Optical spatial dispersion in terms of Jones calculus

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Traditionally, optical spatial dispersion (OSD) is defined as the dependence $\hat{\varepsilon}(\vec{k})$ of the dielectric permittivity tensor $\hat{\varepsilon}$ on the light wave vector $\vec{k}$, similarly to the frequency (ω) dispersion of the dielectric tensor $\varepsilon(\omega)$. We have developed an approach for the description of the OSD phenomena in the framework of Jones calculus. In Jones calculus the differential Jones matrix (DJM) N is the generalization of the light wave-vector $\vec{k}$ in the same sense that $\vec{k}$ is the generalization of the light wave number k. The latter inspires us to expect that there must exist a way to describe the OSD phenomena in terms of the DJM. We show that such a relation between the OSD phenomena and Jones calculus indeed exists. To prove the latter we derive a general relation between the DJM and components of the dielectric permittivity tensor $\hat{\varepsilon}$. We establish the relation of the DJM approach, proposed in this paper, to the traditional OSD approach of the gyration pseudotensor as well as to that developed by Mauguin for light propagation in cholesteric liquid crystals [M. C. Mauguin, Bull. Soc. Fr. Mineral. Crystallogr. **N3**, 71 (1911)]. We demonstrate that both the gyration pseudotensor and Mauguin's approach can be derived as particular cases of the proposed DJM approach. In our approach the integral Jones matrix (IJM) of the medium taking into account OSD is the product of the IJM without taking into account OSD by the correction IJM, which accounts for the OSD effects. In a general case, when all components of the OSD DJM N^D are nonzero, the secular equation for the refractive indices of the eigenwaves is a quartic equation. The coefficient a_3 at the cubic term in the secular equation is nonzero only for nonzero OSD corrections to the average refractive index. For transparent crystals at nonzero OSD correction to the average refractive index and zero to all other correction parameters in N^D, the secular equation has two distinct real and two complex-conjugate roots. We assign the complex-conjugate roots to the forward and backward light scattering. Therefore, taking into account the OSD effect on the refractive index, the Jones calculus becomes capable of describing light scattering. The proposed Jones calculus approach is a general tool for taking into account OSD in optically inhomogeneous media, in which several or all OSD correction parameters are simultaneously nonzero, for example, in liquid-crystal cells with a spatially nonuniform director field, including those containing defects.

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I. INTRODUCTION

Description of light propagation is one of the most developed and still quite intensively developing theories in physics. Several novel concepts such as localization of light [1], light propagation in photonic media [2–4] and optical metamaterials [5,6], mirrorless lasing [7,8], and singular optics [9,10] have been developed recently. The most general approach to the description of new as well as classical problems of light propagation in optical media is based on Maxwell's differential equations, which together with material equations form a closed system of equations allowing for calculation of the electric-field vector for the light propagating in and exiting from the medium. Since the theory is based on a system of second-order differential equations, it bears all the complications related to their solution with respect to four variables: three space coordinates and time. As a result, only in a few cases are exact analytical solutions obtained. In most cases analytical expressions suitable for qualitative analysis are not available; time-demanding numerical computations are needed. However, there are typical cases when considerable simplifications are achieved. For a monochromatic light wave with frequency that does not change on propagation, time derivatives are excluded and thus one deals only with space derivatives. For normal light incidence, when there is no dependence of the electric-field components on the in-plane x and y coordinates, their x and y derivatives are zero. Further simplification achieves taking into account the transverse character of the electric field of the plane light wave, propagating through a flat plate at normal light incidence. For the $\vec{Z}$ axis of the Cartesian coordinate system directed along the light propagation direction, which is along the plate normal, the z component of the vector of dielectric displacement is zero and one deals with the two z derivatives of the in-plane components of the electric field. In such a case the system

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of Maxwell equations can be written in the two-dimensional (2D) matrix form [11]

$$\hat{\varepsilon}\frac{d\vec{E}}{dz} = -\lambdabar^2\frac{d^2\vec{E}}{dz^2}, \tag{1}$$

where $\lambdabar = \frac{\lambda}{2\pi}$, λ is the light wavelength, and $\hat{\varepsilon}$ is the 2×2 tensor of dielectric permittivity. Components of the 2×2 (2D) form $\varepsilon_{ij}^{(2)} = \varepsilon_{ij}$ are related to the components of the conventional 3×3 (3D) dielectric tensor $\varepsilon_{ij}^{(3)}$ as

$$\hat{\varepsilon} = \hat{\varepsilon}^{(2)} = \begin{bmatrix} \varepsilon_{11}^{(3)} - \frac{(\varepsilon_{13}^{(3)})^2}{\varepsilon_{33}^{(3)}} & \varepsilon_{12}^{(3)} - \frac{\varepsilon_{13}^{(3)}\varepsilon_{23}^{(3)}}{\varepsilon_{33}^{(3)}} \\ \varepsilon_{12}^{(3)} - \frac{\varepsilon_{13}^{(3)}\varepsilon_{23}^{(3)}}{\varepsilon_{33}^{(3)}} & \varepsilon_{22}^{(3)} - \frac{(\varepsilon_{23}^{(3)})^2}{\varepsilon_{33}^{(3)}} \end{bmatrix}. \tag{2}$$

Taking into account that the 3D dielectric tensor is symmetric [12], i.e., $\varepsilon_{ij}^{(3)} = \varepsilon_{ji}^{(3)}$, one finds that its 2D form is also symmetric, $\varepsilon_{ij}^{(2)} = \varepsilon_{ji}^{(2)}$.

The convenience of the 2D matrix form of Maxwell equations is that it can be combined with the approach of the 2×2 Jones matrix J, which relates the electric-field vector $\vec{E}^i$ of the incident light wave to the vector $\vec{E}$ of the wave exiting an optical plate in a linear fashion $\vec{E} = J\vec{E}^i$. Thereby, the matrix J describes the plate as a whole; it does not carry any information on the regime of propagation of the light wave inside the plate and of course it gives no indication of the presence of the internal optical inhomogeneity inside the plate. For this reason J is called the integral Jones matrix (IJM) to distinguish it from the differential Jones matrix N, which was introduced by Jones [13] to describe the optical inhomogeneity in a sample. The IJM and differential Jones matrix (DJM) are related via the matrix exponent

$$J = \exp\left(\int N(z)dz\right)J_0, \tag{3}$$

where J_0 is the IJM of the entrance interface of the plate. Since we are dealing with the bulk properties of the media, throughout this paper J_0 is taken to be equal to the identity matrix. For a single crystal N is independent of z and Eq. (3) reduces to the form

$$J = e^{Nz}. \tag{4}$$

Equation (4) describes the light propagation along the coordinate z inside the plate. The regime of light propagation inside the plate is defined by the form of N, which in turn is governed by the crystallographic symmetry of the medium. In a physical sense the matrix N is a generalization of the light wave vector $\vec{k}$ in the same way that $\vec{k}$ is a generalization of the scalar wave number k [13]. In this respect it is worth noting that the $\vec{k}$ dependence of the dielectric tensor $\hat{\varepsilon}$ is at the heart of the phenomena of optical spatial dispersion (OSD) [12,14]. Intuitively, one expects that since the DJM is a generalization of the wave vector, it should be related to the OSD phenomena as well. In this paper we show that such a relation between the OSD phenomena and the DJM indeed exists. We propose an approach for the description of the OSD phenomena via the DJM and show that since the DJM is a generalization of $\vec{k}$, the DJM approach is naturally more general with respect to the traditional so-called approach of the gyration pseudotensor [12,14]. The gyration pseudotensor approach is based on the serial expansion of $\hat{\varepsilon}(\vec{k})$, truncated at the first power of $\vec{k}$ with the expansion coefficient, which is the gyration pseudotensor $\hat{g}$.

An alternative approach, which accounts for the OSD but does not involve the notion of the gyration pseudotensor, was developed by Mauguin [15] for light propagation along the helical axis in a cholesteric liquid crystal. Although this approach results in an exact solution of Maxwell equations without any approximation, traditionally it is considered as a solution for a particular problem. This approach is known to the liquid-crystal community, but has not yet shown any impact on the general treatment of the problem of the OSD phenomena in anisotropic media. We show below that Mauguin's approach is more general than that based on the gyration pseudotensor. We show also that the DJM approach proposed in this paper is more general with respect to both Mauguin's and the gyration pseudotensor approach, both of which can be obtained from the DJM approach as particular cases.

The paper is organized as follows. In Sec. II we show how the DJM can be calculated from the dielectric tensor. As shown by Jones [13], the form of the DJM allows for identification of the optical phenomena possessed by the medium. This statement is explained in detail in Appendix A. In Sec. III we transform the DJM derived in Sec. II to a form which allows for the identification of the contribution of dielectric tensor components to different optical phenomena. To relate the traditional gyration pseudotensor approach to that developed by Mauguin [15] for cholesterics, we discuss them in Secs. IV and V, respectively. In Sec. VI we propose the description of the OSD in terms of Jones matrices and relate this approach to the gyration pseudotensor and Mauguin's approaches. In Sec. VII we discuss the physical meaning of the DJM components responsible for OSD. Section VIII presents some examples of the application of the DJM approach developed in this paper to the description of the OSD phenomena in optical media. Section IX summarizes our results.

II. RELATION BETWEEN THE DJM AND DIELECTRIC TENSOR

Using the Jones equation for a differential Jones matrix N [13],

$$\frac{d\vec{E}}{dz} = N\vec{E}, \tag{5}$$

Eq. (1) can be rewritten in the form

$$\hat{\varepsilon}\vec{E} = -\lambdabar^2 N^2\vec{E}, \tag{6}$$

where it is taken into account that for a uniform crystal $dN/dz = 0$. The latter condition remains valid as long as one remains within the framework of the approach of eigenwaves with spatially uniform refractive indices, propagating in the medium. An example of such a situation is light propagation in a cholesteric, where the local optic axis helically rotates around and along an axis, which is everywhere perpendicular to the local optic axis. Although for such a spatially modulated structure one would expect the refractive index to be spatially modulated, the problem can be formulated and solved in terms of eigenwaves with spatially uniform refractive indices [16], which thereby implies $dN/dz = 0$.

Equation (6) is a symbolic representation of a system of two equations with four unknown variables N_{ij} and thus cannot be solved without an additional assumption. Such an assumption can be introduced concerning the polarization of the light wave propagating in the medium. For the most general case the propagating light wave is elliptically polarized and thus can be written in the form

$$\vec{E} = \vec{E}_0 e^{-i(\eta/\lambdabar)z}, \tag{7}$$

where $\vec{E}_0 = [E_{x0}, iE_{y0}]^{\mathsf{T}}$ (the superscript T denotes the transpose operation such that $\vec{E}_0$ is a column vector) and η is the refractive index for the propagating light wave. For light absorbing crystals $\eta = n - i\kappa$ is complex, with κ standing for the absorption index; for transparent crystals η reduces to the real refractive index n. From Eqs. (1), (6), and (7) one finds

$$\hat{\varepsilon}\vec{E} = -\lambdabar^2 N^2 \vec{E} = \eta^2 \vec{E}. \tag{8}$$

Equation (8) shows that η^2 is the eigenvalue which is common to the matrices $\hat{\varepsilon}$ and $-\lambdabar^2 N^2$. Since eigenvectors and eigenvalues uniquely define a matrix [17], one has

$$\hat{\varepsilon} = -\lambdabar^2 N^2. \tag{9}$$

The eigenvalue η is common to the matrices $\sqrt{\hat{\varepsilon}}$ and $i\lambdabar N$; consequently, these two matrices, $\sqrt{\hat{\varepsilon}}$ and $i\lambdabar N$, are also equal and

$$N = -\frac{i}{\lambdabar}\sqrt{\hat{\varepsilon}} = -\frac{i}{\lambdabar}\frac{1}{\eta_+ + \eta_-}\begin{bmatrix} \varepsilon_{11} + \eta_+\eta_- & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} + \eta_+\eta_- \end{bmatrix}, \tag{10}$$

where η_+ and η_- are square roots of the eigenvalues of the matrix $\hat{\varepsilon}$, namely,

$$\eta_\pm^2 = \bar{\varepsilon} \pm \sqrt{\varepsilon_{12}\varepsilon_{21} + \frac{\Delta\varepsilon^2}{4}}, \tag{11}$$

with $\bar{\varepsilon} = (\varepsilon_{11} + \varepsilon_{22})/2$ and $\Delta\varepsilon = \varepsilon_{11} - \varepsilon_{22}$. Without taking into account OSD, the dielectric tensor is symmetric, i.e., $\varepsilon_{12} = \varepsilon_{21}$. The solution for N, given by Eq. (10), has of course multiple values. However, as remarked by Jones [11], only one of the four square roots is physically meaningful for a light wave propagating in the medium. For this reason, in this paper we do not consider other solutions. Equation (10) shows that without taking into account OSD, the DJM is symmetric, i.e., $N_{12} = N_{21}$, as a consequence of the symmetry $\varepsilon_{12} = \varepsilon_{21}$ of the dielectric tensor $\hat{\varepsilon}$. Taking into account the OSD increases the symmetry of $\hat{\varepsilon}$ such that $\varepsilon_{12} \neq \varepsilon_{21}$. For a transparent crystal, $\hat{\varepsilon}$ should be Hermitian [12].

In Eq. (10) we use the expression for the square root of a matrix given by Jones in Ref. [11]. From Eq. (11) one finds a useful relation

$$\eta_+\eta_- = \sqrt{\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}\varepsilon_{21}}, \tag{12}$$

with $\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}\varepsilon_{21} = \det[\varepsilon]$ the determinant of $\hat{\varepsilon}$. We show below how Eq. (10) can be obtained via matrix diagonalization. The explicit form of the N matrix can be derived using the second part $-\lambdabar^2 N^2 \vec{E} = \eta^2 \vec{E}$ of Eq. (8), which gives

$$N\vec{E} = -\frac{i}{\lambdabar}\eta\vec{E} = -ik\vec{E}. \tag{13}$$

Equation (13) shows that the value $-ik$ is the eigenvalue of the DJM N, with $k = \frac{\eta}{\lambdabar}$ the wave number for the propagating light wave. Equation (13) can also be obtained via substitution of Eq. (7) in Eq. (5), as we recently have shown in Ref. [16]. The eigenvalues η can be found from the third relation of Eq. (8):

$$\hat{\varepsilon}\vec{E} = \eta^2\vec{E}. \tag{14}$$

Two roots of Eq. (14) are given by Eq. (11). Since the roots values η_+ and η_- define the eigenvalues of the matrix N, according to Eq. (13), the diagonal d form of N is

$$N^d = -\frac{i}{\lambdabar}\begin{bmatrix} \eta_+ & 0 \\ 0 & \eta_- \end{bmatrix}. \tag{15}$$

The full form of N can be reconstructed as

$$N = TN^dT^{-1}, \tag{16}$$

where T^{-1} is the inverse matrix of the transform matrix

$$T = [\vec{E}^+, \vec{E}^-], \tag{17}$$

which is composed of the eigenvectors of $\hat{\varepsilon}$. The eigenvectors $\vec{E}^\pm$ of $\hat{\varepsilon}$ are found via substitution of Eq. (11) into Eq. (14),

$$\vec{E}^\pm = \begin{bmatrix} 1 \\ -\frac{\varepsilon_{11} - (\eta_\pm)^2}{\varepsilon_{12}} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{\varepsilon_{12}}{\varepsilon_{22} - (\eta_\pm)^2} \end{bmatrix}, \tag{18}$$

where it is taken into account that the dielectric tensor is symmetric, i.e., $\varepsilon_{12} = \varepsilon_{21}$. It should be noted that the number of eigenvectors of a matrix is infinite, because their x and y components are related through a constant, which can take any value. To resolve this issue, one normalizes either the x or y component to 1. It is the x component in Eq. (18) which is normalized to 1. One can show that all the transform matrices constructed of such eigenvectors give the same full form of N when substituted in Eq. (16); see Ref. [16], where the latter statement is proven for a cholesteric. Substitution of Eqs. (15), (17), and (18) in Eq. (16) indeed gives Eq. (10). Recently, we used the algorithm of the derivation of the matrix N, given by Eq. (16), for derivation of the DJM for a cholesteric liquid crystal [16]. The equivalence of the two algorithms (10) and (16) is based on the matrix property, according to which the square root of a matrix can be calculated via its diagonalization.

We recall that the aim of this paper is to establish how the elements of the matrix $\hat{\varepsilon}$ contribute to N and to trace the contribution of OSD to N. As shown by Jones [11], one of the advantages of the DJM is that its form identifies optical phenomena possessed by the medium. The form of the N matrix (in the following called the template form), in which each of the four components is in the complex rectangular form $N_{ij} = N'_{ij} - iN''_{ij}$, with N'_{ij} and N''_{ij} being split into averages and differences of opposite components standing on the same matrix diagonal, enables the establishment of how the components of $\hat{\varepsilon}$ contribute to optical properties of an anisotropic medium. This statement is explained in Appendix A. In the next section we use Eq. (10) to relate the contribution of the real and imaginary parts of the components of $\hat{\varepsilon}$ to N and thus to identify the contributions of $\hat{\varepsilon}$ components to optical phenomena, predicted by the template form of N [see Eq. (A6)].

III. CONTRIBUTION OF $\hat{\varepsilon}$ COMPONENTS TO OPTICAL PROPERTIES OF TRANSPARENT CRYSTALS: ε_{ij} ARE REAL NUMBERS

For the light transparent medium ε_{ij} components and n_+ are real values, whereas according to Eq. (11) the value of n_- becomes imaginary for nonzero off-diagonal ε_{ij} components if $\bar{\varepsilon} < \sqrt{(\varepsilon_{12})^2 + \frac{\Delta\varepsilon^2}{4}}$. This case is analyzed below. For a light absorbing medium ε_{ij} components are complex numbers with nonzero real and imaginary parts. This case is analyzed in Appendix B.

In the most general case for the light absorbing anisotropic medium the DJM can be written in its template form

$$N = -\frac{i}{\lambda}\begin{bmatrix} \bar{n} - i\bar{\kappa} + \mathcal{B}_{\text{lin}} - i\mathcal{D}_{\text{lin}} & \mathcal{B}_{\text{Jones}} - i\mathcal{D}_{\text{Jones}} + \mathcal{D}_{\text{circ}} - i\mathcal{A}_{\text{opt}} \\ \mathcal{B}_{\text{Jones}} + i\mathcal{D}_{\text{Jones}} - (\mathcal{D}_{\text{circ}} - i\mathcal{A}_{\text{opt}}) & \bar{n} - i\bar{\kappa} - (\mathcal{B}_{\text{lin}} - i\mathcal{D}_{\text{lin}}) \end{bmatrix}, \tag{19}$$

where $\mathcal{B}_{\text{lin}}$, $\mathcal{D}_{\text{lin}}$, $\mathcal{B}_{\text{Jones}}$, $\mathcal{D}_{\text{Jones}}$, $\mathcal{D}_{\text{circ}}$, and $\mathcal{A}_{\text{opt}}$ stand for the contributions of linear birefringence, linear dichroism, Jones birefringence, Jones dichroism, circular dichroism, and optical activity, respectively (see Appendix A for details). For light transparent crystal ε_{ij} are real numbers and Eq. (19) reduces to the form

$$N^0 = -\frac{i}{\lambda}\begin{bmatrix} \bar{n}^{\text{eff}} + \mathcal{B}_{\text{lin}} & \mathcal{B}_{\text{Jones}} \\ \mathcal{B}_{\text{Jones}} & \bar{n}^{\text{eff}} - \mathcal{B}_{\text{lin}} \end{bmatrix}, \tag{20}$$

with $\bar{n}^{\text{eff}} = \frac{\bar{\varepsilon}+n_+n_-}{n_++n_-}$, $\mathcal{B}_{\text{lin}} = \frac{1}{2}\frac{\Delta\varepsilon}{n_++n_-}$, $\mathcal{B}_{\text{Jones}} = \frac{1}{2}\frac{\varepsilon_{12}+\varepsilon_{21}}{n_++n_-}$, and $\mathcal{D}_{\text{circ}} = \frac{1}{2}\frac{\varepsilon_{12}-\varepsilon_{21}}{n_++n_-}$. Accounting for the symmetry of the dielectric tensor $\varepsilon_{12} = \varepsilon_{21}$, one has $\mathcal{B}_{\text{Jones}} = \frac{\varepsilon_{12}}{n_++n_-}$ and $\mathcal{D}_{\text{circ}} = 0$. It is clear that without taking into account OSD, the symmetric form $\varepsilon_{21} = \varepsilon_{12}$ of the dielectric tensor results in $\mathcal{D}_{\text{circ}} = 0$ for all crystals: for transparent as well as for light absorbing crystals, independently of the crystal symmetry. It is also worth noting that Eq. (20) does not contain the $\mathcal{A}_{\text{opt}}$ term, responsible for optical activity. The latter shows that optical activity cannot be described without taking into account OSD, i.e., circular dichroism and optical activity are phenomena of OSD.

Any tensor with real components can be transformed to its diagonal form by the rotation of the coordinate system. For optically uniaxial crystals, i.e., those of high- and middle-symmetry classes, whose symmetry is not lower than the symmetry of orthorhombic crystallographic classes, the axes of the crystal-physical coordinate system (which is attached to the crystal lattice) coincide with the axes of the coordinate system in which the tensor reduces to its diagonal form (called the principal coordinate system). Therefore, for optically uniaxial crystals $\varepsilon_{12} = \varepsilon_{21} = 0$ and thus $\bar{n}^{\text{eff}}$ reduces to $\bar{n} = (\sqrt{\varepsilon_{11}} + \sqrt{\varepsilon_{22}})/2$ and $\mathcal{B}_{\text{lin}}$ reduces to $\Delta n/2 = (\sqrt{\varepsilon_{11}} - \sqrt{\varepsilon_{22}})/2$, whereas $\mathcal{B}_{\text{Jones}}$ and $\mathcal{D}_{\text{circ}}$ are zero. Therefore, as expected, for uniaxial crystals, the average of the square roots of the diagonal components of the dielectric tensor gives the average refractive index $\bar{n}$, whereas their difference gives the linear birefringence Δn of the medium. Equal diagonal components $\varepsilon_{11} = \varepsilon_{22}$ thus imply that the medium is optically isotropic.

For optically biaxial crystals, i.e., those of low-symmetry classes, $\varepsilon_{12} = \varepsilon_{21} \neq 0$. Equation (20) indicates that a medium with real ε_{ij}, n_-, and n_+ possesses linear birefringence $\mathcal{B}_{\text{lin}}$ and Jones birefringence $\mathcal{B}_{\text{Jones}}$. To give an idea on the origin of Jones birefringence in crystals of low-symmetry classes, we remark that nonzero $\varepsilon_{12} = \varepsilon_{21} \neq 0$ can be obtained by rotation of the coordinate system around the Z axis by an angle γ,

$$\begin{aligned}\hat{\varepsilon} &= R(\gamma)\hat{\varepsilon}^{(d)}R^{-1}(\gamma) \\ &= \begin{bmatrix} \bar{\varepsilon}^{(d)} + \frac{1}{2}\Delta\varepsilon^{(d)}\cos 2\gamma & \frac{1}{2}\Delta\varepsilon^{(d)}\sin 2\gamma \\ \frac{1}{2}\Delta\varepsilon^{(d)}\sin 2\gamma & \bar{\varepsilon}^{(d)} - \frac{1}{2}\Delta\varepsilon^{(d)}\cos 2\gamma \end{bmatrix},\end{aligned} \tag{21}$$

where $\hat{\varepsilon}^{(d)}$ is the diagonal form of $\hat{\varepsilon}$, $\bar{\varepsilon}^{(d)} = (\varepsilon_{11}^{(d)} + \varepsilon_{22}^{(d)})/2$, $\Delta\varepsilon^{(d)} = \varepsilon_{11}^{(d)} - \varepsilon_{22}^{(d)}$, and $\varepsilon_{11}^{(d)}$ and $\varepsilon_{22}^{(d)}$ are the principal values of $\hat{\varepsilon}^{(d)}$. The angle γ can be interpreted as the angle of disorientation between the principal coordinate system of the dielectric tensor and the crystal-physical coordinate system of the crystal. Substitution of Eq. (21) in Eqs. (10) and (11) gives

$$\begin{aligned}N &= -\frac{i}{\lambda}\begin{bmatrix} \bar{n} + \frac{1}{2}\Delta n\cos 2\gamma & \frac{1}{2}\Delta n\sin 2\gamma \\ \frac{1}{2}\Delta n\sin 2\gamma & \bar{n} - \frac{1}{2}\Delta n\cos 2\gamma \end{bmatrix} \\ &= R(\gamma)\begin{bmatrix} n_1 & 0 \\ 0 & n_2 \end{bmatrix}R^{-1}(\gamma),\end{aligned} \tag{22}$$

with the refractive indices for the eigenwaves $n_+ = n_1$ and $n_- = n_2$, where $n_1 = \sqrt{\varepsilon_{11}^{(d)}}$ and $n_2 = \sqrt{\varepsilon_{22}^{(d)}}$ are refractive indices of the crystal in the principal coordinate system of the dielectric tensor and $\bar{n}$ and Δn are their average and difference, respectively. From Eq. (22), for such a crystal one finds that $\mathcal{B}_{\text{lin}} = (\Delta n/2)\cos 2\gamma$ and $\mathcal{B}_{\text{Jones}} = (\Delta n/2)\sin 2\gamma$. Therefore, we are led to conclude that the transparent low-symmetry crystals possess Jones birefringence, which originates in the disorientation between the crystal-physical coordinate system and the principal coordinate system of the dielectric tensor. The axes of the DJM for such a crystal are rotated by the same angle γ by which the diagonal dielectric tensor $\hat{\varepsilon}^{(d)}$ is rotated and thus the axes of the DJM for a low-symmetry crystal are along the axes of the crystal-physical coordinate system. Such a crystal plate is equivalent to a crystal plate rotated by the angle γ from the principal coordinate system, with the same effective birefringence $\Delta n^{\text{eff}} = 2\sqrt{\mathcal{B}_{\text{lin}}^2 + \mathcal{B}_{\text{Jones}}^2} = \Delta n$. The substitution of Eq. (22) in Eq. (3) gives the IJM of the crystal in the form

$$J = R(\gamma)\begin{bmatrix} e^{-(i/\bar{\lambda})n_1 z} & 0 \\ 0 & e^{-(i/\bar{\lambda})n_2 z}\end{bmatrix}R^{-1}(\gamma), \quad (23)$$

which shows that in this case the effect of Jones birefringence reduces to the biasing of the coordinate system by the angle γ.

According to Eq. (11) the value of n_- could be imaginary if

$$\bar{\varepsilon} < \sqrt{(\varepsilon_{12})^2 + \frac{\Delta\varepsilon^2}{4}}. \quad (24)$$

It is clear that the condition (24) is never satisfied if $\varepsilon_{12} = 0$. If the off-diagonal component ε_{12} is nonzero due to the disorientation between the crystal-physical and principal coordinate systems of the dielectric tensor, then substitution of Eq. (21) in Eq. (11) gives $n_\pm^2 = \varepsilon^{(d)} \pm \frac{1}{2}\Delta\varepsilon^d$, thereby showing that both values n_+ and n_- are real numbers and thus the condition (24) is not satisfied for any angle γ of disorientation. Therefore, without taking into account OSD in transparent crystals, refractive indices n_+ and n_- of both eigenwaves are real values.

IV. CONTRIBUTION OF OSD TO THE DIELECTRIC TENSOR AND DJM

Optical spatial dispersion is a phenomenon of nonlocal interaction of a light wave with a medium such that the propagating optical field is defined by material parameters not only in a given point but also in its vicinity. In such a case the dielectric tensor is an operator containing spatial derivatives. For the Z axis of the Cartesian coordinate system directed along the light propagation direction, which is along the normal to the sample plate, the dielectric displacement vector of the light wave is perpendicular to the Z axis. In this coordinate system the vector of dielectric displacement can be decomposed into the in-plane and normal components with respect to the plane of the sample plate. Because of the transverse character of the electromagnetic light field, the normal component of the dielectric displacement $D_z = 0$ and thus one can express E_z through E_x and E_y, thereby excluding E_z from consideration. Then in the 2D form the vector of dielectric displacement is

$$\vec{D} = \hat{\varepsilon}^{\rm eff}\begin{bmatrix} E_x \\ E_y \end{bmatrix}, \quad (25)$$

where $\hat{\varepsilon}^{\rm eff}$ is a 2×2 tensor with [11]

$$\begin{aligned}
\varepsilon_{11}^{\rm eff} &= \varepsilon_{11}^{(3)0} - \frac{1}{\varepsilon_{33}^{0}}\left[\left(\varepsilon_{13}^{(3)0}\right)^2 - \left(\bar{\lambda} g_{23}^{(3)}\vec{\nabla}_z\right)^2\right],\\
\varepsilon_{12}^{\rm eff} &= \varepsilon_{12}^{(3)0} - \bar{\lambda} g_{33}^{(3)}\vec{\nabla}_z - \frac{1}{\varepsilon_{33}^{(3)0}}\left(\varepsilon_{13}^{(3)0} + \bar{\lambda} g_{23}^{(3)}\vec{\nabla}_z\right)\left(\varepsilon_{23}^{(3)0} + \bar{\lambda} g_{13}^{(3)}\vec{\nabla}_z\right),\\
\varepsilon_{21}^{\rm eff} &= \varepsilon_{12}^{(3)0} + \bar{\lambda} g_{33}^{(3)}\vec{\nabla}_z - \frac{1}{\varepsilon_{33}^{(3)0}}\left(\varepsilon_{13}^{(3)0} - \bar{\lambda} g_{23}^{(3)}\vec{\nabla}_z\right)\left(\varepsilon_{23}^{(3)0} - \bar{\lambda} g_{13}^{(3)}\vec{\nabla}_z\right),\\
\varepsilon_{22}^{\rm eff} &= \varepsilon_{22}^{(3)0} - \frac{1}{\varepsilon_{33}^{(3)0}}\left[\left(\varepsilon_{23}^{(3)0}\right)^2 - \left(\bar{\lambda} g_{13}^{(3)}\vec{\nabla}_z\right)^2\right],
\end{aligned} \quad (26)$$

where the superscript 0 corresponds to the 3D (superscript 3) dielectric tensor components without taking into account OSD. It can be seen from Eq. (26) that by taking into account the OSD, the 2×2 tensor $\hat{\varepsilon}^{\rm eff}$ becomes nondiagonal and nonsymmetric $\varepsilon_{12}^{\rm eff} \neq \varepsilon_{21}^{\rm eff}$. The values g_{ij} are coefficients of the series expansion of the dielectric tensor and thus $g_{ij} \ll \varepsilon_{ij}$. For this reason, one can neglect the squares and cross products of g_{ij} and consequently Eq. (26) reduces to the form

$$\hat{\varepsilon}^{\rm eff} = \begin{bmatrix} \varepsilon_{11}^{(3)0} - \frac{(\varepsilon_{13}^{(3)0})^2}{\varepsilon_{33}^{(3)0}} & \varepsilon_{12}^{(3)0} - \frac{\varepsilon_{13}^{(3)0}\varepsilon_{23}^{(3)0}}{\varepsilon_{33}^{(3)0}} - \bar{\lambda}\frac{g_{33}^{(3)}\varepsilon_{33}^{(3)0} + \varepsilon_{13}^{(3)0}g_{13}^{(3)} + \varepsilon_{23}^{(3)0}g_{23}^{(3)}}{\varepsilon_{33}^{(3)0}}\vec{\nabla}_z \\ \varepsilon_{12}^{(3)0} - \frac{\varepsilon_{13}^{(3)0}\varepsilon_{23}^{(3)0}}{\varepsilon_{33}^{(3)0}} + \bar{\lambda}\frac{g_{33}^{(3)}\varepsilon_{33}^{(3)0} + \varepsilon_{13}^{(3)0}g_{13}^{(3)} + \varepsilon_{23}^{(3)0}g_{23}^{(3)}}{\varepsilon_{33}^{(3)0}}\vec{\nabla}_z & \varepsilon_{22}^{(3)0} - \frac{(\varepsilon_{23}^{(3)0})^2}{\varepsilon_{33}^{(3)0}} \end{bmatrix}. \quad (27)$$

Equation (27) can be rewritten in the form

$$\hat{\varepsilon}^{\rm eff} = (\hat{\varepsilon}^0)^{\rm eff} + R\left(\frac{\pi}{2}\right)g_{12}^{\rm eff}\frac{d}{dz}, \quad (28)$$

where

$$\hat{\varepsilon}^{\rm eff} = \begin{bmatrix} \varepsilon_{11}^{(3)0} - \frac{(\varepsilon_{13}^{(3)0})^2}{\varepsilon_{33}^{(3)0}} & \varepsilon_{12}^{(3)0} - \frac{\varepsilon_{13}^{(3)0}\varepsilon_{23}^{(3)0}}{\varepsilon_{33}^{(3)0}} \\ \varepsilon_{12}^{(3)0} - \frac{\varepsilon_{13}^{(3)0}\varepsilon_{23}^{(3)0}}{\varepsilon_{33}^{(3)0}} & \varepsilon_{22}^{(3)0} - \frac{(\varepsilon_{23}^{(3)0})^2}{\varepsilon_{33}^{(3)0}} \end{bmatrix} \quad (29)$$

and $g_{12}^{\rm eff} = \bar{\lambda}(g_{33}^{(3)}\varepsilon_{33}^{(3)0} + \varepsilon_{13}^{(3)0}g_{13}^{(3)} + \varepsilon_{23}^{(3)0}g_{23}^{(3)})/\varepsilon_{33}^{(3)0}$. For a medium with point group symmetry which is not lower than orthorhombic, the off-diagonal components $\varepsilon_{ij}^0 = 0$ and $g_{ij} = 0$ at $i \neq j$ and thus Eq. (26) reduces to

$$\hat{\varepsilon}^{\rm eff} = \begin{bmatrix} \varepsilon_{11}^{0(d)} & -\bar{\lambda} g_{33}\vec{\nabla}_z \\ \bar{\lambda} g_{33}\vec{\nabla}_z & \varepsilon_{22}^{0(d)} \end{bmatrix} = \hat{\varepsilon}^{0(d)} + \bar{\lambda} g_{33} R\left(\frac{\pi}{2}\right)\frac{d}{dz}, \quad (30)$$

with $\varepsilon_{11}^{0(d)} = \varepsilon_{11}^{(3)0}$ and $\varepsilon_{22}^{0(d)} = \varepsilon_{22}^{(3)0}$. Equation (30) shows that taking into account OSD leads to nonzero off-diagonal components in $\varepsilon^{\rm eff}$ even if the matrix $\hat{\varepsilon}^0$ is diagonal. From Eq. (7) one has $d\vec{E}/dz = -ik\vec{E}_0$ ($k = n/\bar{\lambda}$ is the wave number) and thus Eq. (30) takes the form

$$\hat{\varepsilon}^{\rm eff} = \hat{\varepsilon}^{0(d)} - ik\bar{\lambda} g_{33} R\left(\frac{\pi}{2}\right). \quad (31)$$

Equation (31) shows that due to the OSD, the effective dielectric permittivity depends on the wave number k of the light wave, which is the key notion of the OSD concept. The k-dependent term in Eq. (31) is responsible for optical activity; it enters ε^{eff} as an imaginary part, which is in agreement with Eq. (19), where the $\mathcal{A}_{\text{opt}}$ term is also the imaginary part of the off-diagonal components of the matrix in square brackets in Eq. (19). Substitution of Eq. (30) in the Maxwell equation (1) gives

$$\left\{\hat{\varepsilon}^{0(d)} + \lambda\!\!\!^{-} g_{33} R\left(\frac{\pi}{2}\right)\frac{d}{dz}\right\}\vec{E} = -\lambda\!\!\!^{-2}\frac{d^2\vec{E}}{dz^2}. \tag{32}$$

Expressions (25)–(32) represent an approach for taking into account OSD in the optics of solid crystals [12,14]. The pseudotensor of gyration is a material tensor, the components of which can be experimentally measured [18]. Despite its approximate character, this approach is commonly accepted in crystal optics, being considered the most general consideration available for OSD phenomena.

An alternative approach, which does not involve the notion of the gyration pseudotensor, was developed by Mauguin [15] for light propagation along the helical axis in a cholesteric liquid crystal. We show below that Mauguin's approach is more general than that based on the gyration tensor (30). Inspired by Mauguin's approach, in this paper we develop a general description of OSD phenomena in an optical medium that is not based on the serial expansion approximation. To be explicit, we first briefly revisit Mauguin's approach for a cholesteric and then, based on it, propose a more generalized approach which allows for tracing of the contribution of OSD phenomena to the dielectric tensor and DJM of an optical medium.

V. LIGHT PROPAGATION IN A CHOLESTERIC ALONG THE HELICAL AXIS IN TERMS OF THE DJM

Liquid crystal is a medium possessing orientational order of anisometric building units (molecules or their aggregates) at the lowered dimensionality of their translation order [19]. In solid crystals the building elements (atoms or molecules) form a 3D crystalline lattice. In liquid crystals at least in one direction the molecules can freely migrate as in a liquid. Liquid crystal media with 2D crystalline order are represented by columnar phases. Columns formed by molecules are arranged in a 2D lattice but can freely move along the third direction, which is along the columns-long axes. Liquid crystals with 1D translational order are layered media, called smectic liquid crystals. The lowest zero dimensionality of the translational order corresponds to a nematic liquid crystal (nematic, for short) for which the ∞-fold rotational axes of elongated (or disklike) building units are on average oriented along a common axis called the nematic director $\vec{n}$; nematic building units can freely migrate as in a true liquid.

A cholesteric is a chiral nematic in which the director $\vec{n}$ is spontaneously twisted around an axis $\vec{Z}$ of a Cartesian coordinate system such that $\vec{n} \perp \vec{Z}$. In the framework of Mauguin's approach [15], a cholesteric is modeled by a stack of parallel uniaxial birefringent (nematic) plates, whose optic axes are parallel to the plate plane and helically rotate around the normal to the plates by a constant angle from plate to plate. A plate at a distance z appears to be rotated by the angle qz with respect to the plate at the origin of the coordinate system, i.e., at $z = 0$. The parameter $q = 2\pi/P$ is the wave number for the helix with the pitch P (a distance at which the optic axis of plates makes a full turn). In liquid-crystal terminology one says that for a nematic plate the optic axis is oriented along the director $\vec{n}$.

It should be noted that the director $\vec{n}$ corresponds to the local optic axis of an elementary nematic model plate but not to the macroscopic optic axis of the cholesteric. Depending on the light propagation regime (see [16] for details), the cholesteric can be considered as either a twisted nematic, uniaxial gyrotropic crystal or as a biaxial gyrotropic crystal. In particular, for a light propagation regime, in which the cholesteric is an optically uniaxial crystal, the cholesteric optic axis is along its helical axis, i.e., it is perpendicular to $\vec{n}$.

If one chooses the local Cartesian coordinate system for each elementary nematic model plate such that the light propagation direction is along the plate normal and at the distance $z = 0$ the $\vec{X}$ axis is along the director $\vec{n}$, then at the distance z the electric-field vector $\vec{E}$ and the dielectric displacement vector $\vec{D}$ in the rotated coordinate system with respect to the nonrotated coordinate system are

$$\vec{E} = R(qz)\vec{E}^0, \tag{33}$$

$$D = R(qz)\vec{D}^0, \tag{34}$$

where $\vec{E}^0$ and $\vec{D}^0$ are the corresponding vectors in the nonrotated (local) coordinate system and $R(qz)$ is the rotation matrix

$$R(qz) = \begin{bmatrix} \cos qz & -\sin qz \\ \sin qz & \cos qz \end{bmatrix}. \tag{35}$$

Vectors $\vec{D}$ and $\vec{D}^0$ are related to $\vec{E}$ and $\vec{E}^0$, respectively, through dielectric tensors $\hat{\varepsilon}$ and $\hat{\varepsilon}^0$ in the rotated and local coordinate systems as

$$\vec{D}^0 = \varepsilon_0 \hat{\varepsilon}^{0(d)}\vec{E}^0, \tag{36}$$

$$\vec{D} = \varepsilon_0 \hat{\varepsilon}\vec{E}, \tag{37}$$

where $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m is the dielectric constant and

$$\hat{\varepsilon}^{0(d)} = \begin{bmatrix} \varepsilon_\parallel & 0 \\ 0 & \varepsilon_\perp \end{bmatrix}, \tag{38}$$

expressed through its principal values, measured along the local coordinate axes in the plane perpendicular to the cholesteric helical axis, which are along and perpendicular to the nematic director $\vec{n}$ in the local coordinate system. Substitution of Eqs. (33), (34), and (36) in Eq. (37) gives the relation between $\hat{\varepsilon}$ and $\hat{\varepsilon}^0$,

$$\hat{\varepsilon} = R(qz)\hat{\varepsilon}^{0(d)}R^{-1}(qz), \tag{39}$$

as it is expected for a tensor. Substituting Eqs. (33) and (39) in the Maxwell equation (1) for a light wave propagating in a cholesteric, one finds the 2D Maxwell equation in the form

$$\left\{\hat{\varepsilon}^{0(d)} - \lambda\!\!\!^{-2} q^2 I + 2\lambda\!\!\!^{-2} q R\left(\frac{\pi}{2}\right)\frac{d}{dz}\right\}\vec{E}^0 = -\lambda\!\!\!^{-2}\frac{d^2\vec{E}^0}{dz^2}. \tag{40}$$

A detailed derivation of Eq. (40) can be found in [16]. Solutions of Eq. (40) for elliptically polarized eigenwaves [Eq. (7)] result in expressions for refractive indices of the two eigenwaves propagating in a cholesteric [16,20]

$$(n^0_\pm)^2 = \bar{n}^2 + \frac{\Delta n^2}{4} + (q\lambda)^2 \pm\sqrt{\bar{n}^2\Delta n^2 + 4\left(\bar{n}^2 + \frac{\Delta n^2}{4}\right)(q\lambda)^2}, \tag{41}$$

where

$$\bar{\varepsilon} = \frac{\varepsilon_\parallel + \varepsilon_\perp}{2} = \bar{n}^2 + \frac{\Delta n^2}{4}, \quad \bar{n} = \frac{n_\parallel + n_\perp}{2} = \frac{\sqrt{\varepsilon_\parallel} + \sqrt{\varepsilon_\perp}}{2},$$
$$\Delta\varepsilon = \varepsilon_\parallel - \varepsilon_\perp = 2\bar{n}\Delta n, \quad \Delta n = n_\parallel - n_\perp = \sqrt{\varepsilon_\parallel} - \sqrt{\varepsilon_\perp}. \tag{42}$$

Comparing Eq. (40) to Eq. (32), one finds that for a cholesteric

$$\hat{\varepsilon}^{\text{eff}} = \hat{\varepsilon}^{0(d)} + 2\bar{\lambda}^2 qR\left(\frac{\pi}{2}\right)\frac{d}{dz} - \bar{\lambda}^2 q^2 I. \tag{43}$$

It is worth noting that Eq. (43) can be written in the form

$$\hat{\varepsilon}^{\text{eff}} = \hat{\varepsilon}^{0(d)} - 2ik\bar{\lambda}^2 qR\left(\frac{\pi}{2}\right) - \bar{\lambda}^2 q^2 I \tag{44}$$

in the same way as it was done above in the pseudogyration approach (31). Again, the k-dependent term appears to be an imaginary part of the off-diagonal components of ε^{eff} as it is for the k-dependent term in the gyration pseudotensor approach. The imaginary character of the k-dependent terms in the gyration pseudotensor and cholesteric approaches is the result of the dependence of $\vec{D}$ on the spatial derivative $d\vec{E}/dz$.

Equations (43) and (30) show that the structure of ε^{eff} in both (gyration pseudotensor and cholesteric) approaches is similar, though only to some extent. Indeed, in both expressions (30) and (43), in addition to the original tensor $\hat{\varepsilon}^0$, which describes the optical properties of the medium without OSD, there is a common term proportional to $R(\frac{\pi}{2})\frac{d}{dz}$, responsible for the OSD. The coefficient $2\bar{\lambda}q$ in Eq. (43) plays the role of a gyration tensor component g_{33} in Eq. (30). Therefore, the g_{33} term from Eq. (30) appears in Mauguin's approach given by Eq. (43). The cholesteric approach appears to be more general with respect to the gyration tensor approach. Indeed, in comparison with Eq. (30), Eq. (43) contains an additional term $-\bar{\lambda}^2q^2I$. In Eq. (43) the term $-\bar{\lambda}^2q^2I$ results from the second derivative of the rotation matrix $R(qz)$, namely, $-\bar{\lambda}^2q^2I = \bar{\lambda}^2\frac{d^2R(qz)}{dz^2}R^{-1}(qz)$.

Equation (33), which is a key hypothesis in Mauguin's approach, describes the rotation of the electric-field vector along the Z-coordinate axis. Below we show that Eq. (33) can be rewritten in terms of the Jones matrices approach such that the gyration pseudotensor and cholesteric approaches can be derived from this general approach as particular cases.

VI. OSD IN TERMS OF JONES MATRICES

The electric-field vector of the light eigenwave propagating in an optical medium in the Jones calculus approach is of the form

$$\vec{E}(z) = J(z)\vec{E}^i, \tag{45}$$

where $\vec{E}^i$ is the electric-field vector of an incident light wave. Let the electric-field vector of the eigenwave at the coordinate z without taking into account OSD be denoted by $\vec{E}^{(0)}(z)$. Then

$$\vec{E}^{(0)} = J^0\vec{E}^i, \tag{46}$$

where J^0 is the IJM of the medium without taking into account OSD. Substitution of $\vec{E}^i$, expressed in Eq. (46), in Eq. (45) gives

$$\vec{E}(z) = J^D(z)\vec{E}^0(z), \tag{47}$$

where

$$J^D = J(J^0)^{-1} \tag{48}$$

plays the role of the IJM responsible for the OSD effects; we call J^D [Eq. (48)] the OSD IJM. Similarly to Eq. (4), the OSD IJM can be expressed via its corresponding OSD DJM N^D,

$$J^D = e^{(N^D)z}. \tag{49}$$

It should be noted that N^D is not restricted to being symmetric as it is for the matrix N^0 without taking into account OSD [see Eq. (B5)]. The explicit form of the matrix N^D and the physical meaning of its components will be given in Sec. VII.

Because both $\vec{E}(z)$ and $\vec{D}(z)$ are vectors they transform along the coordinate axis $\vec{Z}$ by the same rule, given for $\vec{E}(z)$ by Eq. (47), i.e.,

$$\vec{D}(z) = J^D(z)D^0(z), \tag{50}$$

where $\vec{D}^0$ is the vector of dielectric displacement without taking into account OSD. Substitution of Eqs. (36), (37), and (47) in Eq. (50) gives the transformation rule of the dielectric tensor under the action of OSD,

$$\hat{\varepsilon} = J^D\hat{\varepsilon}^0(J^D)^{-1}, \tag{51}$$

where $\hat{\varepsilon}^0$ is the dielectric tensor without taking into account OSD. Substitution of Eqs. (47) and (51) in Eq. (1) gives the Maxwell equation taking into account OSD in the form

$$\left[\hat{\varepsilon}^0 + \lambda^2(J^D)^{-1}\left\{\frac{d^2J^D}{dz^2} + 2\frac{dJ^D}{dz}\frac{d}{dz}\right\}\right]\vec{E}^0 = -\lambda^2\frac{d^2\vec{E}^0}{dz^2}. \tag{52}$$

The term in square brackets in Eq. (52),

$$\hat{\varepsilon}^{\text{eff}} = \hat{\varepsilon}^0 + \bar{\lambda}^2(J^D)^{-1}\left\{\frac{d^2J^D}{dz^2} + 2\frac{dJ^D}{dz}\frac{d}{dz}\right\}, \tag{53}$$

is the effective dielectric permittivity tensor taking into account OSD. Using Eq. (49), one rewrites Eq. (53) in the form

$$\varepsilon^{\text{eff}} = \hat{\varepsilon}^0 + \bar{\lambda}^2\left\{(J^D)^{-1}\frac{d(N^D)}{dz}J^D + (J^D)^{-1}(N^D)^2J^D + 2(J^D)^{-1}N^DJ^D\frac{d}{dz}\right\}. \tag{54}$$

For N^D, which is independent of z, Eq. (54) reduces to

$$\varepsilon^{\text{eff}} = \hat{\varepsilon}^0 + \bar{\lambda}^2\left\{(N^D)^2 + 2N^D\frac{d}{dz}\right\}. \tag{55}$$

Equation (55) was obtained taking into account that the matrices N^D and $e^{(N^D)z}$ commute. The latter can be proven using the definition of the matrix exponent

$$e^{(N^D)z} = 1 + \sum_{k=1}^{k} (N^D)^k \frac{z^k}{k!}. \tag{56}$$

With Eq. (56) the question of the commutation of the matrices N^D and $e^{(N^D)z}$ reduces to the commutation of the matrix N^D with itself. Another proof of Eq. (55) can be done, using the property of the Hausdorff equation [21,22]

$$e^B A e^{-B} = A + \frac{[B, A]}{1!} + \frac{[B, [B, A]]}{2!} + \frac{[B, [B, [B, A]]]}{3!} + \cdots, \tag{57}$$

where A and B are any square matrices and $[B, A]$ is their commutator. For $A = B = N^D$ one has $[B, A] = [N^D, N^D] = 0$. Then Eq. (57) reduces to

$$e^{(N^D)z} N^D e^{-(N^D)z} = N^D \tag{58}$$

and consequently

$$e^{(N^D)z} (N^D)^2 e^{-(N^D)z} = (N^D)^2. \tag{59}$$

Comparison of Eqs. (53) and (55) to Eq. (30) shows that the term $2ƛ^2 (J^D)^{-1} \frac{dJ^D}{dz} \frac{d}{dz}$ in Eq. (53) and the term $2ƛ^2 N^D(z) \frac{d}{dz}$ in Eq. (55) correspond to the traditional form $ƛ g_{33} R(\frac{\pi}{2}) \frac{d}{dz}$ of the OSD term, Eq. (30) in the gyration pseudotensor approach.

Equation (55) demonstrates that even with the condition $\frac{dN^D}{dz} = 0$ the OSD terms appear in ε^{eff}. In fact, by keeping the terms containing $\frac{dN^D}{dz} \neq 0$, one exits the framework of eigenwaves with spatially uniform refractive indices. We recall that even taking into account the OSD effect, one assumes that the refractive indices of the eigenwaves are spatially uniform, though being dependent on the parameter characterizing the OSD. For example, for a cholesteric liquid crystal the nematic director $\vec{n}$ (which is the local optic axis[1]) helically rotates with respect to the axis, which is perpendicular to $\vec{n}$. However the refractive indices of the eigenwaves are spatially uniform, being given by Eq. (41) and thus depending on the ratio of the light wavelength to the cholesteric pitch $\lambda/P = qƛ$, which is a measure of OSD for a cholesteric. At vanishing λ/P, corresponding to long pitches with respect to the light wavelength, the refractive indices n_+ and n_- reduce, respectively, to the values $n_\parallel$ and $n_\perp$ of the untwisted (parent) nematic.

In this paper we do not delve deeper into the DJM approach for the optics of cholesterics; interested readers can find such consideration in our recent paper [16]. A description of the optical properties of the cholesteric in terms of integral Jones matrices can be found in papers by Oldano and co-workers [23–30], Yang and Mi [31], and Gevorgyan [32]. Here we use the DJM approach for cholesterics as a link between the traditional gyration pseudotensor approach and the DJM approach to the description of OSD effects.

VII. PHYSICAL MEANING OF THE PARAMETERS IN N^D

From Eq. (48) one has

$$J = J^D J^0, \tag{60}$$

which shows that J^D is the correction matrix to the IJM J^0. The most general form of $J^D(z)$ is given by Eq. (49), with N^D being of the form

$$N^D = -\frac{i}{ƛ} \begin{bmatrix} \delta\bar{n} - i\delta\bar{\kappa} + \delta\mathcal{B}_{\text{lin}} - i\delta\mathcal{D}_{\text{lin}} & \delta\mathcal{B}_{\text{Jones}} - i\delta\mathcal{D}_{\text{Jones}} + \mathcal{D}_{\text{circ}} - i\mathcal{A}_{\text{opt}} \\ \delta\mathcal{B}_{\text{Jones}} - i\delta\mathcal{D}_{\text{Jones}} - (\mathcal{D}_{\text{circ}} - i\mathcal{A}_{\text{opt}}) & \delta\bar{n} - i\delta\bar{\kappa} - (\delta\mathcal{B}_{\text{lin}} - i\delta\mathcal{D}_{\text{lin}}) \end{bmatrix}. \tag{61}$$

The eight parameters in the matrix N^D are contributions of the OSD to the DJM of the medium; they are akin to those specified in Eq. (19), but of different origin. For diagonal (d) matrices $N^{0(d)}$ and $N^{D(d)}$,

$$N^{(d)} = N^{0(d)} + N^{D(d)}, \tag{62}$$

which shows that $N^{D(d)}$ is an additive correction to $N^{0(d)}$. For this reason we have denoted the parameters in N^D by δ to highlight that they are correction parameters *per se*, and thus can take either positive or negative values. In a general case for nondiagonal N^0 and N^D the DJM of the medium is not equal to their sum. To express N through the components of nondiagonal N^0 and N^D one has to solve the equation

$$e^{Nz} = e^{(N^D)z} e^{(N^0)z} \tag{63}$$

with respect to N. Equation (63) can be rewritten in the form

$$e^{\bar{\eta}z} \Delta J = e^{(\bar{\eta}^0 + \bar{\eta}^D)z} \Delta J^D \Delta J^0 \tag{64}$$

[see Eq. (A8)]. From Eq. (64) one has

$$\bar{\eta} = \bar{\eta}^0 + \bar{\eta}^D, \tag{65}$$

$$\Delta J = \Delta J^D \Delta J^0 \begin{bmatrix} \Delta J^D_{11} \Delta J^0_{11} + \Delta J^D_{12} \Delta J^0_{21} & \Delta J^D_{11} \Delta J^0_{12} + \Delta J^D_{12} \Delta J^0_{22} \\ \Delta J^D_{21} \Delta J^0_{11} + \Delta J^D_{22} \Delta J^0_{21} & \Delta J^D_{21} \Delta J^0_{12} + \Delta J^D_{22} \Delta J^0_{22} \end{bmatrix}. \tag{66}$$

[1]The local optic axis along the director is a modeling issue, which should not be misunderstood as the optic axis of the cholesteric. Depending on the regime of light propagation, the cholesteric can be considered as either a twisted nematic uniaxial cholesteric or a biaxial cholesteric. For the uniaxial regime, the optic axis of the cholesteric is along the helical axis.

Equation (65) shows that $\bar{\eta}^D$ is an OSD correction to the average refractive index $\bar{\eta}^0$ independently of the forms (diagonal or nondiagonal) of N^0 and N^D (equivalently of J^0 and J^D). If the off-diagonal components of N^D are zero, then ΔJ is diagonal,

$$\Delta J^{(d)} = \begin{bmatrix} e^{-(i/\lambda\!\!\bar{})(\Delta\eta^0+\Delta\eta^D)z} & 0 \\ 0 & e^{(i/\lambda\!\!\bar{})(\Delta\eta^0+\Delta\eta^D)z} \end{bmatrix}, \qquad (67)$$

which shows that the diagonal part of N^D corrects the complex phases of the eigenwaves, including their anisotropies, such that the correction of the refraction indices is equivalent to the correction of the length z of the light path in the exponent in Eq. (67). However, the action of nonzero $\bar{\eta}^D$ is not only in the simple biasing of the refractive index. In Sec. VIII we will show that for a light transparent crystal the nonzero OSD correction $\delta\bar{\eta}^D$ to the average refractive index leads to the light scattering.

For nondiagonal forms of N^0 and N^D (equivalently of J^0 and J^D) the average complex refraction indices are corrected additively by $\bar{\eta}^D$, whereas the components of the anisotropic part ΔJ in Eq. (66) appear to be cross coupled such that the diagonal (off-diagonal) components of ΔJ contain off-diagonal (diagonal) components of J^0 and J^D. The latter implies that for nondiagonal forms of N^0 and N^D (equivalently of J^0 and J^D) different anisotropic optical properties are coupled such that, for example, linear birefringence $\mathcal{B}_{\text{lin}}$ (linear dichroism $\mathcal{D}_{\text{lin}}$) is affected not only by $\delta\mathcal{B}_{\text{lin}}$ (by $\delta\mathcal{D}_{\text{lin}}$), but also by other anisotropic corrections: $\mathcal{A}_{\text{opt}}$, $\mathcal{D}_{\text{circ}}$, $\delta\mathcal{B}_{\text{Jones}}$, and $\delta\mathcal{D}_{\text{Jones}}$.

Therefore, the matrix N^D [Eq. (61)] is composed of eight correction parameters due to the OSD effects to the parameters in N^0 describing eight optical phenomena which are possible in an anisotropic medium. It should be noted that the components of the matrix N are solutions of the Maxwell equation, whereas the components of N^D are modeling parameters, which are considered to be known from the experiment. For example, one of the eight optically elementary matrices in Eq. (19),

$$N^{\mathcal{A}_{\text{opt}}} = R\left(\frac{\pi}{2}\right)q, \qquad (68)$$

is the matrix describing rotation of light polarization [for details see Eq. (A6)]. In Eq. (68) we refer to light propagation in a cholesteric and for this reason replace the parameter $\mathcal{A}_{\text{opt}}$, introduced in Eq. (19), by $q\lambda\!\!\bar{}$ [16], where $q = 2\pi/P$ is the wave number of the cholesteric helix with the pitch P. The explicit form of the IJM $J^{\mathcal{A}_{\text{opt}}}$ is obtained by substitution of $N^{\mathcal{A}_{\text{opt}}}$ given by Eq. (68), instead of N^D in Eq. (49), namely,

$$J^{\mathcal{A}_{\text{opt}}} = e^{R(\pi/2)qz} = \begin{bmatrix} \cos qz & -\sin qz \\ \sin qz & \sin qz \end{bmatrix} = R(qz). \qquad (69)$$

Equation (69) shows that for a cholesteric the general form of the OSD IJM (47) reduces to its particular form of the rotation matrix $R(qz)$ [Eq. (35)]. It is worth noting that by replacing J^D in Eq. (51) by $J^{\mathcal{A}_{\text{opt}}}$, given by Eq. (69), one recovers the transformation rule for the dielectric tensor in a cholesteric (39) as a particular case of a general form given by Eq. (51). Replacement of J^D in Eq. (47) by $J^{\mathcal{A}_{\text{opt}}}$, given by Eq. (69), gives Eq. (33), which is the transformation law for the eigenwave on its propagation in a cholesteric. Replacement of J^D by $J^{\mathcal{A}_{\text{opt}}}$ reduces Eqs. (52) and (53) to Eqs. (40) and (43) in the cholesteric approach such that the second term $\lambda\!\!\bar{}^2(J^D)^{-1}\frac{d^2J^D}{dz^2}$ in Eq. (53) and the term $\lambda\!\!\bar{}^2(N^D)^2$ in Eq. (55), respectively, reduce to the term $-\lambda\!\!\bar{}^2q^2I$ in Eq. (43), obtained in the cholesteric approach. There is no term corresponding to the term $-\lambda\!\!\bar{}^2q^2I$ in Eq. (30), obtained in the gyration pseudotensor approach. Equations (30) and (43) are thus particular cases of the general form of the dielectric tensor taking into account OSD, given by Eq. (53). Therefore, for a cholesteric, in a physical sense J^D is the rotation matrix $R(qz)$ and the parameter $q = \mathcal{A}_{\text{opt}}/\lambda\!\!\bar{}$ in N^D [Eq. (61)] is the rotation angle per unit length for the molecules along the helical axis of the cholesteric. For other media the parameter $\mathcal{A}_{\text{opt}}$ stands for modeling of the angle of rotation of light polarization. Another parameter $\mathcal{D}_{\text{circ}}$, which appears in N due to the nonsymmetric condition $\varepsilon_{12} \neq \varepsilon_{21}$ more exactly than from the antisymmetric part of the dielectric tensor, describes circular dichroism. Both these parameters $\mathcal{A}_{\text{opt}}$ and $\mathcal{D}_{\text{circ}}$ are absent in the DJM N^0, obtained without taking into account OSD [see Eq. (B5)].

In a physical sense, the six other δ parameters in N^D are corrections due to OSD to the corresponding parameters found in N^0. For example, $\delta\bar{n}$ is the correction to the average refractive index accounting for the inhomogeneity of the electric field of the light wave caused by the inhomogeneity of the medium.

It is interesting to find N^D, which corresponds to the dielectric tensor (28) taking into account OSD in terms of the pseudogyration tensor. The substitution of Eq. (28) in Eq. (10) followed by the expansion of $\sqrt{\varepsilon^{\text{eff}}}$ in a series of the small parameter containing g^{eff} with the truncation to the first order gives

$$N = N^0 + \frac{1}{2}g^{\text{eff}}(N^0)^{-1}R\left(\frac{\pi}{2}\right)(N^0). \qquad (70)$$

According to Eq. (62), the second term in Eq. (70) is N^D. After matrix multiplications in Eq. (70) one has

$$N^D = \frac{1}{2}g^{\text{eff}}\begin{bmatrix} -\frac{n^0_{12}(n^0_{11}+n^0_{22})}{n^0_{11}n^0_{22}-(n^0_{12})^2} & \frac{1}{2}\frac{(n^0_{11}+n^0_{22})^2}{n^0_{11}n^0_{22}-(n^0_{12})^2} + \left(1-\frac{1}{2}\frac{(n^0_{11}+n^0_{22})^2}{n^0_{11}n^0_{22}-(n^0_{12})^2}\right) \\ \frac{1}{2}\frac{(n^0_{11}+n^0_{22})^2}{n^0_{11}n^0_{22}-(n^0_{12})^2} - \left(1-\frac{1}{2}\frac{(n^0_{11}+n^0_{22})^2}{n^0_{11}n^0_{22}-(n^0_{12})^2}\right) & \frac{n^0_{12}(n^0_{11}+n^0_{22})}{n^0_{11}n^0_{22}-(n^0_{12})^2} \end{bmatrix}, \qquad (71)$$

where according to Eq. (10) $n^0_{ij} = \sqrt{\varepsilon^{(0)\mathrm{eff}}_{ij}}$ $(i, j = 1, 2)$ are components of the DJM $N^0 = -(i/\lambda\!\!\bar{})n^0_{ij}$ without taking into account OSD; $\varepsilon^{(0)\mathrm{eff}}$ is given by Eq. (29). Equation (71) shows that in the framework of the gyration pseudotensor approach the OSD DJM is proportional to g^{eff}, which indicates that without taking into account OSD, i.e., at $g^{\mathrm{eff}} = 0$, one has $N^D = 0$. The diagonal components $N^D_{11} = -N^D_{22}$ are proportional to n^0_{12}, which is zero for media of high- and middle-symmetry classes; the diagonal components are of opposite sign and thus correspond in Eq. (61) to the correction $\delta\Delta\eta = \delta\mathcal{B}_{\mathrm{lin}} - i\delta\mathcal{D}_{\mathrm{lin}}$. The corrections $\delta\bar{\eta} = \delta\bar{\eta} - i\delta\bar{\kappa} = 0$ to the average refraction and absorption indices are zero. The off-diagonal components of N^D contain symmetric and antisymmetric parts, which correspond to the corrections $\delta\bar{g} = \mathcal{B}_{\mathrm{Jones}} - i\mathcal{D}_{\mathrm{Jones}}$ and $\delta\Delta g = \mathcal{D}_{\mathrm{circ}} - i\mathcal{A}_{\mathrm{opt}}$. For light transparent media the symmetric and antisymmetric parts of the off-diagonal components describe Jones birefringence and optical activity.

In the next section we give examples of how the DJM approach developed in this paper can be applied to the description of the OSD phenomena in different optical media.

VIII. EXAMPLES

The description of the medium in the framework of the Jones matrix calculus is considered to be complete if the IJM J of the medium is known. The general scheme for the derivation of the IJM of the medium taking into account OSD can be sketched as follows.

The starting point is the Maxwell equation (1), in which $\hat{\varepsilon}$ has to be replaced by $\hat{\varepsilon}^{\mathrm{eff}}$, given by Eq. (55). The term $\hat{\varepsilon}^0$ in Eq. (55) is the dielectric tensor of the medium in the crystal-physical coordinate system without taking into account OSD; the form of $\hat{\varepsilon}^0$ (number of nonzero components and relations between them) is governed by the symmetry of the medium and can be found in works on crystal physics (see, for example, [33]). Two other terms in Eq. (55), containing N^D, account for OSD. For a given medium the general form of N^D [Eq. (61)] reduces to a form which is specific for the medium. Below we give illustrations of how the matrix N^D can be constructed for different media. Using Eqs. (5), (13), and (55), one transforms the Maxwell equation (1) into an equation for eigenvalues η^0 of the DJM of the medium

$$\{\hat{\varepsilon}^0 - (\eta^0 I + i\lambda\!\!\bar{}\, N^D)^2\}\vec{E}^0 = 0. \tag{72}$$

The characteristic (also called secular) equation (72) has nontrivial solutions for the eigenwaves $\vec{E}^0$ if

$$\det[\hat{\varepsilon}^0 - (\eta^0 I + i\lambda\!\!\bar{}\, N^D)^2] = 0. \tag{73}$$

Explicitly, the secular equation (73) is a quartic equation

$$(\eta^0)^4 + a_3(\eta^0)^3 - a_2(\eta^0)^2 - a_1\eta^0 + a_0 = 0, \tag{74}$$

with

$$\begin{aligned}
a_3 &= 4\delta\bar{\eta},\\
a_2 &= 2\bar{\varepsilon}^0 - 6(\delta\bar{\eta})^2 + 2(\delta\bar{g})^2 - \frac{1}{2}(\Delta\delta g - \Delta\delta\eta),\\
a_1 &= 4\delta\bar{\eta}\{\bar{\varepsilon}^0 + (\delta\bar{\eta})^2 - (\delta\bar{g})^2 + (\Delta\delta\eta)^2/4 - (\Delta\delta g)^2/4\}\\
&\quad - 4\delta\bar{g}\varepsilon^0_{12} - \Delta\delta\eta\Delta\varepsilon^0,\\
a_0 &= \left\{\bar{\varepsilon}^0 - (\delta\bar{\eta})^2 - \frac{1}{4}(\Delta\delta\eta)^2 - (\delta\bar{g})^2 + \frac{1}{4}(\Delta\delta g)^2\right\}^2\\
&\quad - \left\{\frac{\Delta\varepsilon}{2} - \delta\bar{\eta}\Delta\delta\eta\right\}^2 - \{\varepsilon^0_{12} - 2\delta\bar{\eta}\Delta\delta\eta\}^2.
\end{aligned} \tag{75}$$

A quartic equation is a polynomial equation with the highest degree, which can be solved analytically by radicals [34]. There are several different approaches to find four roots of the full quartic equation analytically, but generally the formulas for solutions are rather lengthy and their analysis is beyond the scope of the present paper. Nevertheless, several properties of the solutions can be given even before solving the equation. For real coefficients a_j, which correspond to the case of light transparent media, there are several rules which allow one to predict whether the solutions are rational or complex [35].

For an isotropic transparent medium with nonzero OSD correction $\delta\bar{n}$ to the values of the refractive index and zero to all other parameters in N^D [Eq. (61)], the coefficients a_j [Eqs. (75)] are real. Taking into account that $(\delta\bar{n})^2 \ll \bar{\varepsilon}^0$, we find, for the discriminant of Eq. (74),

$$\Delta = -16384\delta\bar{n}^6(\bar{\varepsilon}^0)^3. \tag{76}$$

Equation (76) shows that Δ is negative, independently of the sign of $\delta\bar{n}$. Negative values of Δ indicate that with the above assumptions, Eq. (74) has two distinct real roots and two complex-conjugate nonreal roots. For such a case of the isotropic medium with the isotropic corrections $\delta\bar{n}$, we find the solutions of Eq. (74) in the form

$$(\bar{\eta}_0)_{1,2} = \sqrt{\bar{\varepsilon}^0} - \delta\bar{n}\left(1 \mp \sqrt{\frac{\delta\bar{n}}{\sqrt{\bar{\varepsilon}^0}}}\right), \tag{77}$$

$$(\bar{\eta}_0)_{3,4} = -\sqrt{\bar{\varepsilon}^0} - \delta\bar{n}\left(1 \mp \sqrt{-\frac{\delta\bar{n}}{\sqrt{\bar{\varepsilon}^0}}}\right). \tag{78}$$

The first pair of roots, given by Eq. (77), corresponds to the transmitting waves, whereas the second pair given by Eq. (78) corresponds to the reflecting waves. In agreement with the prediction following from the negative sign of Δ, there are two real roots and two complex-conjugate roots. In Eqs. (77) and (78) $\delta\bar{n}$ can be either positive or negative. Consequently, the answer to the question of which of the roots are real and which are complex depends on the sign of $\delta\bar{n}$. Nonzero $\delta\bar{n}$ is due to spatial inhomogeneity of the electric field of the light wave propagating in the medium. In turn the spatially inhomogeneous electric field is a result of spatial inhomogeneity of the refractive index such that $\delta\bar{n} = \frac{dn}{dz}dz$. Therefore, the sign of $\delta\bar{n}$ is governed by the sign of the space derivative $\frac{dn}{dz}$ at a given point with the coordinate z. Since the medium under study is assumed to be light transparent, we assign the imaginary parts of the refractive indices in the transmitted waves, for $\delta\bar{n} < 0$, and in the reflected waves, for $\delta\bar{n} > 0$, to the forward and backward scattering depending on the plus or minus sign in Eqs. (77) and (78). Therefore, taking into account OSD allows for the taking into account of light scattering on the refractive index inhomogeneity in the framework of Jones calculus.

It is commonly accepted that the traditional Jones calculus without taking into account OSD is not capable of describing light scattering. A special type of matrix, the so-called Mueller matrix, was designed for this purpose [36]. We have shown above in this section that the light scattering can be described in the framework of Jones calculus taking into account OSD. Spatial inhomogeneity of the average refractive index can be expected, for example, in transparent isotropic media with a spatially nonuniform mass density or concentration of the components, caused, for example, by the spatially inhomogeneous temperature field. Therefore, the origin of the OSD correction $\delta\bar{\eta} = \delta\bar{n} - i\delta\bar{\kappa}$ can be assigned to the spatial inhomogeneity of the average dielectric permittivity. Indeed, the coefficient a_3 of the cubic term in Eq. (74) is nonzero only if the OSD correction $\delta\bar{\eta}$ to the average refractive index is nonzero. Physical origins of the other correction parameters in Eq. (61) can be defined in the same way. Namely, the OSD correction $\Delta\delta\eta = \delta\mathcal{B}_{\rm lin} - i\delta\mathcal{D}_{\rm lin}$ corresponds to the spatial inhomogeneity of the refractive index anisotropy; similarly, other correction parameters in N^D can be interpreted as a result of spatial inhomogeneity of the corresponding optical parameters standing in the DJM (19) [see also Eq. (A6)].

According to Eq. (75), zero OSD correction to the average complex refractive index $\delta\bar{\eta} = 0$ leads to a zero value of the coefficient a_3 of the cubic term in Eq. (74). Equation (74) additionally simplifies for a zero value of the coefficient a_1 of the linear term. For isotropic and uniaxial media (of the high- and middle-symmetry classes, respectively) the off-diagonal components $\varepsilon^0_{12} = \varepsilon^0_{21} = 0$ and according to Eq. (75) the coefficient a_1 of the linear term in Eq. (74) is zero for zero corrections to the average refractive index of the eigenwaves $\delta\bar{\eta}$ and to the refractive index anisotropy $\Delta\delta\eta$. For biaxial crystals $\varepsilon^0_{12} = \varepsilon^0_{21} \neq 0$ and consequently an additional condition for the absence of the linear term in Eq. (74) is zero correction $\delta\bar{g} = 0$ to the Jones birefringence and Jones dichroism.

For $a_3 = a_1 = 0$ the quartic equation (74) reduces to the Fresnel biquadratic equation

$$(\eta^0)^4 - 2\left[\bar{\varepsilon}^0 - \tfrac{1}{4}(\Delta\delta g)^2\right](\eta^0)^2 + \left[\bar{\varepsilon}^0_{11} - \tfrac{1}{4}(\Delta\delta g)^2\right] \times \left[\bar{\varepsilon}^0_{22} - \tfrac{1}{4}(\Delta\delta g)^2\right] = 0, \tag{79}$$

the solutions of which are of the form

$$(\eta^0_\pm)^2 = \bar{\varepsilon}^0 - \frac{1}{4}(\Delta\delta g)^2 \pm \sqrt{\frac{(\Delta\varepsilon^0)^2}{4} - \bar{\varepsilon}^0(\Delta\delta g)^2}. \tag{80}$$

For a light-transparent cholesteric the substitution of $\eta^0_\pm = n^0_\pm$ and $\Delta\delta g = 2iq\lambda\!\!\!^{-}$ in Eq. (80) gives

$$(n^0_\pm)^2 = \bar{\varepsilon}^0 + (\lambda\!\!\!^{-} q)^2 \pm \sqrt{4\bar{\varepsilon}^0(\lambda\!\!\!^{-} q)^2 + \frac{(\Delta\varepsilon^0)^2}{4}}, \tag{81}$$

which can be rewritten in the form of Eq. (41), using the substitutions given by Eqs. (42).

It should be noted that the present approach does not provide instruction for the calculation of the values of the components in the OSD DJM N^D. The correction parameters appearing in N^D are modeling parameters and thus their values are subject to the experimental determination in the same way that the refractive absorption indices and their anisotropies are.

For example, it is known from experiments that the cholesteric possesses optical activity caused by the rotation of the director around the helical axis of the cholesteric. The optically elementary matrix N^D responsible for optical activity is given by Eq. (68). For a light transparent cholesteric, by neglecting all other correction parameters and substituting Eq. (64) in Eq. (55), one recovers Eq. (43), which gives the effective dielectric permittivity tensor $\hat{\varepsilon}^{\rm eff}$ of a cholesteric taking into account OSD.

Once N^d of the medium is constructed then $\hat{\varepsilon}^{\rm eff}$, written using Eq. (55), has to be substituted instead of $\hat{\varepsilon}$ in Eq. (8), the solutions of which give the refractive indices of the eigenwaves taking into account OSD. The obtained refractive indices are eigenvalues of the DJM for the medium and thus give the diagonal form N^d [Eq. (15)] of the DJM of the medium. The full form of the DJM of the medium is obtained by substituting N^d in Eq. (16), in which the transform matrix T is composed of the eigenvectors, obtained from the same equation, from which the eigenvalues $n_\pm$ were derived. To find the IJM of the medium, one substitutes the DJM, obtained from Eq. (16), into Eq. (4). A detailed derivation of the DJM and IJM for a cholesteric can be found in our recent paper [16].

It is worth noting that in the approach of OSD developed in this paper in terms of Jones matrix calculus, the Mauguin solutions for a cholesteric (41) were obtained without the coordinate transformations, which is the key hypothesis of the Mauguin [15] and de Vries [20] approaches. In this respect our approach to OSD in terms of Jones calculus is coordinateless. Another coordinateless approach, the so-called invariant approach, for description of light propagation in anisotropic media was developed by Fedorov [37].

IX. CONCLUSION

Traditionally, optical spatial dispersion is defined as the dependence $\varepsilon(\vec{k})$ of the dielectric permittivity tensor on the light wave vector, similarly to the frequency dispersion of the dielectric tensor $\varepsilon(\omega)$. The dependence $\varepsilon(\vec{k})$ is the result of nonlocal interaction of the electric field $\vec{E}$ of the light wave with a medium such that in the next approximation one accounts for the dependence of the dielectric displacement $\vec{D}$ not only on the local value of $\vec{E}$ in a given point, but also on its values in the vicinity of the given point. As a result, the dielectric tensor $\hat{\varepsilon}$ becomes an operator which includes the $\vec{\nabla}$ vector of space derivatives (of $\vec{E}$ in the expression for $\vec{D}$).

We have developed an approach for the description of the OSD phenomena in the framework of Jones calculus. In Jones calculus the differential Jones matrix N is the generalization for the light wave vector $\vec{k}$ in the same sense that $\vec{k}$ is the generalization for the light wave number $k = n/\lambda\!\!\!^{-}$. The wave number k multiplied by the imaginary unit $-i$ (taken with the negative sign) is the eigenvalue of the DJM N via equation $N\vec{E} = -ik\vec{E}$ [Eq. (13)]. The latter inspires us to expect that there must exist a way to describe the OSD phenomena in terms of the DJM. Since the DJM is a generalization for $\vec{k}$, the DJM approach for OSD is a generalization of the traditional approach based on the k expansion of the dependence $\varepsilon(\vec{k})$. We have shown that such a relation between the OSD phenomena and Jones calculus indeed exists. To prove the latter we have derived a general relation between the DJM and components of the dielectric permittivity tensor $\hat{\varepsilon}$. One of the advantages of the DJM approach is that its template form (19) allows for identification of the optical phenomena which are possible in an anisotropic medium. In the most general case the DJM can be decomposed into the sum of eight elementary matrices describing eight (four pairs of) optical phenomena: refraction and absorption, linear birefringence and linear dichroism, Jones birefringence and Jones dichroism, and circular dichroism and optical activity.

To be explicit, we have established the relation of the approach proposed in this paper to the traditional approach of the gyration pseudotensor as well as to that developed by Mauguin for light propagation in cholesteric liquid crystals. We demonstrate that both the gyration pseudotensor and the Mauguin approach can be derived as particular cases of the proposed DJM approach.

In our approach the IJM $J = e^{Nz}$ of the medium taking into account OSD is the product of the IJM $J^0 = e^{(N^0)z}$ without taking into account OSD by the correction IJM $J^D = e^{(N^D)z}$, which accounts for the OSD effects, i.e., $J = e^{(N^D)z}e^{(N^0)z}$. Because in a general case N^0 and N^D do not commute, $e^{(N^D)z}e^{(N^0)z} \neq e^{(N^0+N^D)z}$ and consequently the DJM N of the medium without taking into account OSD is not the sum of the DJM N^0 and a correction OSD DJM N^D [Eq. (61)]. Equation (62), $N \approx N^0 + N^D$, holds only in an approximation which takes into account the smallness of the components of N^D. In this approximation the components of N^D are correction parameters accounting for the contribution of the OSD to each of the six parameters describing six of eight optical phenomena which are possible in an anisotropic medium without taking into account OSD. The two remaining parameters $\mathcal{A}_{\rm opt}$ and $\mathcal{D}_{\rm circ}$ do not have their counterparts in the DJM N^0 without taking into account OSD. The parameters $\mathcal{A}_{\rm opt}$ and $\mathcal{D}_{\rm circ}$ are not corrections *per se* to the parameters appearing in N^0, as the other six parameters are. They describe properties which do not appear without taking into account OSD, i.e., optical activity and circular dichroism are phenomena of purely OSD origin. Concerning the properties of Jones birefringence and Jones dichroism, which were predicted by Jones in Ref. [13], we show that both these properties originate from the nonzero off-diagonal components of the dielectric tensor. We conclude that the transparent low-symmetry crystals possess Jones birefringence, which originates in the disorientation between the crystal-physical coordinate system and the principal coordinate system of the dielectric tensor.

In a general case when all components of the OSD DJM N^D are nonzero, the secular equation is a quartic equation (74). The coefficient a_3 of the cubic term in the secular equation is nonzero only for nonzero corrections $\bar{\eta}^D$ to the average refractive index. For transparent crystals at $\delta\bar{n}^D \neq 0$ and at zero for all other correction parameters in N^D, the secular equation has two distinct real and two complex-conjugate roots. Two of the solutions describe transmitting waves and the other two describe reflecting waves. In both pairs of solutions there are terms containing either $\sqrt{\delta\bar{n}}$ (transmitting waves) or $\sqrt{-\delta\bar{n}}$ (reflecting waves). Thus, at $\delta\bar{n} > 0$ the solutions are real for transmitting waves, but complex conjugates for reflecting waves and, contrarily, at $\delta\bar{n} > 0$ the solutions are complex conjugates for transmitting waves but real for reflecting waves. For transparent crystals we assign the complex-conjugate terms in the refractive index of the eigenwaves to the forward and backward light scattering depending on the plus or minus sign in Eqs. (77) and (78). Therefore, taking into account the OSD effect on the refractive index, the Jones calculus becomes capable of describing light scattering.

For zero coefficients a_3 and a_1 of the cubic and linear terms in Eq. (74), respectively, which holds for $\delta\bar{\eta} = \delta\Delta\eta = \delta\bar{g} = 0$, the secular equation (74) reduces to a biquadratic equation (79), which is equivalent to the secular equation for a cholesteric liquid crystal [16].

In this paper we have focused on the relation between the proposed approach for description of OSD phenomena in terms of Jones calculus to the traditional approach, based on the gyration pseudotensor approach as well as that developed by Mauguin for cholesteric liquid crystals. We have illustrated how the OSD phenomena can be accounted for in an optically isotropic transparent medium with nonzero OSD correction parameter $\delta\bar{n}$ to the average refractive index $\bar{n}$ and with all other correction parameters zero in the correction matrix N^D as well as in a cholesteric liquid crystal with nonzero parameter $\Delta\delta g = 2iq\lambda$ and other correction parameters zero. However, the proposed Jones calculus approach is a general tool for taking into account OSD in any optically inhomogeneous media, including those in which several or all OSD correction parameters are simultaneously nonzero. For example, taking into account OSD, corrections $\delta\bar{n}$ to the average refractive indices of the eigenwaves and to their anisotropy $\delta\mathcal{B}_{\rm lin}$ should be applied for the description of light propagation in transparent distorted liquid crystal cells with bend and splay distortions of the director field. An example of such a liquid crystal cell is the so-called hybrid nematic cell, in which the director $\vec{n}$ smoothly varies through the cell thickness from planar (parallel to the substrate) at one substrate to homeotropic (perpendicular to the substrate) at the opposite substrate. At nonzero corrections $\delta\bar{n}$ and $\delta\mathcal{B}_{\rm lin}$ the coefficients a_3 and a_1 of the cubic and linear terms in Eq. (74) will be nonzero and thus, taking into account that the discriminant of the secular equation is negative $\Delta < 0$, one can expect that the solutions will be complex conjugates, describing the internal light scattering caused by the spatial variation of the director along the light propagation through the hybrid nematic cell. In liquid-crystal cells the distortion of the director field might be accompanied by orientational singularities, defects in the spatial distribution of the director. In some cases the defects appear spontaneously even if the surface alignment conditions are not singular. Such situations are observed in thin (on the order of several microns) nematic films with degenerate hybrid director alignment [38], in hybrid aligned smectic-*A* films films, in which focal conic domains appear [19,39–41], and in cholesteric films with the same alignment conditions showing stripe and focal conic domain textures [19,42]. In the vicinity of macroscopic defects, where the director distortions are considerably strong, the OSD phenomena should be well expressed. It should also be noted that the light propagation in distorted liquid crystals is a complicated matter: In addition to the polarization properties described in the frame of Jones calculus, one expects also changes in the trajectory of the propagating light waves [43] for which the ray-tracing matrix approach [44,45] should be employed.

If a distorted liquid crystal is light absorbing then four correction parameters $\delta\bar{n}$, $\delta\bar{k}$, $\delta\mathcal{B}_{\text{lin}}$, and $\delta\mathcal{D}_{\text{lin}}$ will be nonzero. All eight correction parameters in N^D will be nonzero, for example, for a light absorbing liquid crystal with splay, twist, and bend deformations of the director field. The OSD phenomena are expected in liquid-crystal phases with a spatially modulated director field such as the nematic twist-bend N_{tb} [46–49] phase and the oblique helicoid state Ch_{OH} of a cholesteric [50,51], the twist grain boundary phases [19,52–55], chiral blue phases [19,56–58], the so-called Q smectics [59–62], and many other thermodynamic phases. The approach developed in this paper is applicable also for nonhomogeneous solid crystals.

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APPENDIX A: THE DJM AS A TEMPLATE FOR IDENTIFICATION OF OPTICAL PHENOMENA IN AN OPTICALLY ANISOTROPIC MEDIUM

According to Eq. (13) the DJM in its most general form can be written as

$$N = -\frac{i}{\lambda\!\!\!^{-}}\begin{bmatrix}\eta_{11} & \eta_{12}\\ \eta_{21} & \eta_{22}\end{bmatrix}. \tag{A1}$$

Though according to Eq. (10) the symmetric form $\hat{\varepsilon}_{ij} = \hat{\varepsilon}_{ji}$ implies the same symmetry $N_{ij} = N_{ji}$ for the DJM, in this Appendix we keep the general form $N_{ij} \neq N_{ji}$. The reason for this is that taking into account OSD increases the symmetry of $\hat{\varepsilon}$ and consequently of N. It is easy to see that by the transformation

$$N = \begin{bmatrix}\frac{N_{11}+N_{22}}{2} + \frac{N_{11}-N_{22}}{2} & \frac{N_{12}+N_{21}}{2} + \frac{N_{12}-N_{21}}{2}\\ \frac{N_{12}+N_{21}}{2} - \frac{N_{12}-N_{21}}{2} & \frac{N_{11}+N_{22}}{2} - \frac{N_{11}-N_{22}}{2}\end{bmatrix}, \tag{A2}$$

the DJM can be rewritten in the form

$$N = -\frac{i}{\lambda\!\!\!^{-}}\begin{bmatrix}\bar{\eta} + \Delta\eta & \bar{g} + \Delta g\\ \bar{g} - \Delta g & \bar{\eta} - \Delta\eta\end{bmatrix} \tag{A3}$$

and thus can be decomposed into four elementary matrices

$$N = -\frac{i}{\lambda\!\!\!^{-}}\left(\bar{\eta}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \Delta\eta\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \bar{g}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \Delta g\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right), \tag{A4}$$

where $\bar{\eta} = (\eta_{11} + \eta_{22})/2$ and $\bar{g} = (\eta_{12} + \eta_{21})/2$ are averages of diagonal and off-diagonal components, respectively, and $\Delta\eta = (\eta_{11} - \eta_{22})/2$ and $\Delta g = (\eta_{12} - \eta_{21})/2$ are their half differences. If the components η_{jl} $(j, l = 1, 2)$ in Eq. (A1) are complex numbers such that $\bar{\eta} = \bar{n} - i\bar{\kappa}$, $\Delta\eta = \mathcal{B}_{\text{lin}} - i\mathcal{D}_{\text{lin}}$, $\bar{g} = \mathcal{B}_{\text{Jones}} - i\mathcal{D}_{\text{Jones}}$, and $\Delta g = \mathcal{D}_{\text{circ}} - i\mathcal{A}_{\text{opt}}$, then Eq. (A3) becomes of the form

$$N = -\frac{i}{\lambda\!\!\!^{-}}\begin{bmatrix}\bar{n} - i\bar{\kappa} + \mathcal{B}_{\text{lin}} - i\mathcal{D}_{\text{lin}} & \mathcal{B}_{\text{Jones}} - i\mathcal{D}_{\text{Jones}} + \mathcal{D}_{\text{circ}} - i\mathcal{A}_{\text{opt}}\\ \mathcal{B}_{\text{Jones}} + i\mathcal{D}_{\text{Jones}} - (\mathcal{D}_{\text{circ}} - i\mathcal{A}_{\text{opt}}) & \bar{n} - i\bar{\kappa} - (\mathcal{B}_{\text{lin}} - i\mathcal{D}_{\text{lin}})\end{bmatrix}. \tag{A5}$$

The matrix form (A5) in which the matrix components are in their complex rectangular forms $N_{ij} = N'_{ij} - iN''_{ij}$ and are transformed according to Eq. (A2) is the template form of the DJM. Equation (A5) shows that in the most general case the DJM contains eight parameters. As shown by Jones [13], these eight DJM parameters describe eight optical phenomena which can take place at light propagation in an anisotropic medium. In other words, the DJM N [Eq. (A5)] is the sum of eight optically elementary DJMs, namely,

$$\begin{aligned}N = -\frac{i}{\lambda\!\!\!^{-}}\bigg(&\bar{n}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \bar{\kappa}\begin{bmatrix}i & 0\\ 0 & i\end{bmatrix} + \mathcal{B}_{\text{lin}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} - \mathcal{D}_{\text{lin}}\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}\\ &+\mathcal{B}_{\text{Jones}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} - \mathcal{D}_{\text{Jones}}\begin{bmatrix}0 & i\\ i & 0\end{bmatrix} + \mathcal{D}_{\text{circ}}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} - \mathcal{A}_{\text{opt}}\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}\bigg),\end{aligned} \tag{A6}$$

each of which is responsible for a specific action of the medium on the propagating light wave (further called optically elementary DJMs). Namely, matrices $N^r = -(i/\lambda\!\!\!^{-})\bar{n}$ $[\{1, 0\}; \{0, 1\}]$ and $N^a = (i/\lambda\!\!\!^{-})\bar{\kappa}[\{i, 0\}; \{0, i\}]$ correspond, respectively, to the refraction (r) and absorption (a), described by the average refractive and absorption indices $\bar{n}$ and $\bar{\kappa}$, respectively; matrices $N^{\mathcal{B}_{\text{lin}}} = -(i/\lambda\!\!\!^{-})(\Delta n/2)[\{1, 0\}; \{0, -1\}]$ and $N^{\mathcal{D}_{\text{lin}}} = (i/\lambda\!\!\!^{-})(\Delta\kappa/2)[\{i, 0\}; \{0, -i\}]$ correspond to the linear birefringence $\mathcal{B}_{\text{lin}}$ and linear dichroism $\mathcal{D}_{\text{lin}}$ (conventional notation $\Delta n/2$ and $\Delta\kappa/2$); matrices $N^{\mathcal{B}_{\text{Jones}}} = -(i/\lambda\!\!\!^{-})\mathcal{B}_{\text{Jones}}$ $[\{0, 1\}; \{1, 0\}]$ and $N^{\mathcal{D}_{\text{Jones}}} = (i/\lambda\!\!\!^{-})\mathcal{D}_{\text{Jones}}[\{0, i\}; \{i, 0\}]$ correspond to the Jones birefringence $\mathcal{B}_{\text{Jones}}$ and Jones dichroism $\mathcal{D}_{\text{Jones}}$; matrices $N^{\mathcal{A}_{\text{opt}}} = (i/\lambda\!\!\!^{-})\mathcal{A}_{\text{opt}}[\{0, i\}; \{-i, 0\}]$ and $N^{\mathcal{D}_{\text{circ}}} = -(i/\lambda\!\!\!^{-})\mathcal{D}_{\text{circ}}[\{0, 1\}; \{-1, 0\}]$ correspond to the circular birefringence (also called optical activity $\mathcal{A}_{\text{opt}}$) and circular dichroism $\mathcal{D}_{\text{circ}}$. Therefore, if the analytical form of the DJM of the medium is known, Eq. (A5) or (A6) can be used as a template for identification of optical properties possessed by an anisotropic medium; we will show that it allows for tracing the contributions from OSD to optical characteristics of the medium.

It should be noted that even though the term $i\mathcal{A}_{\rm opt}$, in the off-diagonal components of the matrix in the square brackets in Eq. (A5), is imaginary, it does not originate from light absorption as is the case for the imaginary terms $i\bar{\kappa}$ and $i\mathcal{D}_{\rm lin}$ on the main diagonal of the matrix. It was shown in Secs. IV and V that the $\mathcal{A}_{\rm opt}$ term appears when the OSD is taken into account.

A remark is in order here concerning the notions of Jones birefringence and Jones dichroism. These two optical properties, predicted by Jones in Ref. [13], are rarely mentioned in research papers [63–72] and their physical sense remains mysterious. In this paper we show that both Jones birefringence and Jones dichroism originate from the nonzero off-diagonal components of the dielectric tensor.

It is worth noting that, according to Eq. (3), in a general case the IJM cannot be decomposed into the product of optically elementary IJMs, each of which is responsible for a single optical property, as it can be for the DJM according to Eq. (A6). Indeed, for a DJM, which is the sum of eight optically elementary DJMs, the decomposition of the IJM (3) into a product of single exponents can be done only if the DJM is diagonal. Indeed, for a diagonal DJM $N^d = N^r + N^a + N^{\mathcal{B}_{\rm lin}} + N^{\mathcal{D}_{\rm lin}}$, one has $J^d = e^{N^r z} e^{N^a z} e^{N^{\mathcal{B}_{\rm lin}} z} e^{N^{\mathcal{D}_{\rm lin}} z}$ because the diagonal matrices N^r, N^a, $N^{\mathcal{B}_{\rm lin}}$, and $N^{\mathcal{D}_{\rm lin}}$ commute. For a nondiagonal DJM the optically elementary DJMs do not commute. In the case of a nondiagonal DJM, which is independent of z, the IJM is of the form

$$J = e^{[(N_{11}+N_{22})/2]z}\begin{bmatrix} \cosh\frac{\Gamma z}{2} + \frac{N_{11}-N_{22}}{\Gamma}\sinh\frac{\Gamma z}{2} & 2\frac{N_{12}}{\Gamma}\sinh\frac{\Gamma z}{2} \\ 2\frac{N_{21}}{\Gamma}\sinh\frac{\Gamma z}{2} & \cosh\frac{\Gamma z}{2} - \frac{N_{11}-N_{22}}{\Gamma}\sinh\frac{\Gamma z}{2} \end{bmatrix}, \tag{A7}$$

where $\Gamma = \sqrt{(N_{11}-N_{22})^2 + 4N_{12}N_{21}}$. Therefore, for the DJM consisting of eight optically elementary matrices, after substitution of Eq. (A5) in Eq. (A7) one has

$$J = e^{-(i/\lambda)\bar{\eta}z}\Delta J, \tag{A8}$$

where $\bar{\eta} = \bar{n} - i\bar{\kappa}$ is the complex average refractive index,

$$\Delta J = \begin{bmatrix} \cosh\left(\frac{i}{\lambda}\Gamma_0 z\right) - \frac{\mathcal{B}_{\rm lin}-i\mathcal{D}_{\rm lin}}{\Gamma_0}\sinh\left(\frac{i}{\lambda}\Gamma_0 z\right) & -\frac{\mathcal{B}_{\rm Jones}-i\mathcal{D}_{\rm Jones}+\mathcal{D}_{\rm circ}-i\mathcal{A}_{\rm opt}}{\Gamma_0}\sinh\left(\frac{i}{\lambda}\Gamma_0 z\right) \\ -\frac{\mathcal{B}_{\rm Jones}-i\mathcal{D}_{\rm Jones}-(\mathcal{D}_{\rm circ}-i\mathcal{A}_{\rm opt})}{\Gamma_0}\sinh\left(\frac{i}{\lambda}\Gamma_0 z\right) & \cosh\left(\frac{i}{\lambda}\Gamma_0 z\right) + \frac{\mathcal{B}_{\rm lin}-i\mathcal{D}_{\rm lin}}{\Gamma_0}\sinh\left(\frac{i}{\lambda}\Gamma_0 z\right) \end{bmatrix}, \tag{A9}$$

and $\Gamma_0 = \sqrt{(\mathcal{B}_{\rm lin}-i\mathcal{D}_{\rm lin})^2 + (\mathcal{B}_{\rm Jones}-i\mathcal{D}_{\rm Jones})^2 - (\mathcal{D}_{\rm circ}-i\mathcal{A}_{\rm opt})^2}$. Equation (A9) shows that in the presence of any of the nonzero off-diagonal elements $\mathcal{B}_{\rm Jones}$, $\mathcal{D}_{\rm Jones}$, $\mathcal{D}_{\rm circ}$, and $\mathcal{A}_{\rm opt}$, the IJM cannot be decomposed into a product of optically elementary IJMs, while the DJM is the sum of optically elementary DJMs. For this reason, the description of light propagation in the DJM approach might be more convenient than in the traditional IJM approach.

APPENDIX B: CONTRIBUTION OF $\hat{\varepsilon}$ COMPONENTS TO OPTICAL PROPERTIES OF LIGHT ABSORBING CRYSTALS: ε_{ij} ARE COMPLEX NUMBERS

For light absorbing crystals both $\hat{\varepsilon}$ and N are complex, $\hat{\varepsilon} = \varepsilon' - i\varepsilon''$ and $N = N' - iN''$. As a result, Eq. (9) splits into a system of two matrix equations for the real N' and imaginary N'' parts of N,

$$(\lambda N')^4 + (\lambda N')^2\varepsilon' - \tfrac{1}{4}(\varepsilon'')^2 = 0,$$
$$(\lambda N'')^2 = (\lambda N')^2 + \varepsilon'. \tag{B1}$$

A general form of the solution of the biquadratic matrix equation for N' is not known [73]. In a special case, when the matrices ε' and ε'' commute and $\sqrt{(\varepsilon')^2 + (\varepsilon'')^2}$ can be diagonalized, the solutions for N' and N'' can be written explicitly [73] and we find

$$(\lambda N')^2 = -\tfrac{1}{2}\left[\varepsilon'^{(d)} + \sqrt{(\varepsilon'^{(d)})^2 + (\varepsilon''^{(d)})^2}\right],$$
$$(\lambda N'')^2 = -\tfrac{1}{2}\left[-\varepsilon'^{(d)} + \sqrt{(\varepsilon'^{(d)})^2 + (\varepsilon''^{(d)})^2}\right]. \tag{B2}$$

Both conditions mentioned above obviously are met if the tensors ε' and ε'' are diagonal (d). This is the case for optically uniaxial crystals, for which $\varepsilon_{12} = \varepsilon_{21} = 0$, and consequently from Eq. (10) one finds that Eq. (A5) reduces to the form

$$N = -\frac{i}{\lambda}\begin{bmatrix} \bar{n} - i\bar{\kappa} + \mathcal{B}_{\rm lin} - i\mathcal{D}_{\rm lin} & 0 \\ 0 & \bar{n} - i\bar{\kappa} - (\mathcal{B}_{\rm lin} - i\mathcal{D}_{\rm lin}) \end{bmatrix}. \tag{B3}$$

Equation (B3) shows that in addition to the isotropic refraction and absorption, a uniaxial light absorbing crystal possesses linear birefringence and linear dichroism. The notation $\bar{n}$, $\bar{\kappa}$, $\mathcal{B}_{\rm lin} = \frac{1}{2}\Delta n$, and $\mathcal{D}_{\rm lin} = \frac{1}{2}\Delta\kappa$ corresponds to the average refractive and absorption indices, birefringence, and dichroism, which can be defined through the dielectric tensor components. Namely, for the light absorbing crystal with complex principal components $\varepsilon_{jj}^{(d)} = \varepsilon_{jj}'^{(d)} - i\varepsilon_{jj}''^{(d)}$ (with $j = 1, 2$) of the diagonal (d) dielectric tensor from Eq. (B2), using

Eqs. (10) and (11), we find

$$\begin{aligned}\bar{n} &= \tfrac{1}{2}(n_1+n_2), \quad \Delta n = n_1 - n_2,\\ n_1 &= \left\{\left[\varepsilon_{11}^{\prime(d)} + \sqrt{\left(\varepsilon_{11}^{\prime(d)}\right)^2 + \left(\varepsilon_{11}^{\prime\prime(d)}\right)^2}\right]/2\right\}^{1/2},\\ n_2 &= \left\{\left[\varepsilon_{22}^{\prime(d)} + \sqrt{\left(\varepsilon_{22}^{\prime(d)}\right)^2 + \left(\varepsilon_{22}^{\prime\prime(d)}\right)^2}\right]/2\right\}^{1/2},\\ \bar{\kappa} &= \tfrac{1}{2}(\kappa_1+\kappa_2), \quad \Delta\kappa = \kappa_1 - \kappa_2,\\ \kappa_1 &= \left\{\tfrac{1}{2}\left[-\varepsilon_{11}^{\prime(d)} + \sqrt{\left(\varepsilon_{11}^{\prime(d)}\right)^2 + \left(\varepsilon_{11}^{\prime\prime(d)}\right)^2}\right]\right\}^{1/2},\\ \kappa_2 &= \left\{\tfrac{1}{2}\left[-\varepsilon_{22}^{\prime(d)} + \sqrt{\left(\varepsilon_{22}^{\prime(d)}\right)^2 + \left(\varepsilon_{22}^{\prime\prime(d)}\right)^2}\right]\right\}^{1/2}.\end{aligned} \tag{B4}$$

For optically isotropic absorbing media $\varepsilon_{11} = \varepsilon_{22}$, and thus Eqs. (B4), derived here in the DJM approach, reduce to those given in Ref. [12]. The latter confirms the compatibility of the DJM approach with Maxwell equations. For biaxial crystals the dielectric tensor is not diagonal, $\varepsilon_{12} = \varepsilon_{21} \neq 0$, and consequently the DJM (A5) is of the form

$$N^0 = -\frac{i}{\bar{\lambda}}\begin{bmatrix} \bar{n} - i\bar{\kappa} + \mathcal{B}_{\text{lin}} - i\mathcal{D}_{\text{lin}} & \mathcal{B}_{\text{Jones}} - i\mathcal{D}_{\text{Jones}} \\ \mathcal{B}_{\text{Jones}} - i\mathcal{D}_{\text{Jones}} & \bar{n} + i\bar{\kappa} - (\mathcal{B}_{\text{lin}} - i\mathcal{D}_{\text{lin}}) \end{bmatrix}. \tag{B5}$$

In light absorbing crystals of lower-symmetry classes the real ε' and imaginary ε'' parts of the dielectric tensor are transformed to their diagonal forms by rotation of the crystal-physical coordinate system by different angles γ' and γ'', i.e., the dielectric tensor can be written in the form

$$\hat{\varepsilon} = R(\gamma')\hat{\varepsilon'}^{(d)}R(\gamma') - iR(\gamma'')\hat{\varepsilon''}^{(d)}R(\gamma''). \tag{B6}$$

Among the crystals of low-symmetry classes there are crystals for which $\gamma' = \gamma'' = \gamma$, i.e., the principal axes of the real and imaginary parts of the dielectric tensor appear to be rotated by the same angle γ with respect to the crystal-physical coordinate system. In such a case we find that Eq. (B5) can be rewritten in the form

$$\begin{aligned} N &= -\frac{i}{\bar{\lambda}}\begin{bmatrix} \bar{\eta} + \tfrac{1}{2}\Delta\eta\cos 2\gamma & \tfrac{1}{2}\Delta\eta\sin 2\gamma \\ \tfrac{1}{2}\Delta\eta\sin 2\gamma & \bar{\eta} - \tfrac{1}{2}\Delta\eta\cos 2\gamma \end{bmatrix}\\ &= -\frac{i}{\bar{\lambda}}R(\gamma)\begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{bmatrix}R^{-1}(\gamma), \end{aligned} \tag{B7}$$

where $\eta_{1,2} = n_{1,2} - i\kappa_{1,2}$ are complex indices of refraction for eigenwaves propagating in the crystal, $\bar{\eta} = (\eta_1 + \eta_2)/2$, and $\Delta\eta = \eta_1 - \eta_2$; values of $n_{1,2}$ and $\kappa_{1,2}$ are given by Eqs. (B4). Equation (B7) shows that for low-symmetry crystals for which $\gamma' = \gamma'' = \gamma$ the DJM can be diagonalized by rotation of the crystal plate around its normal by the angle γ, thereby eliminating simultaneously both Jones birefringence $\mathcal{B}_{\text{Jones}} = (\Delta n/2)\sin 2\gamma$ and Jones dichroism $\mathcal{D}_{\text{Jones}} = (\Delta\kappa/2)\sin 2\gamma$.

For matrix coefficients ε' and ε'' in Eq. (B1), which cannot be diagonalized simultaneously, a method for solving the biquadratic equation (B1) is not available, but for weakly absorbing crystals (most of dielectrics) the relations between components of N and $\hat{\varepsilon}$ can be obtained in the approximation $\varepsilon'' \ll \varepsilon'$. By substitution of Eq. (B6) in Eq. (11), followed by expansion of square roots in a series of ε'' and then neglecting terms with $(\varepsilon'')^2$ and of higher powers, we find the components for N [Eqs. (10) and (B5)] in the form

$$\begin{aligned} \bar{n} - i\bar{\kappa} &= \frac{\bar{\varepsilon} + \eta_+\eta_-}{\eta_+ + \eta_-},\\ \mathcal{B}_{\text{lin}} &= \text{Re}\left\{\frac{\Delta\varepsilon}{\eta_+ + \eta_-}\right\}\\ &\approx \frac{\Delta\varepsilon^{\prime(d)}}{\bar{n}_\pm}\cos 2\gamma' + \frac{\Delta\varepsilon^{\prime\prime(d)}\bar{\kappa}_\pm}{\bar{n}_\pm^2}\cos 2\gamma'',\\ \mathcal{D}_{\text{lin}} &= \text{Im}\left\{\frac{\Delta\varepsilon}{\eta_+ + \eta_-}\right\}\\ &\approx \frac{\Delta\varepsilon^{\prime\prime(d)}}{\bar{n}_\pm}\cos 2\gamma'' - \frac{\Delta\varepsilon^{\prime(d)}\bar{\kappa}_\pm}{\bar{n}_\pm^2}\cos 2\gamma',\\ \mathcal{B}_{\text{Jones}} &= \text{Re}\left\{\frac{\varepsilon_{12}}{\eta_+ + \eta_-}\right\}\\ &\approx \frac{\Delta\varepsilon^{\prime(d)}}{\bar{n}_\pm}\sin 2\gamma' + \frac{\Delta\varepsilon^{\prime\prime(d)}\bar{\kappa}_\pm}{\bar{n}_\pm^2}\sin 2\gamma'',\\ \mathcal{D}_{\text{Jones}} &= \text{Im}\left\{\frac{\varepsilon_{12}}{\eta_+ + \eta_-}\right\}\\ &= \frac{\Delta\varepsilon^{\prime\prime(d)}}{\bar{n}_\pm}\sin 2\gamma'' - \frac{\Delta\varepsilon^{\prime(d)}\bar{\kappa}_\pm}{\bar{n}_\pm^2}\sin 2\gamma', \end{aligned} \tag{B8}$$

with $\bar{\eta} = \frac{\eta_+ + \eta_-}{2}$, $\eta_\pm = n_\pm - i\kappa_\pm$,

$$\begin{aligned} n_+ &= \left\{\left[\varepsilon_{11}^{\prime(d)} + \sqrt{\left(\varepsilon_{11}^{\prime(d)}\right)^2 + \left(\varepsilon_{11}^{\prime\prime(d)} - \Delta\varepsilon^{\prime\prime(d)}\sin^2\Delta\gamma\right)^2}\right]/2\right\}^{1/2},\\ n_- &= \left\{\left[\varepsilon_{22}^{\prime(d)} + \sqrt{\left(\varepsilon_{22}^{\prime(d)}\right)^2 + \left(\varepsilon_{22}^{\prime\prime(d)} + \Delta\varepsilon^{\prime\prime(d)}\sin^2\Delta\gamma\right)^2}\right]/2\right\}^{1/2},\\ \kappa_+ &= \left\{\left[-\varepsilon_{11}^{\prime(d)} + \sqrt{\left(\varepsilon_{11}^{\prime(d)}\right)^2 + \left(\varepsilon_{11}^{\prime(d)} - \Delta\varepsilon^{\prime\prime(d)}\sin^2\Delta\gamma\right)^2}\right]/2\right\}^{1/2},\\ \kappa_- &= \left\{\left[-\varepsilon_{22}^{\prime(d)} + \sqrt{\left(\varepsilon_{22}^{\prime(d)}\right)^2 + \left(\varepsilon_{22}^{\prime\prime(d)} + \Delta\varepsilon^{\prime\prime(d)}\sin^2\Delta\gamma\right)^2}\right]/2\right\}^{1/2}. \end{aligned} \tag{B9}$$

Equations (B8) and (B9) show that the light absorbing crystals of low symmetry possess linear birefringence, linear dichroism, Jones birefringence, and Jones dichroism, which cannot be excluded by rotation of the sample. The latter implies that in light absorbing crystals of lowest symmetry with a nondiagonalizable complex dielectric tensor, the Jones birefringence and Jones dichroism are inherent material properties in the same sense that the linear birefringence and linear dichroism are.

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