W states where $J = 1/2$. These inequalities are derived from the planar spin-uncertainty relation $(\Delta J_x)^2 + (\Delta J_y)^2 \geqslant C_J$ where C_J is a constant for each J. Finally, it is shown how the inequalities detect multipartite entanglement based on Stokes operators. We illustrate with experiments that create entanglement shared among separated atomic ensembles, polarization-entangled optical modes, and the clouds of atoms of an expanding spin-squeezed Bose-Einstein condensate. For each example, we give a criterion to certify the mutual entanglement.

DOI: 10.1103/PhysRevA.100.022126

I. INTRODUCTION

Genuine multipartite quantum entanglement is a resource required for many protocols in the field of quantum information and computation [1–9]. N systems are said to be genuinely N-partite entangled if the systems are mutually entangled in such a way that the entanglement cannot be constructed by mixing entangled states involving fewer than N parties [9–11]. Mathematically, a tripartite system is genuinely tripartite entangled if and only if the density operator characterizing the system cannot be represented in the biseparable form [9–12]

$$\rho_{BS} = P_1 \sum_R \eta_R^{(1)} \rho_1^R \rho_{23}^R + P_2 \sum_{R'} \eta_{R'}^{(2)} \rho_2^{R'} \rho_{13}^{R'} + P_3 \sum_{R'} \eta_{R''}^{(3)} \rho_3^{R''} \rho_{12}^{R''}, \tag{1}$$

where $\sum_{k=1}^{3} P_k = 1$, $P_k \geqslant 0$, and $\sum_R \eta_R^{(k)} = 1$. Here ρ_k^R is an arbitrary density operator for the subsystem k, while ρ_{mn}^R is an arbitrary density operator for the subsystems m and n. The definition of genuine N-partite entanglement follows similarly.

Criteria to certify genuine N-partite entanglement for continuous variable (CV) systems have been derived by Shalm *et al.* [13] and Teh and Reid [14]. These criteria take the form of variance inequalities, similar to those derived for CV bipartite entanglement [15–17]. The work of Refs. [13,14] extended earlier results by van Loock and Furusawa, who developed CV criteria for the related but different concept of full N-partite inseparability [18,19] (see also Refs. [20,21]). Although genuine N-partite entanglement implies full N-partite inseparability, the converse is not true, and full N-partite inseparability is therefore a weaker form of correlation. Nonetheless, for *pure* states, full N-partite inseparability is sufficient to imply genuine N-partite entanglement. Experiments have confirmed both full N-partite inseparability [19,22–25] and genuine N-partite entanglement ($N \geqslant 3$) for CV systems [13,26–29]. Here "continuous variable" (CV) refers to the use of measurements that have continuous-variable outcomes, e.g., field quadrature phase amplitudes X and P, or position and momentum. The CV criteria are derived from the commutation relation $[X, P] = 2i$ and the associated uncertainty relations.

In this paper, we derive criteria for genuine N-partite entanglement that can be applied to discrete variable systems involving spin degrees of freedom. In this case, measurements correspond to spin observables, and it is the spin commutation relation $[J_x, J_y] = iJ_z$ and associated uncertainty relations that are relevant. The criteria we derive involve variances and apply to all physical systems, provided the measurements correspond to operators satisfying spin commutation relations. This approach extends to N systems the treatments of Hofmann and Takeuchi [30] and Raymer *et al.* [31] who used spin-uncertainty relations to derive variance criteria for bipartite entanglement. The question of how to detect genuine N-partite entanglement has been studied previously, but most work has been in the context of qubit (spin 1/2) systems [32–42] or systems of fixed dimension [43–48].

The development of criteria to certify the genuine multipartite entanglement of discrete systems, as in this paper, is motivated by the increasing number of experiments detecting entanglement with atoms. For example, bipartite entanglement has been created between atomic ensembles and separated atomic modes [49–51], and multipartite entanglement has been created among the separated clouds of a Bose-Einstein condensate (BEC) [52]. It is sometimes possible to rewrite the spin commutation relation in a form that resembles the position-momentum commutation relation. This is often true where the spin observables are expressed as Schwinger operators and justifies the use of CV entanglement criteria for the spin system in that case. For instance, Julsgaard *et al.* [49] characterize the entanglement in the collective spins between two atomic ensembles using CV criteria. However, as pointed out by Raymer *et al.* [31], this is valid only in a restricted sense and will not give correct results in general. In other words, the complete spin commutation relation should be used in any derivation of criteria certifying the genuine multipartite entanglement of spin systems.

The program of characterizing entanglement in spin systems has been largely motivated by the observation that a spin-squeezed system exhibits quantum correlations among the spin particles. Sørensen *et al.* [53] derived an N-partite entanglement criterion that implies the presence of an N-partite entangled state. Here, an N-partite entangled state is a state that cannot be expressed in the form

$$\rho_S = \sum_R P_R \rho_R^{(1)} \rho_R^{(2)} \cdots \rho_R^{(N)}, \tag{2}$$

where $\sum_R P_R = 1$. A host of criteria [54–59] were subsequently derived to certify the presence of N-partite entanglement in spin systems. However, these criteria rule out only the possibility of N-partite separable states of the form Eq. (2) and not the more general N-partite biseparable states of the form Eq. (1) (as extended to higher N) where all separable bipartitions (and mixtures of them) are considered. Hence they are not criteria for *genuine* N-partite entanglement, where the entanglement is mutually shared among *all* N parties. An exception is the spin-squeezing criteria of Sørensen and Mølmer (and others like it) which imply a genuine k-particle entanglement shared among k particles of an N-particle system ($k \leqslant N$) [60]. Such criteria differ from those derived in this paper, however, being based on collective spin measurements made on the composite system, rather than local measurements made on separated subsystems, and thus cannot directly test nonlocal models (as described in Ref. [61]).

The task of characterizing genuine multipartite entanglement in spin systems was carried out by Korbicz *et al.* [62,63]. Korbicz and co-workers used the positivity of the partial transpose (PPT) criterion or the Peres-Horodecki criterion [12,64] as the starting point to derive entanglement criteria and showed genuine tripartite entanglement for symmetric states. The PPT criterion, however, is less useful for N-partite separability when N is large [12]. In this paper, we derive criteria for genuine multipartite entanglement for spin systems by ruling out the possibility of the state in a biseparable form as in Eq. (1).

The remainder of the paper is structured as follows. In Sec. II we derive criteria for the detection of the genuine tripartite entanglement using spin measurements. The generalization to genuine N-partite entanglement is given in Sec. IV. These criteria are derived using methods similar to those developed by van Loock and Furusawa [18], Shalm *et al.* [13], and Teh and Reid [14] for CV systems. In Sec. III we extend criteria derived by He and Reid [42], pointing out that these inequalities apply to certify genuine tripartite entanglement as well as Einstein-Podolsky-Rosen (EPR) steering, which is a form of entanglement closely connected with the EPR paradox [61,65,66]. The criteria are derived using planar spin-uncertainty relations [67–71] and apply to subsystems with a fixed spin J. We show that the criteria may be used to detect the genuine tripartite entanglement of Greenberger–Horne–Zeilinger (GHZ) states and the full tripartite inseparability of W states. Finally, in Sec. V we explain how to generate genuinely entangled spin systems based on Stokes operators. We then demonstrate using three examples the application of the criteria derived in Secs. II and IV to certify the genuine N-partite entanglement.

II. CRITERIA FOR GENUINE TRIPARTITE ENTANGLEMENT

The criteria derived in this section involve variances of the sum of spin observables defined for each subsystem. These criteria require only the statistics of a set of observables and, in this sense, are state independent. In this work, all the caret symbols that denote the spin operators are dropped, unless specified otherwise, and we use the symbol $\Delta^2 x$ to denote the variance of x.

A. The sum inequalities

1. Sum of two variances

Consider the sum of $\Delta^2 u$ and $\Delta^2 v$ where

$$u = h_1 J_{x,1} + h_2 J_{x,2} + h_3 J_{x,3},$$
$$v = g_1 J_{y,1} + g_2 J_{y,2} + g_3 J_{y,3}, \tag{3}$$

and h_k and g_k ($k = 1, 2, 3$) are real numbers. Here $J_{x,k}$, $J_{y,k}$, $J_{z,k}$ are the spin operators for subsystem k, satisfying the commutation relation $[J_{x,k}, J_{y,k}] = iJ_{z,k}$. We derive the bound for $\Delta^2 u + \Delta^2 v$ such that the violation of the bound implies genuine tripartite entanglement in the spin degree of freedom. This leads us to the following criterion.

Criterion 1. Violation of the inequality

$$\begin{aligned}\Delta^2 u + \Delta^2 v \geqslant \min\{&|g_1 h_1 \langle J_{z,1}\rangle| + |g_2 h_2 \langle J_{z,2}\rangle + g_3 h_3 \langle J_{z,3}\rangle|,\\ &|g_2 h_2 \langle J_{z,2}\rangle| + |g_1 h_1 \langle J_{z,1}\rangle + g_3 h_3 \langle J_{z,3}\rangle|,\\ &|g_3 h_3 \langle J_{z,3}\rangle| + |g_1 h_1 \langle J_{z,1}\rangle + g_2 h_2 \langle J_{z,2}\rangle|\}\end{aligned} \tag{4}$$

is sufficient to confirm genuine tripartite entanglement.

Proof. First, we assume that the spin state is in a biseparable mixture state $\rho_{BS} = P_1 \sum_R \eta_R^{(1)} \rho_1^R \rho_{23}^R + P_2 \sum_{R'} \eta_{R'}^{(2)} \rho_{13}^{R'} \rho_2^{R'} + P_3 \sum_{R''} \eta_{R''}^{(3)} \rho_{12}^{R''} \rho_3^{R''}$ as in Eq. (1). For any mixture of type $\rho_{\text{mix}} = \sum_R P_R \rho^R$, the variance $\Delta^2 u$ satisfies [30]:

$$\Delta^2 u \geqslant \sum_R P_R \Delta^2 u_R. \tag{5}$$

Hence the biseparable mixture would imply

$$\begin{aligned}\Delta^2 u + \Delta^2 v \geqslant\ & P_1 \sum_R \eta_R^{(1)} [\Delta^2 u_R + \Delta^2 v_R]\\ &+ P_2 \sum_{R'} \eta_{R'}^{(2)} [\Delta^2 u_{R'} + \Delta^2 v_{R'}]\\ &+ P_3 \sum_{R''} \eta_{R''}^{(3)} [\Delta^2 u_{R''} + \Delta^2 v_{R''}].\end{aligned} \tag{6}$$

To proceed, we consider $\Delta^2 u_\zeta + \Delta^2 v_\zeta$ that corresponds to an arbitrary bipartition $\rho_k^\zeta \rho_{lm}^\zeta$:

$$\Delta^2 u_\zeta + \Delta^2 v_\zeta \geqslant |g_k h_k \langle J_{z,k}\rangle| + |g_l h_l \langle J_{z,l}\rangle + g_m h_m \langle J_{z,m}\rangle|. \tag{7}$$

The lower bound given in this inequality is derived in the Appendix A 1, using the uncertainty relations for spin. We can always choose for the lower bound the smallest value of $\Delta^2 u_\zeta + \Delta^2 v_\zeta$ in Eq. (6). Hence, Eq. (6) becomes Eq. (4), where we use the fact that $\sum \eta_R^{(1)} = 1$ and $\sum P_k = 1$. In Eq. (4), the first term in the bracket {} is implied by the biseparable state $\rho_1\rho_{23}$, the second term is implied by the biseparable state $\rho_2\rho_{13}$, and the final term is implied by the biseparable state $\rho_3\rho_{12}$. ■

The optimal values for g_k, h_k depend on the specific spin state. The criterion given by Eq. (4) is a general result that allows us to derive a host of other criteria. Examples of optimal choices for different types of spin states will be given in Sec. V.

2. Van Loock-Furusawa inequalities for spin

We can also derive the spin version of a set of inequalities derived by van Loock and Furusawa [18]. The quantities B_I, B_{II}, and B_{III} are defined as

$$\begin{aligned} B_I &\equiv \Delta^2(J_{x,1} - J_{x,2}) + \Delta^2(J_{y,1} + J_{y,2} + g_3 J_{y,3}),\\ B_{II} &\equiv \Delta^2(J_{x,2} - J_{x,3}) + \Delta^2(g_1 J_{y,1} + J_{y,2} + J_{y,3}),\\ B_{III} &\equiv \Delta^2(J_{x,1} - J_{x,3}) + \Delta^2(J_{y,1} + g_2 J_{y,2} + J_{y,3}). \end{aligned} \tag{8}$$

By choosing the coefficients g_k and h_k in Eq. (4), we obtain a set of inequalities satisfied by B_I, B_{II}, and B_{III}. For example, the left side of the criterion in Eq. (4) is equal to B_I when $h_1 = 1$, $h_2 = -1$, $h_3 = 0$ and $g_1 = g_2 = 1$. The set of inequalities is

$$\begin{aligned} &B_I \geqslant (|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle|), \quad B_{II} \geqslant (|\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|),\\ &B_{III} \geqslant (|\langle J_{z,1}\rangle| + |\langle J_{z,3}\rangle|). \end{aligned} \tag{9}$$

We point out that $B_I \geqslant (|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle|)$ is implied by both the biseparable states $\rho_1\rho_{23}$ and $\rho_2\rho_{13}$, while $B_{II} \geqslant (|\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|)$ is implied by the biseparable states $\rho_2\rho_{13}$ and $\rho_3\rho_{12}$. Finally, $B_{III} \geqslant (|\langle J_{z,1}\rangle| + |\langle J_{z,3}\rangle|)$ is satisfied by the biseparable states $\rho_1\rho_{23}$ and $\rho_3\rho_{12}$. Using the inequalities in Eq. (9), we obtain a criterion that confirms genuine tripartite entanglement.

Criterion 2. Full tripartite inseparability is observed if any two of the inequalities (9) are violated. For a pure state, this is sufficient to imply genuine tripartite entanglement. For arbitrary states, genuine tripartite entanglement is observed if the inequality

$$B_I + B_{II} + B_{III} \geqslant |\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle| \tag{10}$$

is violated.

Proof. Full tripartite inseparability is observed if each one of the inequalities (9) is violated, because this certifies entanglement across all bipartitions. Following van Loock and Furusawa [18], in fact we see that tripartite inseparability is confirmed if any two inequalities are violated. This is so because $B_I \geqslant |\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle|$ is implied by $\rho_1\rho_{23}$ and $\rho_2\rho_{13}$, $B_{II} \geqslant |\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|$ is implied by $\rho_2\rho_{13}$ and $\rho_3\rho_{12}$, and $B_{III} \geqslant |\langle J_{z,1}\rangle| + |\langle J_{z,3}\rangle|$ is implied by $\rho_1\rho_{23}$ and $\rho_3\rho_{12}$. For pure states, the proof of full tripartite inseparability confirms genuine tripartite entanglement. Now we prove the second condition that applies to all states including mixed states. For brevity, we index the biseparable states $\rho_1\rho_{23}$, $\rho_2\rho_{13}$, and $\rho_3\rho_{12}$ by $k = 1, 2, 3$, respectively. Let $B_{I,1}$ be the quantity B_I that is evaluated using the biseparable state $\rho_1\rho_{23}$. Then

$$\begin{aligned} B_I &\geqslant \sum_k P_k B_{I,k} \geqslant P_1 B_{I,1} + P_2 B_{I,2}\\ &\geqslant (P_1 + P_2)(|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle|). \end{aligned}$$

Similarly, $B_{II} \geqslant (P_2 + P_3)(|\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|)$ and $B_{III} \geqslant (P_1 + P_3)(|\langle J_{z,1}\rangle| + |\langle J_{z,3}\rangle|)$. In order to include all possible mixtures, we take the sum of B_I, B_{II}, and B_{III} and use the expansion in Eq. (1). The inequality they satisfy, derived below, provides Criterion 2 for genuine tripartite entanglement:

$$\begin{aligned} B_I + B_{II} + B_{III} &\geqslant (P_1 + P_2 + P_3)(|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|)\\ &\quad + P_1|\langle J_{z,1}\rangle| + P_2|\langle J_{z,2}\rangle| + P_3|\langle J_{z,3}\rangle|\\ &\geqslant (P_1 + P_2 + P_3)(|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|)\\ &= (|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|), \end{aligned}$$

where $\sum_k P_k = 1$. ■

The number of moment measurements in the criterion given by Eq. (10) can be reduced by using a criterion that involves only two of the three quantities B_I, B_{II}, and B_{III}. Setting $g_1 = g_2 = g_3 = 1$, we see that the sum

$$B_I + B_{II} \geqslant |\langle J_{z,1}\rangle| + 2|\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle| \tag{11}$$

is satisfied by any mixture of all tripartite biseparable states. The violation of the inequality in Eq. (11) then implies genuine tripartite entanglement. This is also true for other combinations $B_I + B_{III} \geqslant 2|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|$ and $B_{II} + B_{III} \geqslant |\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle| + 2|\langle J_{z,3}\rangle|$.

B. The product inequalities

1. Product of two variances

Criteria involving products rather than sums can also be derived. Again, we consider the two quantities $\Delta^2 u = \Delta^2(h_1 J_{x,1} + h_2 J_{x,2} + h_3 J_{x,3})$ and $\Delta^2 v = \Delta^2(g_1 J_{y,1} + g_2 J_{y,2} + g_3 J_{y,3})$.

Criterion 3. Genuine tripartite entanglement is observed if the inequality

$$\begin{aligned} \Delta u \Delta v \geqslant \frac{1}{2}\min\{&|g_1 h_1 \langle J_{z,1}\rangle| + |g_2 h_2 \langle J_{z,2}\rangle + g_3 h_3 \langle J_{z,3}\rangle|,\\ &|g_2 h_2 \langle J_{z,2}\rangle| + |g_1 h_1 \langle J_{z,1}\rangle + g_3 h_3 \langle J_{z,3}\rangle|,\\ &|g_3 h_3 \langle J_{z,3}\rangle| + |g_1 h_1 \langle J_{z,1}\rangle + g_2 h_2 \langle J_{z,2}\rangle|\} \end{aligned} \tag{12}$$

is violated.

Proof. The product of two variances $\Delta^2 u$ and $\Delta^2 v$ satisfies the inequality

$$\begin{aligned} \Delta^2 u \Delta^2 v &\geqslant \left[\sum_R P_R \Delta^2 u_R\right]\left[\sum_R P_R \Delta^2 v_R\right]\\ &\geqslant \sum_R P_R \Delta^2 u_R \Delta^2 v_R, \end{aligned} \tag{13}$$

where the Cauchy-Schwarz inequality is used. For an arbitrary bipartition $\rho_k^{\zeta}\rho_{lm}^{\zeta}$, $\Delta^2 u_\zeta \Delta^2 v_\zeta$ satisfies the inequality (see Appendix A 2)

$$\Delta^2 u_\zeta \Delta^2 v_\zeta \geqslant \frac{1}{4}[|g_k h_k \langle J_{z,k}\rangle| + |g_l h_l \langle J_{z,l}\rangle + g_m h_m \langle J_{z,m}\rangle|]^2. \tag{14}$$

Identical to the proof for Criterion 1, we can always choose the bipartition that gives us the smallest value of $\Delta u_\zeta \Delta v_\zeta$ in Eq. (13). Hence, Eq. (13) becomes (12). ■

2. Van Loock-Furusawa product inequalities

The product version of the van Loock-Furusawa inequalities can be obtained, using the criterion in Eq. (12). The quantities involved are S_I, S_{II}, and S_{III}:

$$\begin{aligned} S_I &\equiv \Delta(J_{x,1} - J_{x,2})\Delta(J_{y,1} + J_{y,2} + g_3 J_{y,3}), \\ S_{II} &\equiv \Delta(J_{x,2} - J_{x,3})\Delta(g_1 J_{y,1} + J_{y,2} + J_{y,3}), \\ S_{III} &\equiv \Delta(J_{x,1} - J_{x,3})\Delta(J_{y,1} + g_2 J_{y,2} + J_{y,3}). \end{aligned} \tag{15}$$

By choosing the coefficients g_i and h_i in Eq. (12), we obtain a set of inequalities satisfied by S_I, S_{II}, and S_{III}. For example, the left side of the criterion in Eq. (12) is equal to S_I when $h_1 = 1$, $h_2 = -1$, $h_3 = 0$ and $g_1 = g_2 = 1$. From Eq. (12), S_I, S_{II}, and S_{III} satisfy the following inequalities:

$$\begin{aligned} S_I &\geqslant \tfrac{1}{2}(|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle|), \\ S_{II} &\geqslant \tfrac{1}{2}(|\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|), \\ S_{III} &\geqslant \tfrac{1}{2}(|\langle J_{z,1}\rangle| + |\langle J_{z,3}\rangle|). \end{aligned} \tag{16}$$

Criterion 4. Full tripartite inseparability is observed if any two of the inequalities (16) are violated. Genuine tripartite entanglement is present if the following inequality is violated:

$$S_I + S_{II} + S_{III} \geqslant \tfrac{1}{2}(|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|). \tag{17}$$

Proof. The first result follows as for Criterion 2. Using the same notation as in the proof for Criterion 2, we index the biseparable states $\rho_1\rho_{23}$, $\rho_2\rho_{13}$, and $\rho_3\rho_{12}$ by $k = 1, 2, 3$, respectively. Let $S_{I,1}$ be the quantity S_I that is evaluated using the biseparable state $\rho_1\rho_{23}$. Then

$$\begin{aligned} S_I &\geqslant \sum_k P_k S_{I,k} \geqslant P_1 S_{I,1} + P_2 S_{I,2} \\ &\geqslant \frac{1}{2}(P_1 + P_2)(|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle|). \end{aligned}$$

Similarly, $S_{II} \geqslant (P_2 + P_3)(|\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|)/2$ and $S_{III} \geqslant (P_1 + P_3)(|\langle J_{z,1}\rangle| + |\langle J_{z,3}\rangle|)/2$. In order to include all possible mixtures, we take the sum of S_I, S_{II}, and S_{III}. The inequality they satisfy, derived below, provides a criterion for genuine tripartite entanglement:

$$\begin{aligned} S_I + S_{II} + S_{III} &\geqslant \frac{(P_1 + P_2 + P_3)(|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|)}{2} \\ &\quad + \frac{1}{2}(P_1|\langle J_{z,1}\rangle| + P_2|\langle J_{z,2}\rangle| + P_3|\langle J_{z,3}\rangle|) \\ &\geqslant \frac{(P_1 + P_2 + P_3)(|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|)}{2} \\ &= \frac{1}{2}(|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|), \end{aligned}$$

where $\sum_k P_k = 1$. ■

III. INEQUALITIES INVOLVING PLANAR SPIN UNCERTAINTY RELATIONS

The inequalities in the previous two sections used the canonical spin uncertainty relations. For certain quantum states such as the multipartite spin GHZ state, the right side of these inequalities might be zero, giving the trivial relation that a sum or product of variances should be positive. Here we consider the planar uncertainty relation, where the sum of uncertainties in two of the orthogonal spin observables has a lower bound that is a function of the spin value of the state. The planar uncertainty relation was obtained for spin $J = 1/2$ [72] and $J = 1$ [30] and was later calculated for an arbitrary spin J by He *et al.* [67]. In that work, they minimized $\Delta^2 J_x + \Delta^2 J_y$ for a general quantum state written in the spin-z basis as

$$|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{m=-J}^{J} R_m e^{-i\phi_m} |J, m\rangle. \tag{18}$$

Here R_m, ϕ_m are real numbers characterizing the amplitude and phase of the basis state $|J, m\rangle$, while n is the normalization factor given by $n = \sum_{m=-J}^{J} R_m^2$. He *et al.* found the lower bound C_J ($C_J > 0$) such that for a given J,

$$\Delta^2 J_x + \Delta^2 J_y \geqslant C_J. \tag{19}$$

Also in that work [67], a criterion that verifies N-partite inseparability was derived. Since the total N-partite separable state is a probabilistic sum of the tensor product of N density operators, the planar uncertainty relation can be used. This is not so straightforward for genuine multipartite entanglement where a biseparable state contains partitions that cannot be expressed as a product state of those particles or modes in those partitions.

Nevertheless, the planar uncertainty relation can be used to detect genuine tripartite entanglement, if we use an inference variance method [15,73].

Criterion 5. Consider the inequality given by

$$B_1 + B_2 + B_3 \geqslant C_J, \tag{20}$$

where

$$\begin{aligned} B_1 &= \Delta^2\big(J_{x,1} - O_{23}^{(1)}\big) + \Delta^2\big(J_{y,1} - P_{23}^{(1)}\big), \\ B_2 &= \Delta^2\big(J_{x,2} - O_{13}^{(2)}\big) + \Delta^2\big(J_{y,2} - P_{13}^{(2)}\big), \\ B_3 &= \Delta^2\big(J_{x,3} - O_{12}^{(3)}\big) + \Delta^2\big(J_{y,3} - P_{12}^{(3)}\big), \end{aligned}$$

and $O_{lm}^{(k)}$, $P_{lm}^{(k)}$ are observables defining measurements that can be made on the combined subsystems that we denote by l and m. The violation of this inequality suffices to confirm genuine tripartite entanglement of the three systems denoted 1, 2, and 3. Full tripartite inseparability is observed if

$$B_k \geqslant C_J \tag{21}$$

for each $k = 1, 2, 3$.

Proof. Consider $\Delta^2(J_{x,1} - O_{23}^{(1)})$ and $\Delta^2(J_{y,1} - P_{23}^{(1)})$ where $O_{23}^{(1)}$ and $P_{23}^{(1)}$ are operators for systems 2 and 3. We derive the following inequality that holds for an arbitrary pure state with a separable bipartition $\rho_1^{\zeta}\rho_{23}^{\zeta}$:

$$\begin{aligned} B_1 &= \Delta^2\big(J_{x,1} - O_{23}^{(1)}\big) + \Delta^2\big(J_{y,1} - P_{23}^{(1)}\big) \\ &\geqslant \Delta^2(J_{x,1}) + \Delta^2(J_{y,1}) \geqslant C_J. \end{aligned} \tag{22}$$

This holds also for all mixtures of separable bipartitions $\rho_1^\zeta \rho_{23}^\zeta$. Similarly, the inequalities

$$B_2 \geqslant \Delta^2\big(J_{x,2} - O_{13}^{(2)}\big) + \Delta^2\big(J_{y,2} - P_{13}^{(2)}\big) \geqslant C_J \tag{23}$$

and

$$B_3 \geqslant \Delta^2\big(J_{x,3} - O_{12}^{(3)}\big) + \Delta^2\big(J_{y,3} - P_{12}^{(3)}\big) \geqslant C_J \tag{24}$$

follow from the separable bipartitions $\rho_2^\zeta \rho_{13}^\zeta$ and $\rho_3^\zeta \rho_{12}^\zeta$, respectively. For a pure state, if all three inequalities are violated, we can conclude that the three systems are genuinely tripartite entangled. For a mixed state the conditions change. We require falsifying an arbitrary biseparable mixed state given by $\rho_{BS} = P_1 \sum_R \eta_R^{(1)} \rho_1^R \rho_{23}^R + P_2 \sum_{R'} \eta_{R'}^{(2)} \rho_2^{R'} \rho_{13}^{R'} + P_3 \sum_{R''} \eta_{R''}^{(3)} \rho_3^{R''} \rho_{12}^{R''}$, as defined by Eq. (1). We give a proof similar to those given for Criteria 2 and 4. For brevity, we index the biseparable states $\sum_R \eta_R^{(1)} \rho_1^R \rho_{23}^R$, $\sum_{R'} \eta_{R'}^{(2)} \rho_2^{R'} \rho_{13}^{R'}$, and $\sum_{R''} \eta_{R''}^{(3)} \rho_3^{R''} \rho_{12}^{R''}$ by $k = 1, 2, 3$, respectively. Thus, we denote $B_{1,1}$ to be the quantity B_1 that is evaluated using the biseparable state $\sum_R \eta_R^{(1)} \rho_1^R \rho_{23}^R$. Then, for the biseparable mixture,

$$B_1 \geqslant \sum_k P_k B_{1,k} \geqslant P_1 B_{1,1} \geqslant P_1 C_J.$$

Similarly, for a biseparable mixture, $B_2 \geqslant P_2 C_J$ and $B_3 \geqslant P_3 C_J$. In order to include all possible biseparable mixtures, we consider

$$B_1 + B_2 + B_3 \geqslant (P_1 + P_2 + P_3) C_J = C_J$$

using $\sum_k P_k = 1$. Thus, all biseparable mixtures are excluded when this inequality is violated. ■

This inequality has been derived in Ref. [42] in a similar context to give a condition for genuine tripartite steering. Steering is a form of entanglement linked to the Einstein-Podolsky-Rosen paradox, and hence a steering criterion will also be a criterion for entanglement [65]. The entanglement criterion might be made stronger, if one can make use of uncertainty relations for the operators $O_{lm}^{(k)}$ and $P_{lm}^{(k)}$ once these are established for a given scenario.

It is straightforward to see that the inequality is violated for the GHZ state [74], defined as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle), \tag{25}$$

where $|\uparrow\uparrow\uparrow\rangle$ ($|\downarrow\downarrow\downarrow\rangle$) is the state with z-spins up (down) for all subsystems $k = 1, 2, 3$. This is because, as is well known for the GHZ state, the z-spin, x-spin, and y-spin of any of the three subsystems can be inferred by joint measurements made on the other two subsystems. This result is clear for inferring the value of $J_{z,k}$. The inequality (20) applies for all spin pairs, and if we replace $J_{y,i}$ with $J_{z,i}$, it is clear that by taking $P_{lm}^{(k)} = J_z^{(l)}$, one can achieve $\Delta^2(J_{z,k} - P_{lm}^{(k)}) = 0$ for each k. For inferring $J_{x,k}$, it is also clear, since the GHZ state is an eigenstate of $J_{x,1} J_{x,2} J_{x,3}$ with eigenvalue -1. Thus, $O_{lm}^{(k)}$ is the measurement given as follows: Measure the spin J_x of each of the other subsystems l and m, and assign the value of the measurement by multiplying the spins values together. If the product is $+1$, then the outcome of $O_{lm}^{(k)}$ is -1. If the product is -1, then the outcome of $O_{lm}^{(k)}$ is $+1$. In this way, we see that $\Delta^2(J_{x,k} - O_{lm}^{(k)}) = 0$, for each $k = 1, 2, 3$ with $l \neq m \neq k$. Hence, the inequality (20) is violated, giving a simple method to detect the genuine tripartite entanglement of GHZ states (or approximate GHZ states) in an experiment.

We may ask whether the inequality is also violated for the W state [75] given by

$$|W\rangle = \frac{1}{\sqrt{3}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle). \tag{26}$$

Here we will use the criterion expressed in Pauli spins, so that $B_i = \Delta^2(\sigma_{z,i} - O_{jk}^{(1)}) + \Delta^2(\sigma_{x,i} - P_{jk}^{(1)})$ where $i \neq j \neq k$. The conditions then utilize $C_J = 1$ since $J = 1/2$ [72]. The spin σ_z of system 1 can be inferred by measuring the spin product of 2 and 3. We find that $\Delta^2(\sigma_{z,1} - O_{23}^{(1)}) = 0$. Now consider that the spins σ_x of systems 2 and 3 are simultaneously measured. We consider the measurement $P_{23}^{(1)}$ to have an outcome of 1 if both spins are measured as $+1$, an outcome -1 if the spins are measured as -1, and zero otherwise. Simple calculation tells us that $\Delta^2(\sigma_{x,1} - P_{23}^{(1)}) = \frac{1}{2}$. By symmetry of the W state, this result holds for all permutations of the subsystems. Thus we see that we are able to confirm entanglement across each bipartition, since the condition (22) for Pauli spins reduces to $B_1 \geqslant 1$. Since we find $B_1 = B_2 = B_3 = \frac{1}{2}$, the condition for tripartite inseparability is satisfied. If in an experiment we are able to verify a pure state, then this implies genuine tripartite entanglement. We note the above condition for mixed states, $B_1 + B_2 + B_3 < 1$ is not satisfied. The W state (26) *is* genuinely tripartite entangled. That the condition is not satisfied merely reflects that the criteria we derive are sufficient, but not necessary, to certify genuine tripartite entanglement.

Svetlichny derived conditions to detect the genuine tripartite entanglement of three spin 1/2 systems in the form of Bell inequalities [32]. Further criteria for the certification of the genuine tripartite entanglement of GHZ, W and cluster states have been derived in Refs. [20,35,63]. The method given above is not necessarily advantageous over these earlier methods. It can be readily extended [by applying uncertainty relation (19)], however, to conditions for higher J.

IV. CRITERIA FOR GENUINE *N*-PARTITE ENTANGLEMENT

The method used in Sec. II to derive criteria for genuine tripartite entanglement can be extended to N-partite systems. The complication arises in that the set of possible bipartitions scales as $(2^{N-1} - 1)$, and every bipartition has to be taken into account in the derivation of these criteria that certify genuine N-partite entanglement.

Here, we generalize the criterion in Eq. (4) for N-partite spin systems.

Criterion 6. We denote each bipartition by $S_r - S_s$, where S_r and S_s are two sets of modes in the partitions in a specific bipartition. Then the violation of the inequality

$$\Delta^2 u + \Delta^2 v \geqslant \min\{S_B\} \tag{27}$$

implies genuine N-partite entanglement, where S_B is the set of values of the quantity

$(|\sum_{k_r=1}^{m} h_{k_r} g_{k_r} \langle J_{z,k_r}\rangle| + |\sum_{k_s=1}^{n} h_{k_s} g_{k_s} \langle J_{z,k_s}\rangle|)$ defined for each partition $S_r - S_s$, the indices k_r and k_s summing over the m modes in S_r and the n modes in set S_s respectively. The proof for this inequality follows from the proof for the inequality in Eq. (4).

Criterion 7. Similarly, the violation of the corresponding product inequality

$$\Delta u \Delta v \geqslant \tfrac{1}{2}\min\{S_B\} \tag{28}$$

implies genuine N-partite entanglement.

A. Criteria for genuine four-partite entanglement

1. Sum and product inequalities

Criterion 8. For $N = 4$, there will be $2^{4-1} - 1 = 7$ bipartitions. They are, using the $S_r - S_s$ notation, $1-234$, $2-134$, $3-124$, $4-123$, $12-34$, $13-24$, and $14-23$. The sum inequality in Eq. (27) is then

$$\begin{aligned}\Delta^2 u + \Delta^2 v \geqslant \min\{&|g_1h_1\langle J_{z,1}\rangle| + |g_2h_2\langle J_{z,2}\rangle + g_3h_3\langle J_{z,3}\rangle + g_4h_4\langle J_{z,4}\rangle|,\\ &|g_2h_2\langle J_{z,2}\rangle| + |g_1h_1\langle J_{z,1}\rangle + g_3h_3\langle J_{z,3}\rangle + g_4h_4\langle J_{z,4}\rangle|,\\ &|g_3h_3\langle J_{z,3}\rangle| + |g_1h_1\langle J_{z,1}\rangle + g_2h_2\langle J_{z,2}\rangle + g_4h_4\langle J_{z,4}\rangle|,\\ &|g_4h_4\langle J_{z,4}\rangle| + |g_1h_1\langle J_{z,1}\rangle + g_2h_2\langle J_{z,2}\rangle + g_3h_3\langle J_{z,3}\rangle|,\\ &|g_1h_1\langle J_{z,1}\rangle + g_2h_2\langle J_{z,2}\rangle| + |g_3h_3\langle J_{z,3}\rangle + g_4h_4\langle J_{z,4}\rangle|,\\ &|g_1h_1\langle J_{z,1}\rangle + g_3h_3\langle J_{z,3}\rangle| + |g_2h_2\langle J_{z,2}\rangle + g_4h_4\langle J_{z,4}\rangle|,\\ &|g_1h_1\langle J_{z,1}\rangle + g_4h_4\langle J_{z,4}\rangle| + |g_2h_2\langle J_{z,2}\rangle + g_3h_3\langle J_{z,3}\rangle|\} \equiv \min\{S_{B,4}\}.\end{aligned} \tag{29}$$

Criterion 9. Similarly, the product inequality for genuine four-partite entanglement is given by

$$\Delta u \Delta v \geqslant \tfrac{1}{2}\min\{S_{B,4}\}, \tag{30}$$

where $S_{B,4}$ is defined in Eq. (29). The violation of the inequality in Eq. (29) or Eq. (30) implies the presence of genuine four-partite entanglement.

2. Criteria involving van Loock-Furusawa inequalities

Van Loock and Furusawa [18] derived a set of six inequalities to rule out four-partite inseparability. We can derive similar inequalities to certify genuine four-partite entanglement. The six spin inequalities are given by

$$\begin{aligned}B_I &\equiv \Delta^2(J_{x,1} - J_{x,2}) + \Delta^2(J_{y,1} + J_{y,2} + g_3J_{y,3} + g_4J_{y,4}) \geqslant (|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle|),\\ B_{II} &\equiv \Delta^2(J_{x,2} - J_{x,3}) + \Delta^2(g_1J_{y,1} + J_{y,2} + J_{y,3} + g_4J_{y,4}) \geqslant (|\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|),\\ B_{III} &\equiv \Delta^2(J_{x,1} - J_{x,3}) + \Delta^2(J_{y,1} + g_2J_{y,2} + J_{y,3} + g_4J_{y,4}) \geqslant (|\langle J_{z,1}\rangle| + |\langle J_{z,3}\rangle|),\\ B_{IV} &\equiv \Delta^2(J_{x,3} - J_{x,4}) + \Delta^2(g_1J_{y,1} + g_2J_{y,2} + J_{y,3} + J_{y,4}) \geqslant (|\langle J_{z,3}\rangle| + |\langle J_{z,4}\rangle|),\\ B_V &\equiv \Delta^2(J_{x,2} - J_{x,4}) + \Delta^2(g_1J_{y,1} + J_{y,2} + g_3J_{y,3} + J_{y,4}) \geqslant (|\langle J_{z,2}\rangle| + |\langle J_{z,4}\rangle|),\\ B_{VI} &\equiv \Delta^2(J_{x,1} - J_{x,4}) + \Delta^2(J_{y,1} + g_2J_{y,2} + g_3J_{y,3} + J_{y,4}) \geqslant (|\langle J_{z,1}\rangle| + |\langle J_{z,4}\rangle|).\end{aligned} \tag{31}$$

Criterion 10. The violation of any three of the above inequalities implies that the four-partite system is not in any biseparable states and thus signifies four-partite inseparability (refer Ref. [18] for the proof). Genuine four-partite entanglement is verified if the inequality

$$\sum_{J=1}^{6} B_J \geqslant |\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle| + |\langle J_{z,4}\rangle| \tag{32}$$

is violated. These criteria are sufficient but not necessary conditions for four-partite inseparability or genuine four-partite entanglement.

Proof. For brevity, we index the biseparable states $\rho_1\rho_{234}$, $\rho_2\rho_{134}$, $\rho_3\rho_{124}$, $\rho_4\rho_{123}$, $\rho_{12}\rho_{34}$, $\rho_{13}\rho_{24}$, and $\rho_{14}\rho_{23}$ by $k = 1, 2, \ldots, 7$, respectively. Let $B_{I,1}$ be the quantity B_I that is evaluated using the biseparable state $\rho_1\rho_{234}$. Then

$$B_I \geqslant \sum_k P_k B_{I,k} \geqslant P_1B_{I,1} + P_2B_{I,2} + P_6B_{I,6} + P_7B_{I,7} \geqslant (P_1 + P_2 + P_6 + P_7)(|\langle J_{z,1}\rangle| + |\langle J_{z,2}\rangle|). \tag{33}$$

Similarly, $B_{II} \geqslant (P_2 + P_3 + P_5 + P_6)(|\langle J_{z,2}\rangle| + |\langle J_{z,3}\rangle|)$, $B_{III} \geqslant (P_1 + P_3 + P_5 + P_7)(|\langle J_{z,1}\rangle| + |\langle J_{z,3}\rangle|)$, $B_{IV} \geqslant (P_3 + P_4 + P_6 + P_7)(|\langle J_{z,3}\rangle| + |\langle J_{z,4}\rangle|)$, $B_V \geqslant (P_2 + P_4 + P_5 + P_7)(|\langle J_{z,2}\rangle| + |\langle J_{z,4}\rangle|)$, and $B_{VI} \geqslant (P_1 + P_4 + P_5 + P_6)(|\langle J_{z,1}\rangle| + |\langle J_{z,4}\rangle|)$. In order to include all possible mixtures, we take the sum of B_I, B_{II}, B_{III}, B_{IV}, B_V, and B_{VI}. The inequality they satisfy, derived below, provides a criterion for genuine four-partite entanglement. The violation of the following inequality implies genuine four-partite entanglement:

$$\sum_{J=1}^{6} B_J \geqslant (P_1+P_2+P_3+P_4+P_5+P_6+P_7)(|\langle J_{z,1}\rangle|+|\langle J_{z,2}\rangle|+|\langle J_{z,3}\rangle|+|\langle J_{z,4}\rangle|)+(2P_1+P_5+P_6+P_7)|\langle J_{z,1}\rangle| \\ +(2P_2+P_5+P_6+P_7)|\langle J_{z,2}\rangle|+(2P_3+P_5+P_6+P_7)|\langle J_{z,3}\rangle|+(2P_4+P_5+P_6+P_7)|\langle J_{z,4}\rangle| \\ \geqslant (P_1+P_2+P_3+P_4+P_5+P_6+P_7)(|\langle J_{z,1}\rangle|+|\langle J_{z,2}\rangle|+|\langle J_{z,3}\rangle|+|\langle J_{z,4}\rangle|)=|\langle J_{z,1}\rangle|+|\langle J_{z,2}\rangle|+|\langle J_{z,3}\rangle|+|\langle J_{z,4}\rangle|, \tag{34}$$

where $\sum_k P_k = 1$. ■

V. APPLICATIONS

We now show how one may create N-partite entangled states satisfying the criteria derived in Secs. II and IV. In Sec. V A we outline optical experiments involving polarization entanglement, where the measured observables at each site are the Stokes operators for two polarization modes. We then consider in Sec. V B experiments that entangle spatially separated atomic ensembles. In Sec. V C we analyze recent experiments that generate entanglement between spatially separated clouds of atoms formed from a spin-squeezed Bose-Einstein condensate. Here, for each separated subsystem, the measured observable is a Schwinger operator involving two internal atomic levels. The Schwinger and Stokes operators satisfy the same commutation relation as spin operators, and hence all the criteria derived in Secs. II–IV are applicable.

A. Polarization entanglement

The polarization of a quantum state can be characterized by the Stokes operators defined as [76]

$$S_x = a_H^\dagger a_H - a_V^\dagger a_V, \quad S_y = a_H^\dagger a_V e^{i\theta} + a_V^\dagger a_H e^{-i\theta}, \\ S_z = i a_V^\dagger a_H e^{-i\theta} - i a_H^\dagger a_V e^{i\theta}, \tag{35}$$

where a_H and a_V are the annihilation operators of the horizontal and vertical polarization modes, respectively, and θ is the phase difference between these polarization modes. In the work of Bowen *et al.* [76], bipartite polarization entanglement was created by first generating CV bipartite entanglement in the quadrature degree of freedom, and then transferring the entanglement into the polarization degree of freedom.

This scheme can be extended to generate genuine tripartite polarization entanglement. Genuine CV tripartite entanglement in the quadratures is first created in an optical setup involving squeezed vacuums and beam splitters, as shown in Figs. 1 and 2. The three entangled modes from the outputs of these beam splitters are horizontally polarized. Each of these modes is subsequently mixed with a bright coherent beam with vertical polarization using a polarizing beam splitter. At each site $i = 1, 2, 3$ prior to mixing, one can define pairs of orthogonally polarized modes (with annihilation operators a_{Hi}, $a_{V,i}$). The choice of polarizer angle determines which Stokes observable is measured, after a number difference is taken. The final readout is given as a difference current. After the mixing, the genuine CV entanglement has been transformed into genuine tripartite polarization entanglement, as illustrated in Fig. 3.

To verify the tripartite polarization entanglement, we consider the sum inequality of Criterion 1 (Eq. (4)):

$$\Delta^2[S_{y,1}+h(S_{y,2}+S_{y,3})]+\Delta^2[S_{z,1}+g(S_{z,2}+S_{z,3})] \\ \geqslant 2\min\{\alpha_v^2+2|gh|\alpha_v^2, |gh|\alpha_v^2+\alpha_v^2|1+gh|\}, \tag{36}$$

where α_v is the coherent amplitude of the vertically polarized coherent beam. The variances are

$$\Delta^2[S_{y,1}+h(S_{y,2}+S_{y,3})]=\alpha_v^2\Delta^2[P_{H,1}+h(P_{H,2}+P_{H,3})], \\ \Delta^2[S_{z,1}+g(S_{z,2}+S_{z,3})]=\alpha_v^2\Delta^2[X_{H,1}+g(X_{H,2}+X_{H,3})]. \tag{37}$$

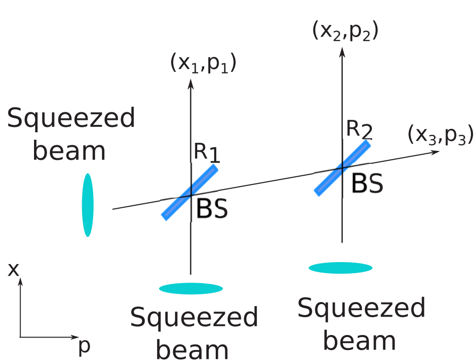

FIG. 1. Generation of the tripartite-entangled CV Greenberger–Horne–Zeilinger (GHZ) state. The configuration uses three squeezed-vacuum inputs and two beam splitters (BS) with reflectivities $R_1 = 1/3$ and $R_2 = 1/2$. The x_i and p_i are the two orthogonal quadrature-phase amplitudes of the spatially separated optical modes i ($i = 1, 2, 3$).

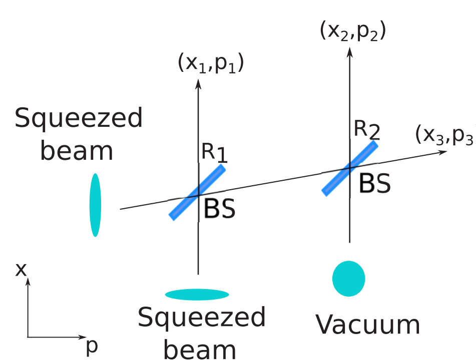

FIG. 2. Generation of the tripartite entangled CV Einstein–Podolsky–Rosen (EPR)-type state. The configuration uses two squeezed-vacuum inputs, a coherent-vacuum input, and two beam splitters (BS) with reflectivities $R_1 = R_2 = 1/2$. x_i and p_i are the two orthogonal quadrature-phase amplitudes of the spatially separated optical modes i ($i = 1, 2, 3$).

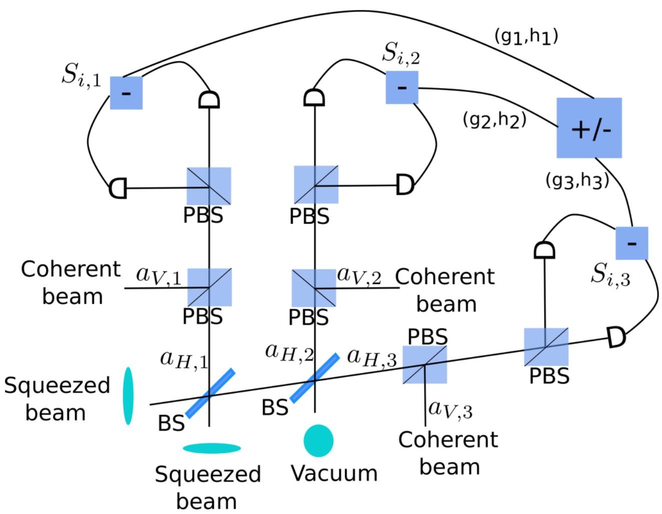

FIG. 3. The experimental setup to generate genuine tripartite polarization entanglement from genuine tripartite CV entanglement. In this schematic diagram, an EPR-type genuine tripartite-entanglement is generated as shown in Fig. 2. The outputs are mixed with coherent fields, as described in the text. The $S_{i,k}$ denotes the polarization $S_{x,k}$, $S_{y,k}$, or $S_{z,k}$ for the site k ($k = 1, 2, 3$). The (g_k, h_k) are the gains used in the criteria (1 and 3) and are introduced in the final currents. By using a third squeezed input state at the second beam splitter instead of the vacuum input, the CV GHZ genuine tripartite entanglement (refer to Fig. 1) can be transformed into an equivalent genuine tripartite polarization entanglement. Alternatively, by using only one squeezed input, one can transfer the genuine tripartite entanglement depicted in Fig. 4.

Here $S_{x,k}$, $S_{y,k}$, and $S_{z,k}$ are the Stokes operators defined in (35) for each mode pair at site k. $X_{H,k}$ and $P_{H,k}$ are the X and P quadratures for beam k, and h and g are gain factors defined in the criteria of Eq. (4), where we take $h_1 = 1$, $h_2 = h_3 = h$ and $g_1 = 1$, $g_2 = g_3 = g$. Note that the commutation relations satisfied by these Stokes operators are $[S_i, S_j] = 2i\varepsilon_{ijk}S_k$, which differ from the spin commutation relations by a factor of 2. As a result, the sum and product inequalities below have an extra factor of 2 compared to the sum and product inequalities in Eqs. (4) and (12), respectively. With these variances (Eq. (37)), the sum inequality Eq. (4) and the product inequality Eq. (12) are, respectively, transformed into a continuous-variable genuine tripartite entanglement sum and product criterion given in Ref. [14], according to

$$\frac{\Delta^2[S_{y,1} + h(S_{y,2} + S_{y,3})] + \Delta^2[S_{z,1} + g(S_{z,2} + S_{z,3})]}{2\alpha_v^2\min\{1 + 2|gh|, |gh| + |1 + gh|\}}$$
$$= \frac{\Delta^2[X_{H,1} + g(X_{H,2} + X_{H,3})] + \Delta^2[P_{H,1} + h(P_{H,2} + P_{H,3})]}{2\min\{1 + 2|gh|, |gh| + |1 + gh|\}}$$
$$\geqslant 1 \tag{38}$$

and

$$\frac{\Delta[S_{y,1} + h(S_{y,2} + S_{y,3})]\Delta[S_{z,1} + g(S_{z,2} + S_{z,3})]}{\min\{\alpha_v^2 + 2|gh|\alpha_v^2, |gh|\alpha_v^2 + \alpha_v^2|1 + gh|\}}$$
$$= \frac{\Delta[X_{H,1} + g(X_{H,2} + X_{H,3})]\Delta[P_{H,1} + h(P_{H,2} + P_{H,3})]}{\min\{1 + 2|gh|, |gh| + |1 + gh|\}}$$
$$\geqslant 1. \tag{39}$$

TABLE I. Values of the gains g and h that minimize the variance-sum and variance-product terms in Criteria 1 and 3 for each configuration, CV GHZ and CV EPR. Here, r is the squeezing parameter. Each squeezed vacuum input has a quadrature variance of $\Delta x = e^{\mp r}$ and $\Delta p = e^{\pm r}$ (the sign depends on the orientation of the squeezing). For simplicity, we take all the squeezed vacuum inputs to have the same squeezing strength r. The CV GHZ and CV EPR configurations are depicted in Figs. 1 and 2 respectively.

	CV GHZ		CV EPR	
r	g	h	g	h
0	0	0	0	0
0.25	0.36	−0.27	0.33	−0.33
0.50	0.68	−0.40	0.54	−0.54
0.75	0.86	−0.46	0.64	−0.64
1.00	0.95	−0.49	0.68	−0.68
1.50	0.99	−0.50	0.70	−0.70
2.00	1.00	−0.50	0.70	−0.70

Hence, any CV genuine tripartite quadrature entanglement of the original fields then implies genuine tripartite polarization entanglement of the final output fields.

There are two types of states that show genuine tripartite entanglement in the quadratures. These are the CV GHZ- and CV EPR-type states, defined in Refs. [18] and [14], and illustrated in Figs. 1 and 2, respectively. It has been shown previously that these two states violate both the quadrature sum inequality of Eq. (38) and the product inequality of Eq. (39) with specific values for the gains, $g_1 = h_1 = 1$ and $g_{i>1} = g$, $h_{i>1} = h$ [14]. The gains g, h are chosen such that the variance sum and product are minima, and are given in Table I. With these gain values, as shown in Ref. [14], inequalities (38) and (39) (and hence Inequalities (4) and (12) of Criteria 1 and 3) are always violated for any nonzero squeezing of the squeezed vacuum inputs, implying the presence of genuine tripartite entanglement. By the same transformation of quadrature into polarization entanglement, the inequalities of Criteria 2 and 4 are also useful in showing genuine tripartite entanglement. The optimal gains for these inequalities can be found in Ref. [14].

Genuine tripartite entanglement is also created using a third configuration involving only one squeezed input, as shown in Fig. 4. Often, strong Einstein–Podolsky–Rosen (EPR) correlations are created between output modes by combining two squeezed vacuum inputs across a beam splitter [19,73]. It is also possible to create EPR-entangled modes using only one squeezed vacuum input [15]. While the EPR correlations are weaker, the entanglement is sufficiently strong that a subsequent beam-splitter interaction with a nonsqueezed vacuum input can create genuine tripartite entanglement. A summary of this calculation is given in Appendix A 3, where we show how the entanglement that is generated can be detected by Criterion 5 of Ref. [14] with the gains $h = -1/2$ and $g = 1/2$. This tripartite entanglement is not sufficiently strong to generate tripartite EPR-steering correlations, but can be transformed into genuine tripartite polarization-entanglement using the configuration of Fig. 3. The spin sum inequality given by Criterion 1 is then useful to detect the genuine tripartite entanglement.

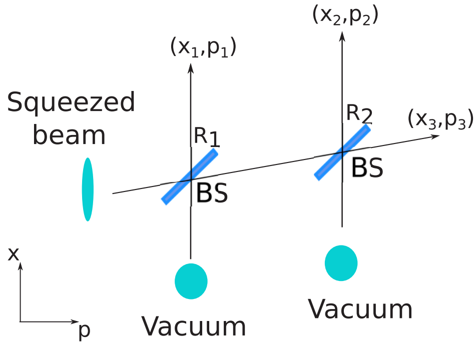

FIG. 4. Generation of tripartite entanglement using a squeezed vacuum beam with squeezing along the P (or X) quadrature. All other beam-splitter ports have vacuum inputs. The reflectivities for the first and second beam splitters are $R_1 = 1/3$ and $R_2 = 1/2$, respectively. A calculation of the genuine tripartite entanglement generated from this configuration is given in Appendix A 3.

B. Tripartite entanglement of atomic ensembles

Tripartite entanglement can also be created among three atomic ensembles by successively passing polarized light through the ensembles. Here we outline a generalization of the scheme of Julsgaard *et al.* that creates bipartite entanglement between two atomic ensembles [49]. The observables for the atomic ensembles are the collective Schwinger spins defined by the following operators:

$$J_x = \tfrac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-), \quad J_y = \tfrac{1}{2}(a_+^\dagger a_- e^{i\theta} + a_-^\dagger a_+ e^{-i\theta}),$$
$$J_z = \tfrac{1}{2}(i a_-^\dagger a_+ e^{-i\theta} - i a_+^\dagger a_- e^{i\theta}), \tag{40}$$

which satisfy the commutation relations $[J_i, J_j] = i\varepsilon_{ijk} J_k$. Here a_+, a_- are the (destruction) boson operators for atomic states corresponding to "spin-up" and "spin-down" along the spin-x axis, respectively. We label the operators for each ensemble by the subscript k ($k = 1, 2, 3$).

First, three atomic ensembles are prepared such that the mean collective spins for these atomic ensembles are pointing along the x axis: $J_{x1} = -2J_{x2} = -2J_{x3} = J_x$. Following [49], an off-resonant polarized pulse of light (described by the Stokes operators of Eq. (35)) is transmitted through the atomic ensembles. The light-spin interaction is given by the Hamiltonian $H_{\rm int} = \omega S_z J_{z123}$, where $J_{z123} = J_{z1} + J_{z2} + J_{z3}$. The light variable then evolves in terms of the inputs to give an output of

$$S_y^{\rm out} = S_y^{\rm in} + \alpha J_{z123}, \tag{41}$$

while the spin variables evolve as

$$J_{y1}^{\rm out} = J_{y1}^{\rm in} + \beta S_z, \quad J_{y2}^{\rm out} = J_{y2}^{\rm in} - \tfrac{1}{2}\beta S_z, \quad J_{y2}^{\rm out} = J_{y2}^{\rm in} - \tfrac{1}{2}\beta S_z. \tag{42}$$

By measuring $S_y^{\rm out}$, $J_{z1} + J_{z2} + J_{z3}$ can be inferred. Also, $J_{y1} + J_{y2} + J_{y3}$ can be measured using another light pulse without affecting the measured value of $J_{z1} + J_{z2} + J_{z3}$. This is possible because $[J_{z1} + J_{z2} + J_{z3}, J_{y1} + J_{y2} + J_{y3}] = 0$. Hence, the quantity $\Delta^2(J_{z1} + J_{z2} + J_{z3}) + \Delta^2(J_{y1} + J_{y2} + J_{y3})$ can be arbitrarily small. Using the sum inequality Eq. (4) and product inequality Eq. (12) with gain values $g_i = h_i = 1$, $(i = 1, 2, 3)$, a genuine tripartite entanglement is certified among the atomic ensembles if $\Delta^2(J_{z1} + J_{z2} + J_{z3}) + \Delta^2(J_{y1} + J_{y2} + J_{y3}) < 2J_x$ for the sum inequality and $\Delta(J_{z1} + J_{z2} + J_{z3})\Delta(J_{y1} + J_{y2} + J_{y3}) < J_x$ for the product inequality.

C. Entangled Bose-Einstein condensate clouds

In the experiment of Kunkel *et al.* [52], a ^{87}Rb Bose-Einstein condensate is first generated in the magnetic substate $m_F = 0$ of the $F = 1$ hyperfine manifold, before a spin-squeezing operation coherently populates the $m_F = \pm 1$ atomic states and entangles all the atoms in the condensate. The condensate is then released from the trap and expands during a time-of-flight. After a time, N spatially separated regions (partitions) of the atomic cloud are identified. The N-partite entanglement among these partitions is verified by measuring $F_{0,k}$ and $F_{\pi/2,k}$ for each partition k, where $F_{\phi,k} = [(a_{+1}^\dagger + a_{-1}^\dagger)a_0 e^{i\phi} + \text{H.c.}]/\sqrt{2}$, $a_j^\dagger$ is the creation operator for a state $m_F = j$ (H.c. refers to hermitian conjugate). These operators satisfy the commutation relation $[F_{0,k}, F_{\pi/2,k}] = 2i\hat{N}_k$, where $\hat{N}_k$ is the number operator for the partition k [52]. By applying $\pi/2$ pulses and rotations, these observables are measured by reading out the population difference between the states $m_F = \pm 1$. If the number of atoms in group $m_F = 0$ is large, then the measurement becomes similar to a homodyne detection of the amplitudes $[(a_{+1}^\dagger + a_{-1}^\dagger)e^{i\phi} + \text{H.c.}]$ associated with the atoms of each of the partitions, carried out with the second larger group of atoms (given by a_0) acting as the local oscillator, as explained in Refs. [77,78]. More generally, spin relations must be used. In the atomic experiment of Kunkel *et al.*, the genuine N-partite entanglement (up to $N = 5$) mutually shared among the N clouds is certified using criteria similar to that derived in Ref. [14], for quadrature phase amplitudes, but properly accounting for the spin and number operators that apply in this case.

In another experiment based on the two hyperfine states $|1\rangle = |F = 1, m_F = -1\rangle$ and $|2\rangle = |F = 2, m_F = 1\rangle$ of a ^{87}Rb BEC, Fadel *et al.* [50] prepare the system in an atomic spin-squeezed state and allow the condensate to expand into two well-separated partitions (which we denote A and B). This creates a bipartite entanglement between the two clouds, which is detected using the entanglement criterion [17,50]:

$$\Delta(g_z S_{z,A} + S_{z,B})\Delta(g_y S_{y,A} + S_{y,B}) < \tfrac{1}{2}(|g_z g_y||\langle S_{x,A}\rangle| + |\langle S_{x,B}\rangle|). \tag{43}$$

Here, $S_{z,A/B}$ and $S_{y,A/B}$ are the collective Schwinger spin operators [79,80] along the z axis and y axis, respectively, for partition A/B. Explicitly the collective spin operator $S_{z,A/B}$ is given as the number difference

$$S_{z,A/B} = \tfrac{1}{2}\big(N_{z,A/B}^1 - N_{z,A/B}^2\big), \tag{44}$$

where $N_{z,A/B}^1$ and $N_{z,A/B}^2$ are the number of atoms in the internal spin states $|1\rangle$ and $|2\rangle$, respectively, along the spin z axis, for partition A/B. The collective spin operators along the y axis $S_{y,A/B}$ are defined accordingly following Eq. (40), but noting the switching of the labels x, y, z. Other proposals exist to create a similar bipartite entanglement that can be detected using a similar spin criterion [81–83].

The experiment of Fadel *et al.* observed bipartite entanglement and EPR steering but did not investigate tripartite entanglement. It is likely, however, that one could detect a genuine tripartite entanglement for clouds generated by further splitting the BEC. This would seem possible, given the result obtained in the Appendix A 3 and depicted in Fig. 4, where tripartite entanglement is generated using only one squeezed input, followed by a sequence of "splitting" of the modes using beam-splitter interactions. This works, because entangled modes can be created from a beam splitter with only one squeezed vacuum input [15]. The tripartite entanglement created in the three modes of Fig. 4 can be detected using Criterion 5 of Ref. [14] with the gains $h = -1/2$ and $g = 1/2$. If one considers transforming into an equivalent tripartite entanglement in the Schwinger operators, then the suitable criterion would be Criterion 3 in Eq. (12) with the gains $h = -1/2$ and $g = 1/2$.

A realization of a beam-splitter interaction for the BEC can be obtained in several ways. An analogy of optical beam splitters with the splitting of a condensate (which is envisaged to be a realization of the final beam splitter of Fig. 4) is explained by Killoran *et al.* [84]. The splitting into two modes is described by the interaction Hamiltonian

$$H_{I+} = e^{i\phi}a_{+1}^{\dagger}a_{+2} + e^{-i\phi}a_{+1}a_{+2}^{\dagger}, \tag{45}$$

where a_{+1}, a_{+2} are the annihilation operators for modes labeled $A_{+,1}$ and $A_{+,2}$ respectively, and ϕ is the phase difference between these two modes. The transformation is equivalent to the beam splitter relations

$$a_{+1,\text{out}} = a_{+1}\cos\tau - ie^{i\phi}a_{+2}\sin\tau,$$
$$a_{+2,\text{out}} = a_{+2}\cos\tau - ie^{-i\phi}a_{+1}\sin\tau, \tag{46}$$

where τ is the interaction time and $a_{+1,\text{out}} = a_{+1}(\tau)$, $a_{+2,\text{out}} = a_{+2}(\tau)$. One can adjust the effective transmission to reflection ratio by adjusting the interaction time between the two modes.

We thus consider two separated clouds A and B that show spin entanglement with respect to the difference operators $g_zS_{z,A} + S_{z,B}$ and $g_yS_{y,A} + S_{y,B}$ so that the criterion of Eq. (43) is satisfied. These two clouds are analogous to the entangled outputs after the first beam splitter BS of the configuration shown in Fig. 4. Each cloud is identified with Schwinger spin observables. For example, $S_{z,A}$ and $S_{y,A}$ are measurements that can be made on cloud A, where $S_{z,A} = \frac{1}{2}(a_+^{\dagger}a_+ - a_-^{\dagger}a_-)$ and a_+, a_- correspond to the two atomic hyperfine states, of atoms in the cloud A. To generate the tripartite entanglement, the system A is transformed (split into two) according to a beam-splitter interaction modeled as Eq. (45). Since the splitting is insensitive to the internal spin degrees of freedom, there is a similar independent interaction for a_-. We denote operators for the outputs (given according to the transformation (46)) by $a_{\pm1}$ and $a_{\pm2}$, dropping the subscript "out" for simplicity. The output fields $a_{\pm1}$ and $a_{\pm2}$, associated respectively with $A1$ and $A2$, are spatially separated, so that three separate clouds are created, labeled $A1$, $A2$, and B, these being analogous to the three outputs of the configuration of Fig. 4. The final Schwinger operators at A_1 and A_2 are defined by the $a_{\pm1}$ at A_1, and the $a_{\pm2}$ at A_2. The different Schwinger components can be measured using Rabi rotations or equivalent [50,78]. The calculation carried out in Appendix A 5 predicts a tripartite entanglement between the three clouds that could be detected by Criteria 1 and 3. Using Eqs. (A8) and (A9) in Appendix A 5, the inequality of Criterion 3 is then

$$\Delta[g_z(S_{z,A1} + S_{z,A2}) + S_{z,B}]\Delta[g_y(S_{y,A1} + S_{y,A2}) + S_{y,B}]$$
$$\geqslant \frac{1}{2}\min\{|g_zg_y||\langle S_{x,A1}\rangle + \langle S_{x,A2}\rangle| + |\langle S_{x,B}\rangle|,\ |(g_zg_y\langle S_{x,A1}\rangle + \langle S_{x,B}\rangle)| + |g_zg_y||\langle S_{x,A2}\rangle|\}. \tag{47}$$

The violation of this inequality implies genuine tripartite entanglement. We show in Appendix A 5 that, assuming the number of atoms is large, $S_{z,A1} + S_{z,A2} \approx S_{z,A}$, $S_{y,A1} + S_{y,A2} \approx S_{y,A}$, and $S_{x,A1} + S_{x,A2} \approx S_{x,A}$. The criterion for genuine tripartite entanglement will therefore be satisfied if there is sufficient entanglement as measured by the bipartite criterion given in Eq. (43). Assuming $S_{x,A}$ and $S_{x,B}$ correspond to the Bloch vectors, with the directions of axes being chosen to ensure $\langle S_{x,A}\rangle$ and $\langle S_{x,B}\rangle$ are positive, we see that the beam-splitter transformation (refer to Appendix A 5) ensures the signs of $S_{x,A1}$ and $S_{x,A2}$ are also positive. The right-side of the inequality is then either precisely that given by the right-side of Eq. (43) (if $g_zg_y > 0$) or is less than this value (if $g_zg_y < 0$).

We note from the results reported in Refs. [14,18,24] that we can generate N-partite entangled states ($N > 3$) by successive use of beam splitters with vacuum inputs, once an initial entangled state is created from two squeezed inputs or some other means. This has been implemented for a BEC (for $N = 5$) [52]. We show in Appendix A 4 that one can also create genuinely four-partite entangled states from a single squeezed input (refer to Fig. 5), followed by multiple beam-splitter combinations and vacuum inputs (with no squeezing). This may provide an avenue (using successive splittings) for the generation of multipartite entanglement in experiments such as reported in Ref. [50].

VI. CONCLUSION

In summary, we have derived several different criteria for certifying genuine N-partite entanglement using spin measurements. The criteria are inequalities expressed in terms of variances of spin observables measured at each of the N sites.

In Secs. II and IV, we derive criteria based on the standard spin uncertainty relation, involving $|\langle J_z\rangle|$. These criteria are valid for any systems, provided at each site the outcomes are reported faithfully, as results of accurately calibrated quantum measurements [9,85]. We present in Sec. V three examples of application of these criteria. In these examples,

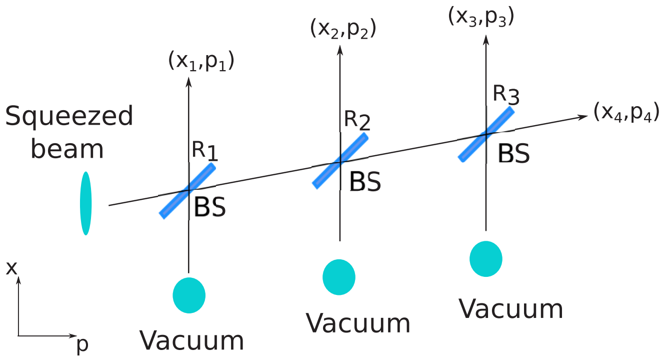

FIG. 5. Generation of four-partite entanglement using a squeezed beam along the P (or X) quadrature. All other beam-splitter ports have vacuum inputs. The reflectivities for the first, second, and third beam splitters are $R_1 = 1/4$, $R_2 = 1/3$, and $R_3 = 1/2$, respectively.

entanglement is created that can be detected using Stokes or Schwinger operators defined at each site. These observables arise naturally in atomic ensembles, where the creation and detection of multipartite entanglement is important for testing the quantum mechanics of massive systems. The criteria we develop may be useful for this purpose. In particular, we specifically propose how to extend the experiments of Julsgaard *et al.* [49] and Fadel *et al.* [50], to generate three or more genuinely entangled spatially-separated ensembles of atoms. The experiment of Kunkel *et al.* [52] succeeded in generating genuine five-partite entanglement.

Where Stokes operators are defined for atomic systems, it is possible to introduce a normalization with respect to total atom number. This concept was introduced by He *et al.* [68,81] and Żukowski *et al.* [86–89]. These authors show how the detection of entanglement and nonlocality can be enhanced using such a normalization. It is likely that the criteria derived in Secs. II and IV may also be further improved using this technique.

In Sec. III we outlined criteria derived from the planar spin uncertainty relation $\Delta^2 J_x + \Delta^2 J_y \geqslant C_J$ valid for a system of fixed spin J. This is useful for states where $\langle J_z \rangle = 0$, such as the GHZ states. Such criteria were developed previously for genuine tripartite steering. Although genuine tripartite steering implies genuine tripartite entanglement, we have extended the results of the earlier work by giving details of the application of these criteria to certify the genuine tripartite entanglement and the full tripartite inseparability of the GHZ and W states, respectively. While other methods exist to detect the genuine tripartite entanglement of these states (for example, Refs. [33,35,63]), the criteria we present in Sec. III are readily extended to higher spin J.

ACKNOWLEDGMENTS

This research has been supported by the Australian Research Council Discovery Project Grants schemes under Grant No. DP180102470. This work was performed in part at Aspen Center for Physics, which is supported by National Science Foundation Grant No. PHY-1607611.

APPENDIX

1. Lower bound of the sum inequality for an arbitrary bipartition

We derive the inequality in Eq. (7) for an arbitrary pure state bipartition $\rho_k^\zeta \rho_{lm}^\zeta$:

$$\begin{aligned}\Delta^2 u_\zeta + \Delta^2 v_\zeta &= \Delta^2(h_k J_{x,k}) + \Delta^2(h_l J_{x,l} + h_m J_{x,m}) + \Delta^2(g_k J_{y,k}) + \Delta^2(g_l J_{y,l} + g_m J_{y,m})\\ &\geqslant |g_k h_k [J_{x,k}, J_{y,k}]| + |g_l h_l [J_{x,l}, J_{y,l}] + g_m h_m [J_{x,m}, J_{y,m}]|\\ &= |g_k h_k \langle J_{z,k}\rangle| + |g_l h_l \langle J_{z,l}\rangle + g_m h_m \langle J_{z,m}\rangle|. \end{aligned} \tag{A1}$$

Here the uncertainty relation $\Delta^2(hJ_x) + \Delta^2(gJ_y) \geqslant \langle |gh[J_x, J_y]| \rangle$ is used to obtain the first inequality in Eq. (A1). The spin commutation relation $[J_x, J_y] = iJ_z$ is used in the last line.

2. Lower bound of the product inequality for an arbitrary bipartition

We derive the inequality in Eq. (14) for an arbitrary bipartition $\rho_k^\zeta \rho_{lm}^\zeta$:

$$\begin{aligned}\Delta^2 u_\zeta \Delta^2 v_\zeta &= [\Delta^2(h_k J_{x,k}) + \Delta^2(h_l J_{x,l} + h_m J_{x,m})][\Delta^2(g_k J_{y,k}) + \Delta^2(g_l J_{y,l} + g_m J_{y,m})]\\ &= \Delta^2(h_k J_{x,k})\Delta^2(g_k J_{y,k}) + \Delta^2(h_l J_{x,l} + h_m J_{x,m})\Delta^2(g_l J_{y,l} + g_m J_{y,m})\\ &\quad + \Delta^2(h_l J_{x,l} + h_m J_{x,m})\Delta^2(g_k J_{y,k}) + \Delta^2(h_k J_{x,k})\Delta^2(g_l J_{y,l} + g_m J_{y,m})\\ &\geqslant \Delta^2(h_k J_{x,k})\Delta^2(g_k J_{y,k}) + \Delta^2(h_l J_{x,l} + h_m J_{x,m})\Delta^2(g_l J_{y,l} + g_m J_{y,m})\\ &\quad + 2\Delta(h_l J_{x,l} + h_m J_{x,m})\Delta(g_k J_{y,k})\Delta(h_k J_{x,k})\Delta(g_l J_{y,l} + g_m J_{y,m})\\ &= [\Delta(h_k J_{x,k})\Delta(g_k J_{y,k}) + \Delta(h_l J_{x,l} + h_m J_{x,m})\Delta(g_l J_{y,l} + g_m J_{y,m})]^2\\ &\geqslant \left[\frac{|g_k h_k \langle J_{z,k}\rangle|}{2} + \frac{|g_l h_l \langle J_{z,l}\rangle + g_m h_m \langle J_{z,m}\rangle|}{2}\right]^2. \end{aligned} \tag{A2}$$

In going from the second equality to the first inequality, the inequality for two real numbers x and y, $x^2 + y^2 \geqslant 2xy$, is used. The uncertainty relation in the final line is $\Delta(hJ_x)\Delta(gJ_y) \geqslant \langle|gh[J_x, J_y]|\rangle/2$.

3. Generating genuine tripartite entangled states using three beam splitters and one squeezed input

Here we consider the configuration of Fig. 4. The output mode operators $a_{\rm out}$, $b_{\rm out}$, and $c_{\rm out}$ are

$$a_{\rm out} = \frac{1}{\sqrt{3}}a_{\rm in} + \sqrt{\frac{2}{3}}b_{\rm in}, \quad b_{\rm out} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{3}}a_{\rm in} - \frac{1}{\sqrt{3}}b_{\rm in}\right) + \frac{1}{\sqrt{2}}c_{\rm in}, \quad c_{\rm out} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{3}}a_{\rm in} - \frac{1}{\sqrt{3}}b_{\rm in}\right) - \frac{1}{\sqrt{2}}c_{\rm in}. \tag{A3}$$

Now we consider $X_{a,\rm out} - (X_{b,\rm out} + X_{c,\rm out})/2$ and $P_{a,\rm out} + (P_{b,\rm out} + P_{c,\rm out})/2$. Their variances are

$$\Delta^2\left[X_{a,\rm out} - \frac{(X_{b,\rm out} + X_{c,\rm out})}{2}\right] = \frac{3}{2}\Delta^2 X_{b,\rm in} = \frac{3}{2}, \quad \Delta^2\left[P_{a,\rm out} + \frac{(P_{b,\rm out} + P_{c,\rm out})}{2}\right] = \frac{2}{3}\Delta^2 P_{a,\rm in} + \frac{1}{6}\Delta^2 P_{b,\rm in} = \frac{2}{3}e^{-2r} + \frac{1}{6}, \tag{A4}$$

and their sum is

$$\Delta^2\left[X_{a,\rm out} - \frac{(X_{b,\rm out} + X_{c,\rm out})}{2}\right] + \Delta^2\left[P_{a,\rm out} + \frac{(P_{b,\rm out} + P_{c,\rm out})}{2}\right] = \frac{10}{6} + \frac{2}{3}e^{-2r}, \tag{A5}$$

giving a minimum of $10/6 = 1.6667$ for large squeezing parameter r. The sum inequality for those variances is $\Delta^2[X_{a,\rm out} - (X_{b,\rm out} + X_{c,\rm out})/2] + \Delta^2[P_{a,\rm out} + (P_{b,\rm out} + P_{c,\rm out})/2] \geqslant 2$, as shown in Criterion 5 of Ref. [14] with the gains $h = -1/2$ and $g = 1/2$. This inequality is violated, and hence the final output state in Fig. 4 is genuinely tripartite entangled. We can also consider the input to be squeezed along X, in which case the gains g and h will have an opposite sign.

4. Generating genuine four-partite entangled states using four beam splitters and one squeezed input

Here we consider the configuration of Fig. 5. The output mode operators $a_{\rm out}$, $b_{\rm out}$, $c_{\rm out}$, and $d_{\rm out}$ are

$$a_{\rm out} = \frac{1}{\sqrt{4}}a_{\rm in} + \sqrt{\frac{3}{4}}b_{\rm in}, \quad b_{\rm out} = \frac{1}{\sqrt{3}}\left(\sqrt{\frac{3}{4}}a_{\rm in} - \frac{1}{\sqrt{4}}b_{\rm in}\right) + \sqrt{\frac{2}{3}}c_{\rm in}, \quad c_{\rm out} = \frac{1}{\sqrt{3}}\left(\sqrt{\frac{3}{4}}a_{\rm in} - \frac{1}{\sqrt{4}}b_{\rm in}\right) - \frac{1}{\sqrt{6}}c_{\rm in} + \frac{1}{\sqrt{2}}d_{\rm in},$$
$$d_{\rm out} = \frac{1}{\sqrt{3}}\left(\sqrt{\frac{3}{4}}a_{\rm in} - \frac{1}{\sqrt{4}}b_{\rm in}\right) - \frac{1}{\sqrt{6}}c_{\rm in} - \frac{1}{\sqrt{2}}d_{\rm in}. \tag{A6}$$

Now we consider $X_{a,\rm out} - (X_{b,\rm out} + X_{c,\rm out} + X_{d,\rm out})/3$ and $P_{a,\rm out} + (P_{b,\rm out} + P_{c,\rm out} + P_{d,\rm out})/3$. Their variances are

$$\Delta^2\left[X_{a,\rm out} - \frac{(X_{b,\rm out} + X_{c,\rm out} + X_{d,\rm out})}{3}\right] = \frac{4}{3}\Delta^2 X_{b,\rm in} = \frac{4}{3},$$
$$\Delta^2\left[P_{a,\rm out} + \frac{(P_{b,\rm out} + P_{c,\rm out} + P_{d,\rm out})}{3}\right] = \Delta^2 P_{a,\rm in} + \frac{1}{3}\Delta^2 P_{b,\rm in} = e^{-2r} + \frac{1}{3}, \tag{A7}$$

and their sum is $5/3 + e^{-2r}$, giving a minimum of $5/3 = 1.6667$ for large squeezing parameter r. The sum inequality for those variances is $\Delta^2[X_{a,\rm out} - (X_{b,\rm out} + X_{c,\rm out} + X_{d,\rm out})/3] + \Delta^2[P_{a,\rm out} + (P_{b,\rm out} + P_{c,\rm out} + P_{d,\rm out})/3] \geqslant 16/9$, as shown in Criterion 8 of Ref. [14] for $N = 4$ and with the gains $h = -1/3$ and $g = 1/3$. This inequality is violated, and hence the final output state in Fig. 5 is genuinely four-partite entangled. We note we can also consider the input to be squeezed along X, in which case the gains g and h will have an opposite sign.

5. Beam splitter operation as a model for splitting BEC clouds

We define the mode operators $a_+ = (a_{+1} - ia_{+2})/\sqrt{2}$ and $a_- = (a_{-1} - ia_{-2})/\sqrt{2}$, and their corresponding auxiliary mode operators $a_{\rm vac+} = (a_{+1} - ia_{+2})/\sqrt{2}$ and $a_{\rm vac-} = (a_{+1} - ia_{+2})/\sqrt{2}$. This allows us to model the splitting of a BEC cloud with the beam-splitter operations where the mode operators a_+, $a_{\rm vac+}$ are the operators for inputs of a beam splitter. Since the different spin species do not interact, the mode operators a_-, $a_{\rm vac-}$ are also the operators for inputs of a beam splitter, these inputs being split independently of the other spin species. With these mode operators, the Schwinger spin operators after splitting are

$$S_{z,A1} = \tfrac{1}{2}(a^\dagger_{+1}a_{+1} - a^\dagger_{-1}a_{-1}) = \tfrac{1}{4}(a^\dagger_+ a_+ - a^\dagger_- a_-) + F(a_{\rm vac+}, a_{\rm vac-}), \tag{A8}$$

$$S_{z,A2} = \tfrac{1}{2}(a^\dagger_{+2}a_{+2} - a^\dagger_{-2}a_{-2}) = \tfrac{1}{4}(a^\dagger_+ a_+ - a^\dagger_- a_-) + G(a_{\rm vac+}, a_{\rm vac-}). \tag{A9}$$

Here we take the orientation of x, y, z so that S_z corresponds to the number difference. $S_{z,A1}$ and $S_{z,A2}$ are the Schwinger spin operators along the z axis for clouds $A1$ and $A2$, respectively, and F and G are terms containing $a_{\rm vac+}$, $a^\dagger_{\rm vac+}$, $a_{\rm vac-}$, $a^\dagger_{\rm vac-}$. Similar Schwinger spin operators along the x and y axes have the same expressions as Eqs. (A8) and (A9), but the spin up and down are relative to their respective axis. From Eqs. (A8) and (A9), we see that $S_{z,A1} + S_{z,A2} = S_{z,A} + F + G \approx S_{z,A}$. Here we assume the terms F and G involving the incoming unoccupied modes can be neglected in the calculation of the variances, relative to the leading terms which come from the incoming modes with a high occupation (the number of atoms being assumed large). Using a similar argument, we consider $S_\theta = \frac{1}{2}(a_+^\dagger a_- e^{i\theta} + a_-^\dagger a_+ e^{-i\theta})$:

$$S_{\theta,A1} = \tfrac{1}{2}(a_{+1}^\dagger a_{-1} e^{i\theta} + a_{-1}^\dagger a_{+1} e^{-i\theta}) = \tfrac{1}{4}(a_+^\dagger a_- e^{i\theta} + a_-^\dagger a_+ e^{-i\theta}) + F(a_{\text{vac}+}, a_{\text{vac}-}),$$

$$S_{\theta,A2} = \tfrac{1}{2}(a_{+2}^\dagger a_{-2} e^{i\theta} + a_{-2}^\dagger a_{+2} e^{-i\theta}) = \tfrac{1}{4}(a_+^\dagger a_- e^{i\theta} + a_-^\dagger a_+ e^{-i\theta}) + G(a_{\text{vac}+}, a_{\text{vac}-}).$$

Thus, for large numbers of atoms, $S_{y,A1} + S_{y,A2} = S_{y,A} + F + G \approx S_{y,A}$ and similarly $S_{x,A1} + S_{x,A2} \approx S_{x,A}$.

[1] D. Gottesman, Phys. Rev. A **54**, 1862 (1996).

[2] S. Bose, V. Vedral, and P. L. Knight, Phys. Rev. A **57**, 822 (1998).

[3] M. Hillery, V. Bužek, and A. Berthiaume, Phys. Rev. A **59**, 1829 (1999).

[4] P. van Loock and S. L. Braunstein, Phys. Rev. Lett. **84**, 3482 (2000).

[5] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).

[6] J. Jing, J. Zhang, Y. Yan, F. Zhao, C. Xie, and K. Peng, Phys. Rev. Lett. **90**, 167903 (2003).

[7] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).

[8] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, Nat. Phys. **5**, 19 (2009).

[9] J.-D. Bancal, N. Gisin, Y.-C. Liang, and S. Pironio, Phys. Rev. Lett. **106**, 250404 (2011).

[10] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **87**, 040401 (2001).

[11] B. Jungnitsch, T. Moroder, and O. Gühne, Phys. Rev. Lett. **106**, 190502 (2011).

[12] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[13] L. K. Shalm, D. R. Hamel, Z. Yan, C. Simon, K. J. Resch, and T. Jennewein, Nat. Phys. **9**, 19 (2012).

[14] R. Y. Teh and M. D. Reid, Phys. Rev. A **90**, 062337 (2014).

[15] M. D. Reid, Phys. Rev. A **40**, 913 (1989).

[16] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).

[17] V. Giovannetti, S. Mancini, D. Vitali, and P. Tombesi, Phys. Rev. A **67**, 022320 (2003).

[18] P. van Loock and A. Furusawa, Phys. Rev. A **67**, 052315 (2003).

[19] T. Aoki, N. Takei, H. Yonezawa, K. Wakui, T. Hiraoka, A. Furusawa, and P. van Loock, Phys. Rev. Lett. **91**, 080404 (2003).

[20] J. Sperling and W. Vogel, Phys. Rev. Lett. **111**, 110503 (2013).

[21] S. Gerke, W. Vogel, and J. Sperling, Phys. Rev. X **8**, 031047 (2018).

[22] A. S. Coelho, F. A. S. Barbosa, K. N. Cassemiro, A. S. Villar, M. Martinelli, and P. Nussenzveig, Science **326**, 823 (2009).

[23] M. Pysher, Y. Miwa, R. Shahrokhshahi, R. Bloomer, and O. Pfister, Phys. Rev. Lett. **107**, 030505 (2011).

[24] S. Armstrong, J.-F. Morizur, J. Janousek, B. Hage, N. Treps, P. K. Lam, and H.-A. Bachor, Nat. Commun. **3**, 1026 (2012).

[25] S. Gerke, J. Sperling, W. Vogel, Y. Cai, J. Roslund, N. Treps, and C. Fabre, Phys. Rev. Lett. **114**, 050501 (2015).

[26] S. Armstrong, M. Wang, R. Y. Teh, Q. Gong, Q. He, J. Janousek, H.-A. Bachor, M. D. Reid, and P. K. Lam, Nat. Phys. **11**, 167 (2015).

[27] M. R. Barros, O. J. Farías, A. Keller, T. Coudreau, P. Milman, and S. P. Walborn, Phys. Rev. A **92**, 022308 (2015).

[28] L. Wu, Z. Yan, Y. Liu, R. Deng, X. Jia, C. Xie, and K. Peng, Appl. Phys. Lett. **108**, 161102 (2016).

[29] C. W. Sandbo Chang, M. Simoen, J. Aumentado, C. Sabín, P. Forn-Díaz, A. M. Vadiraj, F. Quijandría, G. Johansson, I. Fuentes, and C. M. Wilson, Phys. Rev. Appl. **10**, 044019 (2018).

[30] H. F. Hofmann and S. Takeuchi, Phys. Rev. A **68**, 032103 (2003).

[31] M. G. Raymer, A. C. Funk, B. C. Sanders, and H. de Guise, Phys. Rev. A **67**, 052104 (2003).

[32] G. Svetlichny, Phys. Rev. D **35**, 3066 (1987).

[33] D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Phys. Rev. Lett. **88**, 170405 (2002).

[34] M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **92**, 087902 (2004).

[35] G. Tóth and O. Gühne, Phys. Rev. A **72**, 022340 (2005).

[36] P. Lougovski, S. J. van Enk, K. S. Choi, S. B. Papp, H. Deng, and H. J. Kimble, New J. Phys. **11**, 063029 (2009).

[37] S. B. Papp, K. S. Choi, H. Deng, P. Lougovski, S. J. van Enk, and H. J. Kimble, Science **324**, 764 (2009).

[38] J. Lavoie, R. Kaltenbaek, and K. J. Resch, New J. Phys. **11**, 073051 (2009).

[39] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[40] O. Gühne and M. Seevinck, New J. Phys. **12**, 053002 (2010).

[41] H.-X. Lu, J.-Q. Zhao, X.-Q. Wang, and L.-Z. Cao, Phys. Rev. A **84**, 012111 (2011).

[42] Q. Y. He and M. D. Reid, Phys. Rev. Lett. **111**, 250403 (2013).

[43] M. Huber, F. Mintert, A. Gabriel, and B. C. Hiesmayr, Phys. Rev. Lett. **104**, 210501 (2010).

[44] Z.-H. Ma, Z.-H. Chen, J.-L. Chen, C. Spengler, A. Gabriel, and M. Huber, Phys. Rev. A **83**, 062325 (2011).

[45] C. Spengler, M. Huber, A. Gabriel, and B. C. Hiesmayr, Quant. Info. Proc. **12**, 269 (2013).

[46] M. Li, J. Wang, S. Shen, Z. Chen, and S.-M. Fei, Sci. Rep. **7**, 17274 (2017).

[47] B. Regula, M. Piani, M. Cianciaruso, T. R. Bromley, A. Streltsov, and G. Adesso, New J. Phys. **20**, 033012 (2018).

[48] W. Kłobus, W. Laskowski, T. Paterek, M. Wieśniak, and H. Weinfurter, Eur. Phys. J. D **73**, 29 (2019).

[49] B. Julsgaard, A. Kozhekin, and E. S. Polzik, Nature (London) **413**, 400 (2001).

[50] M. Fadel, T. Zibold, B. Décamps, and P. Treutlein, Science **360**, 409 (2018).

[51] K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt, Science **360**, 416 (2018).

[52] P. Kunkel, M. Prüfer, H. Strobel, D. Linnemann, A. Frölian, T. Gasenzer, M. Gärttner, and M. K. Oberthaler, Science **360**, 413 (2018).
[53] A. Sørensen, L. M. Duan, J. I. Cirac, and P. Zoller, Nature (London) **409**, 63 (2001).
[54] G. Tóth, Phys. Rev. A **69**, 052327 (2004).
[55] G. Tóth, C. Knapp, O. Gühne, and H. J. Briegel, Phys. Rev. Lett. **99**, 250405 (2007).
[56] G. Tóth, J. Opt. Soc. Am. B **24**, 275 (2007).
[57] G. Tóth, C. Knapp, O. Gühne, and H. J. Briegel, Phys. Rev. A **79**, 042334 (2009).
[58] G. Vitagliano, P. Hyllus, I. L. Egusquiza, and G. Tóth, Phys. Rev. Lett. **107**, 240502 (2011).
[59] Q. Y. He, P. D. Drummond, and M. D. Reid, Phys. Rev. A **83**, 032120 (2011).
[60] A. S. Sørensen and K. Mølmer, Phys. Rev. Lett. **86**, 4431 (2001).
[61] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Phys. Rev. A **80**, 032112 (2009).
[62] J. K. Korbicz, J. I. Cirac, and M. Lewenstein, Phys. Rev. Lett. **95**, 120502 (2005).
[63] J. K. Korbicz, O. Gühne, M. Lewenstein, H. Häffner, C. F. Roos, and R. Blatt, Phys. Rev. A **74**, 052319 (2006).
[64] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[65] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007).
[66] S. J. Jones, H. M. Wiseman, and A. C. Doherty, Phys. Rev. A **76**, 052116 (2007).
[67] Q. Y. He, S.-G. Peng, P. D. Drummond, and M. D. Reid, Phys. Rev. A **84**, 022107 (2011).
[68] Q. Y. He, T. G. Vaughan, P. D. Drummond, and M. D. Reid, New J. Phys. **14**, 093012 (2012).
[69] G. Vitagliano, I. Apellaniz, M. Kleinmann, B. Lücke, C. Klempt, and G. Tóth, New J. Phys. **19**, 013027 (2017).
[70] G. Vitagliano, G. Colangelo, F. Martin Ciurana, M. W. Mitchell, R. J. Sewell, and G. Tóth, Phys. Rev. A **97**, 020301(R) (2018).
[71] G. Puentes, G. Colangelo, R. J. Sewell, and M. W. Mitchell, New J. Phys. **15**, 103031 (2013).
[72] R. W. Finkel, Phys. Rev. A **35**, 1486 (1987).
[73] M. D. Reid, P. D. Drummond, W. P. Bowen, E. G. Cavalcanti, P. K. Lam, H. A. Bachor, U. L. Andersen, and G. Leuchs, Rev. Mod. Phys. **81**, 1727 (2009).
[74] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[75] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[76] W. P. Bowen, N. Treps, R. Schnabel, and P. K. Lam, Phys. Rev. Lett. **89**, 253601 (2002).
[77] A. J. Ferris, M. K. Olsen, E. G. Cavalcanti, and M. J. Davis, Phys. Rev. A **78**, 060104(R) (2008).
[78] C. Gross, H. Strobel, E. Nicklas, T. Zibold, N. Bar-Gill, G. Kurizki, and M. K. Oberthaler, Nature (London) **480**, 219 (2011).
[79] J. Hu, W. Chen, Z. Vendeiro, H. Zhang, and V. Vuletić, Phys. Rev. A **92**, 063816 (2015).
[80] L. Pezzè, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Rev. Mod. Phys. **90**, 035005 (2018).
[81] Q. Y. He, M. D. Reid, T. G. Vaughan, C. Gross, M. Oberthaler, and P. D. Drummond, Phys. Rev. Lett. **106**, 120405 (2011).
[82] Q. Y. He, P. D. Drummond, M. K. Olsen, and M. D. Reid, Phys. Rev. A **86**, 023626 (2012).
[83] B. Opanchuk, Q. Y. He, M. D. Reid, and P. D. Drummond, Phys. Rev. A **86**, 023625 (2012).
[84] N. Killoran, F. E. S. Steinhoff, and M. B. Plenio, Phys. Rev. Lett. **116**, 080402 (2016).
[85] B. Opanchuk, L. Arnaud, and M. D. Reid, Phys. Rev. A **89**, 062101 (2014).
[86] M. Żukowski, M. Wieśniak, and W. Laskowski, Phys. Rev. A **94**, 020102(R) (2016).
[87] M. Żukowski, W. Laskowski, and M. Wieśniak, Phys. Scr. **91**, 084001 (2016).
[88] M. Żukowski, W. Laskowski, and M. Wieśniak, Phys. Rev. A **95**, 042113 (2017).
[89] J. Ryu, B. Woloncewicz, M. Marciniak, M. Wieśniak, and M. Żukowski, arXiv:1903.03526.