Generic singularities of scattering coefficients and a paradox of resonant wave scattering

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We show that in the exact solution describing the resonant scattering of a plane monochromatic electromagnetic wave by a lossless spherical or cylindrical particle of radius R, the scattering coefficients in the multipolar expansion have the generic singularities at vanishing R and certain values of the particle permittivity. At these singularities, the linewidths of the corresponding resonances turn to zero, while the partial cross sections do not have definite limits and may take any value from a certain domain. The value depends on the trajectory along which one approaches the singularities in the plane of the problem parameters. To resolve the singularity it is required to go beyond the commonly used monochromatic approximation, to take into account the finite linewidth of the incident wave, and then to perform the correct sequence of limit transitions, while the straightforward application of the monochromatic approximation may give rise to erroneous results. The effects of finite dissipation are discussed too. Our study may be important for a broad class of resonant wave scattering phenomena associated with high-Q resonances, in particular to the problem of the incident wave interaction with the bound states in the continuum.

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I. INTRODUCTION

Despite the more than hundred-year history of the study of light scattering by obstacles, the problem still remains among the most important issues of electrodynamics. In addition to the purely academic interest (plenty of new effects have been discovered recently, see, e.g., the discussions in Refs. [1,2]) there is great demand from numerous technologies, ranging from medicine and biology [3,4] to telecommunications, data storage and processing, etc. [5]. Nowadays, the frontier of this study has shifted to the nanoscale [6–8]. It makes the problem of light scattering by subwavelength particles more important than ever.

Meanwhile, some fundamental properties of the resonant light scattering by these particles still remain unrevealed. The goal of the present paper is to draw the attention of the community to a problem of that kind. Specifically, we discuss generic singularities of the scattering coefficients at the scattering of a linearly polarized plane electromagnetic wave by a spherical or cylindrical lossless particle of radius R. The singularities correspond to the point $R = 0$ and certain resonant values of the particle permittivity ε (the particle is regarded as nonmagnetic so that its permeability $\mu = 1$). The same singularities should exist in acoustics and other wave scattering phenomena.

We show that the scattering coefficients, and hence, the corresponding partial cross sections, do not have definite limits at the singular points and may take any values from a certain domain, depending on the shape of the trajectories along which one approaches the singular points in the plane of the problem parameters. We reveal the grounds for such unusual behavior. We point out that in each particular case the physical constraints may reduce the continuum of possible trajectories to a single one possessing physical meaning. It allows finding a definite value of the scattering coefficient (partial cross section) in the points where mathematically it does not have any limit.

As an example of this approach, we consider the resonant light scattering of a plane wave by a cylinder. We show that the conventional monochromatic approximation may give rise to the paradoxical result—the divergence of the total scattering cross section (per unit length of the cylinder) at $R \to 0$. The correct approach requires us to consider small but finite R; to go beyond the monochromatic approximation, taking into account the finite linewidths of the incident wave and the resonance lines; then, to integrate the spectral partial cross sections over the entire spectrum, and only after that to turn R to zero. The application of this routine gives rise to the physically meaningful results: the vanishing scattering cross section at $R \to 0$.

The physical grounds for that are simple: the smaller R, the narrower the resonance lines, and the higher the Q factor of the resonances. Then, no matter how small the source linewidth is, at small enough R all resonance lines become narrower than the source one. As a result, only a certain fraction of the spectrum of the incident wave is scattered resonantly. The smaller R, the smaller the contribution of this fraction to the overall scattering process. Eventually, the overall scattering transforms into the nonresonant Rayleigh scattering and vanishes at $R = 0$.

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Strictly speaking, in full these arguments are solely valid in the nondissipative limit. Any finite dissipation smoothes the singularity since in this case the resonant linewidths are bounded from below by the dissipative processes and cannot vanish at $R = 0$. However, the point of the crossover from approximately nondissipative to the essentially dissipative scattering occurs at such value of R when the linewidth associated with the radiative damping becomes of the same order of magnitude as that related to the dissipative processes. At larger values of R the nondissipative approximation is quite accurate [9]. This makes it possible to extend the approach developed in the present paper to weakly lossy scatterers too.

It should be stressed that though the imaginary part of ε, which is responsible for the dissipation, and its real part are connected to each other by the Kramers-Kronig formulas [10], the corresponding expressions are *integral*—they do not impose any fundamental restrictions to a value of $\mathrm{Im}\,\varepsilon$ at a given frequency but the condition $\mathrm{Im}\,\varepsilon > 0$ for a passive scatterer. It means that in some practically important cases the value of $\mathrm{Im}\,\varepsilon$ at the resonant frequency may be extremely small and the nondissipative approximation is valid until very small values of R. For example, the resonant wave scattering by a lossless cylinder discussed in the present paper may find a direct application in the old problem of radio-wave scattering by ionized meteor trails [11].

In a broader context, we would like to emphasize that in high-Q resonances the finiteness of the source linewidth may play an important role, no matter which specific mechanisms define the shapes of the resonance lines. In these cases, the monochromatic approximation gives rise to erroneous and even paradoxical results. Nonetheless, so far, the theoretical treatments of such problems (including our own previous studies, e.g., [4,12]) mostly are based on this approximation and, therefore, may require revision.

Note also that similar effects may be important for the bound states in the continuum, which in the ideal case of a non-dissipative system are characterized by the infinite Q factor, see, e.g., [13–17] and references therein. Thus, the application of the general results discussed below may be extended far beyond the problem of the cylinder resonant scattering, presented here just as a typical example.

The paper has the following structure. First, we discuss the physical grounds for the singularities to come into being. Then, we show that the values of the scattering coefficients at the singular points do depend on the shape of the trajectories along which we approach the singularities in the plane of the problem parameters. Next, we reveal the reason for this unusual behavior. Then, we discuss the manifestation of the singularities at the resonant scattering of a plane wave by a cylinder, show invalidity of the conventional monochromatic approximation, and correct it. Next, the effects of finite dissipation are inspected. The discussion ends with conclusions highlighting the main points of the study.

II. GENERIC SINGULARITY OF SCATTERING COEFFICIENTS AND ITS GROUNDS

To begin with, we recall certain basic points of the problem of light scattering by a cylindrically or spherically symmetric particle [18]. The exact solution of the problem is built up as the infinite series in the cylindrical (spherical) harmonics weighted by the appropriate dimensionless complex *scattering coefficient*: a_ℓ or b_ℓ, where integer ℓ is the number of the corresponding harmonic (multipole). Usually, the modes weighted by a_ℓ are called electric, the ones weighted by b_ℓ are named magnetic. Since all harmonics are orthogonal, the contribution of each of them to the scattered (absorbed) power is independent. This makes possible introducing the partial cross sections associated with each multipole so that the total cross section is the sum of the partial ones. For the cylindrical and spherical particles the partial extinction ($\sigma_{\text{ext}}^{(\ell)}$) and scattering ($\sigma_{\text{sca}}^{(\ell)}$) cross sections associated with the set $\{a_\ell\}$ are shown in Table I. Here $k = \omega/c$ stands for the wave number of the incident wave with the angular frequency ω and c is the speed of light in a vacuum. The replacement $a_\ell \to b_\ell$ gives the corresponding expressions for the set $\{b_\ell\}$.

TABLE I. Partial cross sections associated with $\{a_\ell\}$.

Symmetry	$\sigma_{\text{ext}}^{(\ell)}$	$\sigma_{\text{sca}}^{(\ell)}$
Cylinder	$\frac{4\pi}{k}\mathrm{Re}\,a_\ell$	$\frac{4\pi}{k}\lvert a_\ell\rvert^2$
Sphere	$\frac{2\pi(2\ell+1)}{k^2}\mathrm{Re}\,a_\ell$	$\frac{2\pi(2\ell+1)}{k^2}\lvert a_\ell\rvert^2$

By definition, the extinction cross section is the sum of the scattering and absorption ones. The fact that $\sigma_{\text{ext}}^{(\ell)} \propto \mathrm{Re}\,a_\ell$ follows from the optical theorem [18], which itself is the consequence of the energy conservation law.

For a lossless passive scatterer the absorption cross section vanishes. Then, the extinction and scattering cross sections become identical. It means $|a_\ell|^2 = \mathrm{Re}\,a_\ell$, see Table I. Presenting a_ℓ as $a'_\ell + ia''_\ell$, we may rewrite this condition in the following form: $a''^2_\ell = a'_\ell - a'^2_\ell \geqslant 0$, resulting in the constraints $0 \leqslant a'_\ell \leqslant 1$, $a''^2_\ell \leqslant a'_\ell$. Then, since a'_ℓ is a non-negative quantity no greater than unity, it always may be written as

$$a'_\ell = \frac{F_\ell^2}{F_\ell^2 + G_\ell^2}, \tag{1}$$

where F and G are purely real. The substitution of this expression into the formula $a''^2_\ell = a'_\ell - a'^2_\ell$ yields

$$a''_\ell = \pm\frac{F_\ell G_\ell}{F_\ell^2 + G_\ell^2}. \tag{2}$$

In the right-hand-side (RHS) of Eq. (2) any sign may be selected. Without loss of generality, we may take minus. For the opposite choice, the replacement $F \to -F$ (or $G \to -G$) returns the problem to the previous one.

Collecting all together, we arrive at the following expression for the structure of the scattering coefficient:

$$a_\ell = \frac{F_\ell^2}{F_\ell^2 + G_\ell^2} - i\frac{F_\ell G_\ell}{F_\ell^2 + G_\ell^2} \equiv \frac{F_\ell}{F_\ell + iG_\ell}. \tag{3}$$

The same is true for b_ℓ, though the quantities F_ℓ and G_ℓ for the set $\{b_\ell\}$ will be different. Thus, even though the dependence of F and G on the problem parameters is individual for each set of them, Eq. (3) is generic and valid for any case [19]. Note, the quantities F_ℓ, G_ℓ are defined up to an arbitrary common multiplier. We employ this fact later.

Equation (3) and Table I make it possible to arrive at several important conclusions:

(i) $\sigma_{\text{ext, sca}}^{(\ell)}$ are maximized at $G_\ell = 0$, i.e., this condition defines the trajectories of the Mie resonances in the space of the problem parameters (to obtain the trajectories explicitly we have to know the explicit dependence of G on these parameters, see below).

(ii) The scattering must vanish at the vanishing size of the particle. Therefore, $F \to 0$ at $R \to 0$.

(iii) Introducing $Z \equiv F + iG$ we may rewrite Eq. (3) as $a_\ell = (\text{Re}\, Z)/Z$. However, despite the fact that at $Z \neq 0$, this function is infinitely differentiable with respect to both the real part of Z and its imaginary part (F and G), *it is not differentiable with respect to the complex variable Z as a whole*, since the derivative $d(\text{Re}\, Z)/dZ$ cannot be defined as a single-valued function of Z. In other words, $a_\ell(Z)$ *is not an analytic function in the entire plane of complex* Z. To illustrate that, let us check whether $a_\ell(Z)$ satisfies any of the equivalent analyticity criteria, e.g., the Cauchy-Riemann equations. According to them, if $a_\ell(Z)$ were an analytic function, the following identities would have held: $\partial a'_\ell/\partial F = \partial a''_\ell/\partial G$ and $\partial a'_\ell/\partial G = -\partial a''_\ell/\partial F$. It is straightforwardly seen from Eqs. (1) and (2) that none of them is satisfied.

$Z = 0$ is a singular point of $a_\ell(Z)$. Owing to the nonanalyticity of a_ℓ on the entire Z plane, to analyze the type of the singularity we cannot apply the standard approaches of complex analysis dealing with isolated singular points of analytic functions. Nonetheless, it is easy to show that $a_\ell(Z)$ does not have any definite limit at this point. In a certain sense the case is analogous to the essential singularity of analytic functions, e.g., the one which $\exp(1/Z)$ has at $Z = 0$: if Z is purely real it turns to zero at $Z \to 0$, provided Z remains negative, and tends to infinity if the point $Z = 0$ is reached from the opposite direction of the Z axis. At complex Z it exhibits even more complicated oscillatory behavior.

To clarify the nature of the singularity of a_ℓ at $F = G = 0$ let us parametrize the trajectory along which we approach the coordinate origin in the Z plane by the family $|G| = A|F|^\alpha$, where A and α are real free parameters. Then, it is trivial to see from Eq. (3) that at $F \to 0$ coefficient $a_\ell \to 1$ at $\alpha > 1$; $a_\ell \to 0$ at $\alpha < 1$; and $a_\ell \to 1/(1 \pm iA)$ at $\alpha = 1$, where sign plus is taken if both F and G have the same sign, and minus, if their signs are opposite.

Thus, $a_\ell(Z)$ does not have a definite limit at Z tends to zero—*the limit depends on the trajectory along which one approaches this point.* To understand why such a simple function has this unusual singularity let us look at the plot $|a_\ell(F_\ell, G_\ell)|^2$, in the proximity of the singular point, see Fig. 1. We see that the two wings of the surface $|a_\ell(F_\ell, G_\ell)|^2$ intersect each other along the straight line $F_\ell = G_\ell = 0$ originated in the singular point in the plane (F_ℓ, G_ℓ) and perpendicular to this plane. For this reason, though the points of the intersections of various trajectories lying on this surface with this line are different, their projections onto the plane (F_ℓ, G_ℓ) collapse in one and the same point $F_\ell = G_\ell = 0$. For this reason, the actual value of $|a_\ell(F_\ell, G_\ell)|^2$ in this point may be any from the domain [0,1] depending on the specific trajectory lying on the three-dimensional surface $|a_\ell(F_\ell, G_\ell)|^2$ along which the straight line $F_\ell = G_\ell = 0$ is achieved.

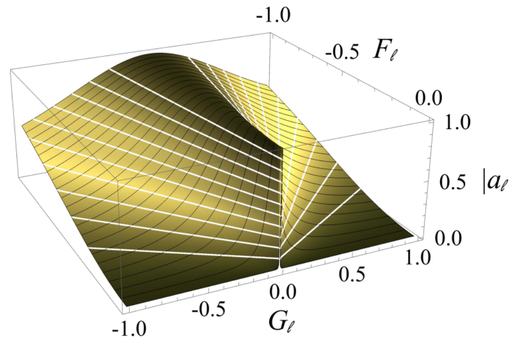

FIG. 1. Surface $|a_\ell(F_\ell, G_\ell)|^2 = \frac{F_\ell^2}{F_\ell^2 + G_\ell^2}$ and a family of lines $F_\ell = \text{const}$ on it (thin black lines). The closer F_ℓ to zero, the steeper the maximum of the lines at $G_\ell = 0$. Eventually, the two wings of the surface intersect each other along the line $F_\ell = G_\ell = 0$. Though various trajectories lying in the surface may intersect this line at different values of $|a_\ell|^2$, all points of the intersections are projected onto one and the same point $F_\ell = G_\ell = 0$ in the (F_ℓ, G_ℓ) plane. As an example of these trajectories, a set of the level lines $|a_\ell|^2 = \text{const}$ for different values of the constant are shown as thick white lines.

To conclude the discussion of this issue, we have to stress two important aspects of the problem. First, the above discussion is based just on the orthogonality of the problem eigenfunctions, the optical theorem, and the energy conservation law. The explicit form of the solution of the Maxwell equations is not used. For these reasons, *the singularity is generic for the elastic wave scattering by a lossless passive particle and any problem formulation characterized by orthogonal eigenmodes.*

Second, the fact that according to Table I and the aforementioned arguments *mathematically* the partial cross sections do not have a definite limit in the singular point, does not mean that the same is true for the corresponding *physical* problem. The fact of the matter is that the physical problem may impose additional constraints. Then, it may happen that among the continuum of possible trajectories, there is just a single one satisfying the constraints. However, we should be very careful in the formulation of the constraints and in the corresponding treatment of the singular point; otherwise, it is easy to make a mistake.

III. PARADOX OF RESONANT LIGHT SCATTERING BY LOSSLESS SCATTERER

Let us illustrate these general arguments discussing the resonant Mie scattering by a spatially uniform nonmagnetic ($\mu = 1$) infinite right circular cylinder of infinitesimal radius. Though the same effect is exhibited at any mutual orientation of the incident wave and cylinder, for the sake of simplicity, we consider the normal incidence of the so-called TE-polarized plane wave when the wave vector $\mathbf{k}$ is perpendicular to the axis of the cylinder and vector $\mathbf{E}$ oscillates in the plane of the base of the cylinder, see Fig. 2. The extension of our consideration to the general case of an arbitrary incidence is quite straightforward.

The exact solution of the formulated problem is well known, see, e.g., [18]. For the problem in question, the

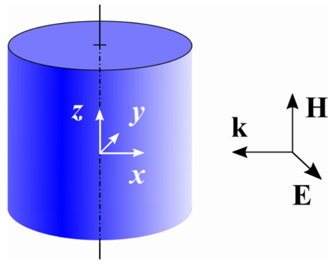

FIG. 2. Mutual orientation of the cylinder, coordinate frame, and vectors **k**, **E**, and **H** of the incident linearly polarized plane wave.

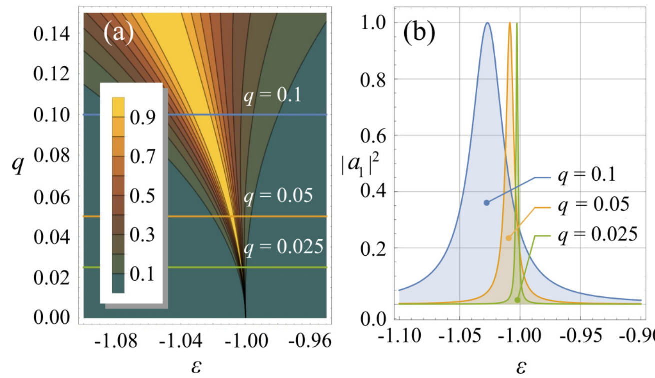

FIG. 3. (a) Contour plot of $|a_1(q,\varepsilon)|^2$ calculated according to the exact solution, Eq. (4). (b) Its cross sections by the planes $q = 0.1$ (blue), $q = 0.05$ (dark yellow), and $q = 0.025$ (green), i.e., the corresponding line shapes as the functions of ε. Note the similarity with Fig. 1 and the contraction of the linewidths at $q \to 0$.

scattering field outside the cylinder is described by the single set of scattering coefficients $\{a_\ell\}$ since all b_ℓ identically vanish. The coefficients a_ℓ are expressed in terms of the Bessel functions. Employing the connection between the Bessel functions of different kinds it is possible to rewrite the conventional expressions for a_ℓ in the form of Eq. (3). The corresponding formulas read

$$F_\ell = [mJ_\ell(mq)J_\ell'(q) - J_\ell'(mq)J_l(q)]/m^{\ell-1},$$
$$G_\ell = [mJ_\ell(mq)N_\ell'(q) - J_\ell'(mq)N_\ell(q)]/m^{\ell-1}, \tag{4}$$

where $J_\ell(z)$, $N_\ell(z)$ stand for the Bessel and Neumann functions, respectively; the prime indicates differentiation with respect to the entire argument of the corresponding function; $m \equiv \sqrt{\varepsilon}$ is the refractive index; and the size parameter q equals kR.

At $\varepsilon < 0$ the refractive index is purely imaginary. Therefore, to make F and G purely real we have made use of the fact that they are defined up to any common factor and multiplied the RHS of Eqs. (4) by $1/m^{\ell-1}$. Employing the well-known identity $J_\ell(iz) \equiv i^\ell I_\ell(z)$, where $I_\ell(z)$ stands for the modified Bessel functions of the first kind, it is easy to see that the quantities given by Eqs. (4) indeed remain purely real, no matter whether m is purely real or purely imaginary.

The total scattering cross section ($\sigma_{\rm sca}^{\rm tot}$) is given by the following expression, see Table I:

$$\sigma_{\rm sca}^{\rm tot} = \sum_{\ell=-\infty}^{\infty} \sigma_{\rm sca}^{(\ell)} \equiv \sigma_{\rm sca}^{(0)} + 2\sum_{\ell=1}^{\infty}\sigma_{\rm sca}^{(\ell)}, \tag{5}$$

the latter owing to the identity $a_\ell \equiv a_{-\ell}$. We are interested in the limit $q \to 0$. Then, the expansion of the functions in the RHS of Eq. (4) in powers of small q yields [20]

$$F_\ell \cong \begin{cases} \frac{(\varepsilon-1)\varepsilon}{16}q^3 + \cdots & \text{at } \ell = 0,\\ \frac{(\varepsilon-1)}{2^{2\ell}\ell!(\ell-1)!}q^{2\ell-1} + \cdots & \text{at } \ell \neq 0, \end{cases} \tag{6}$$

$$G_\ell \cong \begin{cases} \frac{2\varepsilon}{\pi q} + \cdots & \text{at } \ell = 0,\\ \frac{\varepsilon+1}{\pi q} - q\frac{\varepsilon-1}{8\pi}\left[2+\varepsilon - 4\ln\frac{qC}{2}\right] + \cdots & \text{at } \ell = 1,\\ \frac{(\varepsilon+1)}{\pi q} - q\frac{(\varepsilon-1)}{4\pi}\left[\frac{\varepsilon}{\ell+1} + \frac{1}{\ell-1}\right] + \cdots & \text{at } \ell > 1, \end{cases} \tag{7}$$

Here $C \equiv \exp(\gamma) = 1.78107\ldots$ and $\gamma = 0.577\ldots$ stands for Euler's constant.

The equation $G_\ell(\varepsilon, q) = 0$ determines the resonance trajectories $\varepsilon_\ell(q)$. Along the trajectories $a_\ell = 1$ and $\sigma_{\rm sca}^{(\ell)} = 4\pi/k$, see Table I. Note, as it follows from Eqs. (7), at $q \to 0$ all trajectories but the one with $\ell = 0$ merge in the same point $q = 0$, $\varepsilon = -1$. Seemingly, it results in the paradoxical conclusion—the divergence of $\sigma_{\rm sca}^{\rm tot}$ at $R = 0$. In what follows we are going to show that, actually, $\sigma_{\rm sca}^{\rm tot}$ vanishes at $R = 0$ and that the divergence is caused by the too straightforward treatment of Eqs. (6) and (7).

Since for the mode with $\ell = 0$ the point $q = 0$, $\varepsilon = -1$ does not belong to the resonant trajectory, this mode is not interesting for us and will be excluded from further consideration. Regarding the other modes, this point is the mapping of the singular point $F_\ell = 0$, $G_\ell = 0$, when for each ℓ the change of the independent variables from F_ℓ, G_ℓ to q, ε maps the planes $(F_\ell,\ G_\ell)$ to the plane $(q,\ \varepsilon)$. For this reason the point $q = 0$, $\varepsilon = -1$ exhibits the same type of singularity as that discussed above for the plane $(F_\ell,\ G_\ell)$. To see that, let us consider a departure $\Delta\varepsilon$ of ε from $\varepsilon_\ell(q)$. Once again, assuming $\Delta\varepsilon = Aq^\alpha$, we obtain at $\ell \neq 0$: $a_\ell = 1$ at $\alpha > 2\ell - 1$, $a_\ell = 0$ at $\alpha < 2\ell - 1$, and $a_\ell = 1/(1 + i\tilde{A})$ at $\alpha = 2\ell - 1$. Here $\tilde{A} \equiv 2^{2\ell}\ell!(\ell-1)!A/(\varepsilon - 1)$. Assigning A and α various values we may obtain different values of a_ℓ at one and the same point $q = 0$, $\varepsilon = -1$.

To illustrate these arguments, the contour plot of $|a_1(q,\varepsilon)|^2$ and the evolution of the corresponding line shape as a function of ε at $q \to 0$ in the vicinity of the point $q = 0$, $\varepsilon = -1$ are shown in Fig. 3. The shapes of $|a_\ell(q,\varepsilon)|^2$ at $\ell > 1$ are similar to that.

In addition to the discussed generic singularities, the merging of all resonant trajectories with $\ell \neq 0$ at the same point means that for the problem under consideration, overlaps of the infinite number of resonances may take place. Whether the overlaps occur indeed depends on the answer to the question if at $q \to 0$ the distances between different resonance lines contract sharper than their linewidths.

Thus, to proceed further, we have to study the line shapes of the resonance lines. To this end let us select any resonant trajectory, take in the trajectory a point $\varepsilon = \varepsilon_\ell$, $q = q_\ell$ in the proximity of the singular one but different from it, and consider small departures from this point: $\varepsilon = \varepsilon_\ell + \delta\varepsilon$ and $q = q_\ell + \delta q$. Then, expanding the RHS of Eqs. (6) and (7) in small $\delta\varepsilon$, δq, keeping just the leading terms, and bearing in mind that $\varepsilon_\ell \approx -1$ so that in the final expressions it may be replaced by -1, we obtain

At $\ell = 1$,

$$a_1 \cong \frac{\pi q_1^2}{\pi q_1^2 - 2i(\delta\varepsilon - 2[q_1 \ln q_1]\delta q)}. \tag{8}$$

At $\ell > 1$,

$$a_\ell \cong \frac{\pi\ell(\ell^2-1)q_\ell^{2\ell}}{\pi\ell(\ell^2-1)q_\ell^{2\ell} - i2^{2\ell-1}(\ell!)^2[\delta\varepsilon(\ell^2-1) + 2q_\ell\delta q]}. \tag{9}$$

Now, recall that the limit we are interested in depends on the trajectory. The "natural" set of parameters in Eqs. (8) and (9) are $q \equiv R\omega/c$ and ε, while the varying parameters in any actual physical experiment usually are R and ω. Therefore, to describe such a case we have to transfer to new independent variables R and ω. Moreover, since we discuss the light scattering by *a given* particle, we have to fix R so that the only remaining varying parameter is ω. Then, $\delta q = (\partial q/\partial\omega)\delta\omega \equiv (R/c)\delta\omega$ and $\delta\varepsilon \cong (\partial\varepsilon/\partial\omega)_{\omega_0}\delta\omega$. Here ω_0 is defined by the condition $\varepsilon(\omega_0) = -1$. In this case, Eqs. (8) and (9) give rise to a usual Lorentzian profile for $|a_\ell|^2$:

$$|a_\ell|^2 \cong \frac{(\Gamma_\ell/2)^2}{(\delta\omega)^2 + (\Gamma_\ell/2)^2}, \tag{10}$$

where

$$\Gamma_\ell \cong \frac{\pi}{(\ell-1)!\ell!(\partial\varepsilon/\partial\omega)_{\omega_0}}\left(\frac{R\omega_0}{2c}\right)^{2\ell} \tag{11}$$

(it is regarded that $0! = 1$).

As followed from Eqs. (10) and (11), the linewidths of the resonance lines vanish very sharply at $R \to 0$. If so, we must take into account that any actual physical source of light (even a very good laser, to say nothing about noncoherent sources) has a fixed finite linewidth. How does it affect the calculation of the total integral scattering cross section?

The general answer to the question implies the study of long transient processes at nonsteady resonant scattering associated with excitation of high-Q resonant modes [21,22]. It is a complicated separate problem lying beyond the scope of the present paper. Here we consider just the steady-state scattering realized at long enough laser pulses or CW beams so that the broadening of the source line is explained by effects different from the finiteness of the pulse duration.

Let us first consider the simplest case of a superposition of just two modes with close frequencies ω_1 and $\omega_2=\omega_1+\Delta\omega$, where $|\Delta\omega| \ll \omega_1$ and calculate the total scattering flux through a closed remote surface. To this end, we have to calculate the Poynting vector

$$\mathbf{S} = \frac{c}{16\pi}[(\mathbf{E}+\mathbf{E}^*)\times(\mathbf{H}+\mathbf{H}^*)], \tag{12}$$

where the asterisk means complex conjugation. Here $\mathbf{E} = \mathbf{E}_1(\mathbf{r})\exp(-i\omega_1 t) + \mathbf{E}_2(\mathbf{r})\exp(-i\omega_2 t)$ and similar for $\mathbf{H}$.

Expanding the vector product, we get the time-independent terms of the type $\mathbf{E}_1 \times \mathbf{H}_1^*$, $\mathbf{E}_2 \times \mathbf{H}_2^*$, and c.c. as well as oscillatory ones. To get rid of the latter we have to average the Poynting vector over time. It seems that here we encounter a problem: the slowest oscillations correspond to $\exp(\pm i\Delta\omega t)$. The period of them diverges at $\Delta\omega \to 0$. Seemingly, to get rid of these oscillations we also have to increase the time of the averaging up to infinity.

Fortunately, this is not the case. In the above arguments, we have not taken into account the finiteness of the coherence time, which cannot exceed the inverse linewidth of the source. Owing to losses of coherency the temporal average of the Poynting vector makes the oscillatory terms vanish at the averaging over the time larger than the coherence one, no matter how small $\Delta\omega$ is.

After the time averaging, only the time-independent terms remain. These terms correspond to the sum of the intensities scattered at frequencies $\omega_{1,2}$ independently. Then, the integral partial cross section of the ℓth mode is just the sum $\sigma_{\rm sca}^{(\ell)}(\omega_1) + \sigma_{\rm sca}^{(\ell)}(\omega_2)$. The generalization of these arguments to a continuous spectrum is trivial—to find the integral ℓth partial cross sections at any given value of R we have to take the corresponding integrals over the spectrum. The total integral scattering cross section, in this case, is the sum of these integral partial cross sections over ℓ. If we are interested in the limit $R \to 0$, first we have to complete these calculations for a fixed finite value of R and only after that to turn R to zero.

The rest is simple. According to Eq. (11), $2|a_\ell|^2/(\pi\Gamma_\ell)$ converges to the δ function at $R \to 0$. Then, the integral partial scattering cross section is (see Table I)

$$\begin{aligned}\sigma_{\rm sca}^{(\ell)\,\rm int} &\equiv \frac{1}{I}\int \sigma_{\rm sca}^{(\ell)} S_\omega d\omega \equiv \int \frac{2\pi^2 c\Gamma_\ell}{\omega I} S_\omega \frac{2|a_\ell|^2}{(\pi\Gamma_\ell)} d\omega \\ &\cong \int \frac{2\pi^2 c\Gamma_\ell}{\omega I} S_\omega \delta(\omega-\omega_\ell) d\omega \cong \frac{2\pi^2 c S_{\omega_0}\Gamma_\ell}{\omega_0 I}.\end{aligned} \tag{13}$$

Here S_ω and $I \equiv \int S_\omega d\omega$ are the spectral and total energy flux densities of the source, respectively. Note that if ω_0 corresponds to the center of the Lorentzian source line with the linewidth $\Gamma_{\rm s}$ Eq. (13) may be rewritten in the following simple and physically transparent form:

$$\sigma_{\rm sca}^{(\ell)\,\rm int} \cong \frac{4\pi}{k}\frac{\Gamma_\ell}{\Gamma_{\rm s}}, \tag{14}$$

compare with Table I.

Next,

$$\begin{aligned}\sigma_{\rm sca}^{\rm int}(R) &= \sigma_0 \sum_{\ell=1}^{\infty} \frac{1}{(\ell-1)!\ell!}\left(\frac{R\omega_0}{2c}\right)^{2\ell} \\ &< \sigma_0 \sum_{\ell=1}^{\infty} \frac{1}{\ell!}\left(\frac{R\omega_0}{2c}\right)^{2\ell} = \sigma_0\left(\exp\left[\left(\frac{R\omega_0}{2c}\right)^2\right] - 1\right),\end{aligned} \tag{15}$$

where $\sigma_0 \equiv 4\pi^3 c S_{\omega_0}/[\omega_0 I(\partial\varepsilon/\partial\omega)_{\omega_0}]$. Thus, $\sigma_{\rm sca}^{\rm int}(R) \to 0$ at $R \to 0$, in agreement with common sense.

To conclude this section let us highlight the differences between the resonant scattering by the lossless cylinder and sphere. The only essential difference is that while for the cylinder at $R \to 0$ the resonant ε for all modes with $\ell \neq 0$ tend to one and the same value $\varepsilon = -1$, for the sphere the corresponding limit for each mode is individual and equal to $-(\ell+1)/\ell$ [9]. It removes the problem of the multiple overlaps of resonance lines, reducing the case to a single resonant mode. All the rest remains the same; though, of course, the explicit expressions for F and G are different from those given by Eq. (4).

IV. LOSSY SCATTERER

What happens if the dissipation is small but finite, i.e., $\varepsilon = \varepsilon' + i\varepsilon''$, $0 < \varepsilon'' \ll 1$? To answer the question, calculations are not required—the result can be readily obtained from Eqs. (8) and (9) by means of the replacement $\delta\varepsilon \to \delta\varepsilon' + i\varepsilon''$. We see that any finite ε'' removes the singularity so that the point $R = 0$, $\varepsilon' = -1$ becomes a regular point of a_ℓ, where $a_\ell = 0$.

Geometrically it is explained by the fact that the two wings of the surfaces $|a_\ell(\varepsilon', q)|^2$ do not anymore intersect each other along the vertical line $\varepsilon' = -1$, $q = 0$ but smoothly transfer to each other, making a sharp but smooth cusp tilted with respect to the horizontal plane. Then, the points where different level lines pass through the cusp are projected into the different points of the horizontal plane, which removes the ambiguity.

Thus, any finite dissipation removes the singularity. Regarding the problem of the calculations of the scattering and extinction cross sections at small but finite R, the case is more complicated. The replacement $\varepsilon = \varepsilon' + i\varepsilon''$ transforms Γ_ℓ given by Eq. (11) into

$$\Gamma_\ell \cong \frac{\pi}{(\ell-1)!\ell!(\partial\varepsilon/\partial\omega)_{\omega_0}}\left(\frac{R\omega_0}{2c}\right)^{2\ell} + \frac{\varepsilon''}{(\partial\varepsilon/\partial\omega)_{\omega_0}}. \quad (16)$$

However, Eq. (13) still remains valid as long as this new Γ_ℓ is much smaller than the source linewidth, Γ_s. As for the calculation of $\sigma_{\rm sca}^{\rm int}(R)$, the problem becomes even simpler than before since the series in Eq. (15) may be truncated at ℓ satisfying the condition

$$\frac{\pi}{(\ell-1)!\ell!}\left(\frac{R\omega_0}{2c}\right)^{2\ell} \sim \varepsilon'', \quad (17)$$

owing to the rapid decay of the partial cross sections with the larger values of ℓ, when the second term in the RHS of Eq. (16), describing the dissipative damping, prevails over the first (radiative damping), see Table I and Eqs. (10) and (16).

Note also that in a lossy scatter the extinction and scattering cross sections are not equal to each other. However, the problem of calculation of the extinction cross section is completely identical to that discussed above for $\sigma_{\rm sca}$ and will not be presented here.

V. CONCLUSIONS

Highlighting the main results obtained in this study we may say the following:

(1) The scattering coefficients in the exactly solvable problems of the scattering of a plane monochromatic electromagnetic wave by a lossless cylindrically, or spherically, symmetric particle of radius R have the generic singular points at $R = 0$ and the certain resonant value(s) of ε.

(2) Mathematically, the scattering coefficients do not have definite limits at these singularities. The limits depend on the shape of the trajectories along which one approaches the singular points in the space of the problem parameters.

(3) However, the physical formulation of the problem may impose additional constraints. In this case, it may happen that among the continuum of possible trajectories, there is just a single one satisfying the constraints.

(4) Then, the limits become definite. To obtain them, one has to consider the behavior of the scattering coefficients at small but finite R, to make the corresponding calculations, and after that to turn R to zero.

(5) The application of this approach to resonant wave scattering requires the generalization of the conventional monochromatic approximation for the incident wave, taking into account the finiteness of the linewidth of the source and redefining the resonant partial cross sections as those that are integrated over the spectrum.

(6) Any finite dissipation smoothes the singularities out, making the singular points regular. However, at small but finite R the problem of the calculation of the cross sections, actually, remains the same as long as the resonant linewidths are small with respect to the source one.

The realization of this approach gives rise to the resonant integral cross section as a single-valued function of R, which tends to zero at vanishing R even for a lossless scatterer. Meanwhile, the straightforward application of the monochromatic approximation to this problem may bring about erroneous results. This point is important for a much broader class of problems where high-Q resonances with extremely narrow linewidths may occur.

Our study indicates that the old and well-known problem of the elastic wave scattering by subwavelength particles may exhibit novel and rather unusual properties. The obtained results open a door to new interesting effects, coming into being at small but finite R. These effects are associated with different positions of the centers of the resonant lines of various multipoles with respect to the center of the line of the source and different scales of the corresponding linewidths. In this case, an increase in R may push resonant lines outside the source line, which may give rise to repeating sharp drops in the scattering. This phenomenon should affect both the far-field zone (unusual, nonmonotonic dependence of the cross section on R) and in the near-field zone (sharp changes in the topological structure of the field caused by small variations in R). However, these effects lie beyond the scope of the present paper and will be discussed in detail elsewhere.

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