Operator coherence dynamics in Grover's quantum search algorithm

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Coherence is one of the most basic concepts and resources in quantum information. To clear the role coherence plays on the essential operator level in Grover's search algorithm, here we discuss the coherence dynamics of the state after each basic operator is applyied. As it is known, Grover's search algorithm repeats the application of Grover operator G, which can be decomposed into $G = H^{\otimes n}PH^{\otimes n}O$, where H is Hadamard operator, P is the condition phase-shift operator, and O is the oracle operator. First, we show that O and P are incoherent operators while $H^{\otimes n}$ is coherent. Second, we prove that the amount of the operator coherence of the first $H^{\otimes n}$ and the operator coherence produced or depleted by $H^{\otimes n}$ depends not only on the size of the database and the success probability, but also on target states. Moreover, the amount of operator coherence is larger when the superposition state of targets is entangled rather than product. Third, we show that the two $H^{\otimes n}$ have different effects on coherence that one produces coherence and the other depletes coherence, and the depletion plays a major role. Therefore, the coherence is vibrating during the search process and the overall effect is that coherence is in depletion.

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I. INTRODUCTION

Coherence, as one of the essential quantum properties which derived from the superposition principle of quantum states, plays an important role in quantum physics [1,2] and quantum information [3–5]. Coherence not only exists in bipartite and multipartite systems, but also exists in a single system. However, a rigorous framework for quantifying coherence was proposed by Baumgratz *et al.* recently [6]. From the viewpoint of the resource theory, the authors established the quantitative theory of coherence following the approach that has been established for entanglement [7–9]. Following their footsteps, the characterization and quantification of coherence aroused a great deal of interest [10–15]. It is worth noting that coherence can be converted to other quantum resources, such as entanglement and discord, by suitable operations [16–19]. Egloff *et al.* [20] unified entanglement, discord, and coherence as different aspects of a single underlying resource theory.

The role of coherence in quantum algorithms has attracted people's attention [21–27]. For the Deutsch-Jozsa algorithm, Hillery showed that coherence can be viewed as a resource in the sense that a bigger amount of coherence decreases the failure of this algorithm [21]. In deterministic quantum computation with one qubit (DQC1), Matera *et al.* [22] displayed that the precision is directly related to the recoverable coherence.

Grover's search algorithm (GSA) [28], as one of the famous quantum algorithms, was introduced to speed up the search process which is a quadratic improvement over its classical one [29]. To achieve the speedup, it was proven that multipartite entanglement is necessary for GSA operating on pure state [30]. Therefore, a great deal of research works have been done to investigate the properties of entanglement in GSA [31–36]. These works showed that entanglement plays an important role in GSA and relates to the success probability. As an important quantum resource, coherence in GSA has been investigated [24–26]. Anand and Pati [24] studied the relation between coherence and success probability in the analog GSA, which was based on adiabatic Hamiltonian evolution. They found that the success probability of the algorithm is related to the amount of coherence. Shi *et al.* [25] investigated the role of quantum coherence dynamics in GSA. They showed that the behavior about the quantum coherence depletion enhances the success probability. Reference [26] introduced a discrete coherence monotone named the coherence number and showed the similar conclusion that the enhancement of success probability consumes coherence as iteration number increases. These works discussed the coherence of the state after each iteration of G in GSA.

GSA repeats the application of Grover iteration G which consists of $G = H^{\otimes n}PH^{\otimes n}O$, where H is Hadamard operator, P is the condition phase-shift operator, and O is the oracle operator. To investigate how these operators contribute to coherence and the relationship among them and the success probability, in this paper, we would investigate the coherence dynamics of each basic operator in GSA. We show that O and P are incoherent operators while two $H^{\otimes n}$ are coherent

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operators. The two $H^{\otimes n}$ have different effects on coherence that one produces coherence and the other depletes coherence in terms of the l_1 norm of coherence. Therefore, in the whole process, the coherence is vibrating. With the help of O, P which are incoherent but entangling operators, the vibration of the coherence is weakened. We discuss different cases of the target states and show that the coherence of the state after the first $H^{\otimes n}$ is lower when the target states are product states than that when the target states are entangled states. In addition, the amount of the operator coherence depends directly on the success probability and also on the target states.

The paper is structured as follows. In Sec. II, we review GSA and conceptions about incoherent states, incoherent operators, and the previous works on coherence dynamics in GSA. Then, in Sec. III, we investigate the coherence dynamics of the states after the four basis operators applied in GSA. In particular, in Sec. IV, we discuss the operator coherence for different cases of the target states. In Sec. V, we study the coherence production and depletion by the changes before and after the basis operators are applied. In addition, in Sec. VI, we compare our works with previous works on coherence dynamics in GSA. Finally in Sec. VII, we conclude with a summary and future works.

II. PRELIMINARIES

In this section, we recall Grover's search algorithm (GSA) and definitions about quantum coherence, and then review the coherence dynamics in GSA.

A. Grover's search algorithm

Suppose we search through a set of $N = 2^n$ elements and the task is to find out one of t targets which satisfy some special conditions. Grover's search algorithm employs n-qubit pure states and begins with the state $|0\rangle^{\otimes n}$. Then, Hadamard operator H, which the corresponding matrix representation is $H = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$, is applied to each qubit $|0\rangle$. Therefore, $H^{\otimes n}$ is applied to $|0\rangle^{\otimes n}$ and then the state is transferred into an equal superposition state of all computational basis states $|\psi_0\rangle = 1/\sqrt{N}\sum_{x=0}^{N-1}|x\rangle$, which can be more conveniently written as

$$|\psi_0\rangle = \sqrt{\frac{N-t}{N}}\,|\chi_0\rangle + \sqrt{\frac{t}{N}}\,|\chi_1\rangle\,, \tag{1}$$

where $|\chi_0\rangle = \frac{1}{\sqrt{N-t}}\sum_{x_n}|x_n\rangle$ denotes the superposition of all the states $|x_n\rangle$ that are not-target states, and $|\chi_1\rangle = \frac{1}{\sqrt{t}}\sum_{x_s}|x_s\rangle$ represents the superposition of all the states $|x_s\rangle$ that are targets (i.e., the solutions of search problem).

The algorithm then repeats the application of a quantum subroutine, which is named as *Grover iteration* or *Grover operator* and denoted as G. The Grover iteration consists of the application of four basis operators:

(i) Apply the oracle O, where the oracle consists of a function $f(x)$: $f(x) = 1$ if $x \in \{x_s\}$, else $f(x) = 0$. It inverts the target states and leaves the not-target states unchanged, i.e.,

$$O = \sum_x (-1)^{f(x)}\,|x\rangle\,\langle x| = I - 2\sum_{x_s}|x_s\rangle\,\langle x_s|\,.$$

(ii) Apply the Hadamard operator $H^{\otimes n}$.

(iii) Apply the conditional phase shift operator P, with every computational basis state except $|0\rangle$ receiving a phase shift of -1,

$$P\,|x\rangle \rightarrow -(-1)^{\delta_{x0}}|x\rangle.$$

It was shown that $P = 2\,|0\rangle\,\langle 0| - I$.

(iv) Apply the Hadamard operator $H^{\otimes n}$.

In other words, Grover operator $G = H^{\otimes n}PH^{\otimes n}O$.

For the equal superposition state $|\psi_0\rangle$, k iterations of Grover operator give

$$|\psi_k\rangle \equiv G^k\,|\psi_0\rangle = \cos\theta_k\,|\chi_0\rangle + \sin\theta_k|\chi_1\rangle, \tag{2}$$

with $\theta_k = (2k+1)\theta$ and $\theta = \arcsin\sqrt{t/N}$ when $1 \leqslant t \ll N$. Accordingly, the success probability of $|\psi_k\rangle$ to find out one of the targets is $p_k = \sin^2\theta_k$. In the limit $t \ll N$, the optimal number of iterations is $k_{\rm opt} = \lfloor\frac{\pi}{4}\sqrt{N/t}\rfloor$.

B. Incoherent states, incoherent operators, and coherence measures

The definitions of incoherent states and incoherent operations were presented by Baumgratz *et al.* in Ref. [6], where quantum operations are specified by a set of Kraus operators. Since the operators in Grover's algorithm are unitary, we present the notion of incoherent unitary operator following the definition of incoherent operations as follows.

Definition 1 (Incoherent states). In d-dimensional Hilbert space $\mathcal{H}$, fix a particular basis $\{|i\rangle\}_{i=1,2...d}$. If the density matrix of a state is diagonal in this basis, then this state is called incoherent, and we label the set of incoherent states by $\mathcal{I} \subset \mathcal{H}$.

Definition 2 (Incoherent unitary operator). Let U be a unitary operator. Then, it is called an incoherent operator if it fulfils $U\mathcal{I}U^\dagger \subset \mathcal{I}$.

According to Refs. [10,11,20], all the unitary incoherent operators have the form as presented in Lemma 1.

Lemma 1. All the unitary incoherent operators take the form $U = \sum_i e^{j\alpha_i}\,|\beta(i)\rangle\,\langle i|$, where $\beta(i)$ is relabeling of {i}.

According to Lemma 1, it is easy to learn that a diagonal unitary operator is incoherent. In the view of quantum resource theory [37], an incoherent unitary operator is "free" of coherence.

Based on the work in Ref. [6], a number of coherence measures, such as the relative entropy of coherence [6], the l_1 norm of coherence, geometric coherence [17], and the coherence of formation [19], have been proposed. Among these quantifiers, the l_1 norm of coherence C_{l_1} is the simplest and direct, which is defined as all the nonzero off-diagonal elements of a density matrix. Hence, we choose it to investigate the coherence in GSA.

The expression of the l_1 norm of coherence for a given density operator ρ is defined as [6]

$$C_{l_1}(\rho) \equiv \sum_{i\neq j}|\rho_{ij}|, \tag{3}$$

where i, j are the ith row and jth column in ρ.

C. Coherence in Grover's search algorithm

Let the density operator of state $|\psi_k\rangle$ be $\rho_k = |\psi_k\rangle\langle\psi_k|$ in GSA. By employing Eqs. (2) and (3), the coherence dynamics in GSA is [25]

$$C_{l_1}(\rho_k) = (\sqrt{t}\sin\theta_k + \sqrt{N-t}\cos\theta_k)^2 - 1 \tag{4}$$

when $0 \leqslant k \leqslant k_{\text{opt}}$. Using Eq. (4), the l_1 norm of coherence can be rewritten as a function of the success probability $p_k = \sin^2\theta_k$ as

$$C_{l_1}(\rho_k) = [\sqrt{tp_k} + \sqrt{(N-t)(1-p_k)}]^2 - 1. \tag{5}$$

According to Eq. (5), $C_{l_1}(\rho_k)$ is indeed a smooth function of p_k in the interval [0,N]. In the asymptotic limits $1 \leqslant t \ll N$, the l_1 norm of coherence $C_{l_1}(\rho_k)$ takes the simple form [25]

$$C_{l_1}(\rho_k) \simeq -p_k N + N. \tag{6}$$

Here, $A \simeq B$ means that A asymptotically equals to B under the condition $t \ll N$. Obviously, the coherence is monotone decreasing with the success probability increasing. From another perspective, the success probability depends on the quantum coherence depletion in terms of l_1 norm of coherence measure [25].

III. COHERENCE DYNAMICS OF OPERATORS IN GROVER'S SEARCH ALGORITHM

To clarify the characters of coherence of the state after each basic operator applied in GSA, we give the following definition.

Definition 3 (Operator coherence). Let U be a unitary operator. Operator coherence of U of a state $|\psi\rangle$ is the coherence of the state after U operating on $|\psi\rangle$ that $C(\rho_U) = C(U\rho U^+)$, where $\rho = |\psi\rangle\langle\psi|$.

Now, we discuss the characters of coherence of the state after the four basic operators applied in each application of one Grover iteration in GSA.

A. Operator coherence of oracle and conditional phase-shift operators

According to the action of the oracle, the state $|\psi_k\rangle$ after O applied on becomes

$$|\psi_{k_O}\rangle \equiv O|\psi_k\rangle = \cos\theta_k|\chi_0\rangle - \sin\theta_k|\chi_1\rangle. \tag{7}$$

Denote $\rho_{k_O} = |\psi_{k_O}\rangle\langle\psi_{k_O}|$, according to Eqs. (3) and (7), we have

$$C_{l_1}(\rho_{k_O}) = C_{l_1}(\rho_k). \tag{8}$$

Obviously, the oracle does not change the coherence.

Moreover, we can show that the oracle and the conditional phase-shift operator are incoherent operators in the following theorems.

Theorem 1. The oracle $O = I - 2\sum_{x_s}|x_s\rangle\langle x_s|$ is an incoherent operator, where $\{x_s\}$ is the set of target states.

Proof. The expression of oracle is

$$O = I - 2\sum_{x_s}|x_s\rangle\langle x_s| = \sum_x (-1)^{f(x)}|x\rangle\langle x|,$$

where $f(x) = 1$ if $x \in \{x_s\}$, else $f(x) = 0$. Observably, the oracle is a diagonally unitary. It is clear that the oracle is an incoherent operator according to Lemma 1. ■

Theorem 2. The condition phase-shift operator $P = 2|0\rangle\langle 0| - I$ is an incoherent operator.

Proof. For the condition phase-shift operator P, we have

$$P = 2|0\rangle\langle 0| - I = \sum_x -(-1)^{\delta_{x0}}|x\rangle\langle x|.$$

According to Lemma 1, it is easy to obtain that the conditional phase-shift operator P is an incoherent operator. ■

B. Operator coherence of Hadamard operator

It is known that coherence depends on the preselected basis states. For convenience, we choose the computational basis states as the preselected basis states. For N-dimension Hadamard matrix, aside from the common coefficient $1/\sqrt{N}$, it has the following properties:

(i) The elements of first row and first column are 1.

(ii) In any other row or column, a half of the elements are 1, and the others are −1.

When $H^{\otimes n}$ applies on a superposition state $|\psi\rangle = \sum_{x=0}^{N-1} a_x|x\rangle$, we have

$$H^{\otimes n}|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} a_x(-1)^{xy}|y\rangle. \tag{9}$$

Let $A_k = \cos\theta_k$ and $B_k = \sin\theta_k$, $k \in [0, k_{\text{opt}}]$. For clarity, denote the first $H^{\otimes n}$ following O as H_O and the second $H^{\otimes n}$ following P as H_P in one Grover iteration. Denote $|\psi_{k_{H_O}}\rangle$ the state after H_O applied that after k iterates of G and cascades O applied and $|\psi_{k_{H_P}}\rangle$ the state after H_P applied that after k iterates of G and cascades $PH^{\otimes n}O$ applied:

$$\begin{aligned}|\psi_{k_{H_O}}\rangle &\equiv H^{\otimes n}|\psi_{k_O}\rangle \\ &= \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}\left[\frac{A_k}{\sqrt{N-t}}\sum_{x\in\{x_n\}}(-1)^{xy} - \frac{B_k}{\sqrt{t}}\sum_{x\in\{x_s\}}(-1)^{xy}\right]|y\rangle\end{aligned} \tag{10}$$

and

$$\begin{aligned}|\psi_{k_{H_P}}\rangle &\equiv H^{\otimes n}(PH^{\otimes n}O|\psi_k\rangle) = |\psi_{k+1}\rangle \\ &= A_{k+1}|\chi_0\rangle + B_{k+1}|\chi_1\rangle.\end{aligned} \tag{11}$$

According to Eqs. (6) and (11), it is easy to obtain

$$C_{l_1}(\rho_{k_{H_P}}) \simeq -p_{k+1}N + N. \tag{12}$$

In the following part, we will discuss the properties of the coherence of $|\psi_{k_{H_O}}\rangle$. Denote $H_{y,x}$ the element of y row x column in Hadamard matrix and t_y is as

$$t_y = |\{H_{y,x}|H_{y,x} = 1, x \in \{x_s\}\}|, \tag{13}$$

where $\{x_s\}$ denotes the set of all target states. We can derive that

$$C_{l_1}(\rho_{k_{H_O}}) = \frac{1}{N}\left[2\sum_{y\neq 0}|(t-2t_y)(A_k\sqrt{N-t}-B_k\sqrt{t})| \times\left(\frac{A_k}{\sqrt{N-t}}+\frac{B_k}{\sqrt{t}}\right)+\sum_{\substack{y,y'\neq 0\\ y\neq y'}}|(t-2t_y)(t-2t_{y'})| \times\left(\frac{A_k}{\sqrt{N-t}}+\frac{B_k}{\sqrt{t}}\right)^2\right]. \quad (14)$$

The derivation of Eq. (14) is given in Appendix A. According to Eq. (14), we can obtain that $C_{l_1}(\rho_{k_{H_O}})$ relates to $|t-2t_y|$. Since t, t_y depend on the target states, the coherence $C_{l_1}(\rho_{k_{H_O}})$ is easy to calculate once the target states are given. However, it is difficult to obtain a universal theoretical result without knowing the target states. In the asymptotic limits $t \ll N$, we can derive the following theorem.

Theorem 3. The coherence of the state $|\psi_{k_{H_O}}\rangle$ can be expressed as

$$C_{l_1}(\rho_{k_{H_O}}) \simeq \gamma(t,t_y)p_kN$$

when $1 \leqslant t \ll N$, where $t_y \in [0,t]$ is the number of 1 in y row x_s columns in Hadamard matrix, $\gamma(t,t_y)$ is a positive coefficient about t and t_y, and p_k is the success probability after k iterations of Grover operator.

Proof. For $\sum_{y\neq 0}|(t-2t_y)|$ in Eq. (14), the summation is over all $y \in (0, N-1]$ and $t_y \in [0,t]$, hence, $\sum_{y\neq 0}|(t-2t_y)| = \gamma'(t,t_y)N$, where $\gamma'(t,t_y)$ is a positive coefficient about t and t_y.

Therefore, when $t \ll N$, Eq. (14) becomes

$$\begin{aligned} C_{l_1}(\rho_{k_{H_O}}) &\simeq \frac{1}{N}\sum_{\substack{y,y'\neq 0\\ y\neq y'}}\left[|(t-2t_y)(t-2t_{y'})|\left(\frac{B_k}{\sqrt{t}}\right)^2\right] \\ &= \frac{1}{N}\sum_{\substack{y,y'\neq 0\\ y\neq y'}}\frac{|(t-2t_y)(t-2t_{y'})|}{t}B_k^2 \\ &\simeq \gamma(t,t_y)p_kN, \end{aligned} \quad (15)$$

where $p_k = B_k^2$ and $\gamma(t,t_y) = \gamma'(t,t_y)\gamma'(t,t_{y'})/t$, i.e., $\gamma(t,t_y)$ is a positive coefficient about t and t_y. ∎

Clearly, the coherence of the state $|\psi_{k_{H_O}}\rangle$ is a positive function of the success probability p_k and the size N of the database; meanwhile, it is also a positive function of target states (t, t_y).

IV. COHERENCE IN GROVER'S ALGORITHM FOR SPECIAL TARGET STATES

In this section, we discuss some properties of the operator coherence in GSA by supposing the target states are known as the following. Since the operator coherence of O, P, H_P are clear in Sec. III, we focus on the operator coherence of H_O.

A. For product states

For the special cases that $|\chi_1\rangle$ is a product state, we have the following theorem.

Theorem 4. When the superposition state $|\chi_1\rangle$ of t target states is a product state, the coherence of the state $|\psi_{k_{H_O}}\rangle$ is $C_{l_1}(\rho_{k_{H_O}}) \simeq \frac{N}{t}p_k$ when $1 \leqslant t \ll N$.

Proof. For an n-qubits system, when $|\chi_1\rangle$ is a product state, it can be written as

$$|\chi_1\rangle = |\phi_1\rangle|\phi_2\rangle\dots|\phi_n\rangle, \quad (16)$$

where $|\phi_i\rangle$ is a single-qubit state. Since $|\chi_1\rangle = \frac{1}{\sqrt{t}}\sum_{x_s}|x_s\rangle$ is an equal superposition state of t targets, when it is a product state, $|\phi_i\rangle$ will be $|0\rangle$, $|1\rangle$, and $(|0\rangle+|1\rangle)/\sqrt{2}$ and there are $l=\log_2 t$ items of $(|0\rangle+|1\rangle)/\sqrt{2}$. Denote $|\chi_{1H}\rangle = H^{\otimes n}|\chi_1\rangle$. It is easy to obtain that $|\chi_{1H}\rangle$ is also a product state. Moreover, there are $n-l$ items of $(|0\rangle \pm |1\rangle)/\sqrt{2}$ in $|\chi_{1H}\rangle$ which means that there are $2^{(n-l)} = N/t$ states in the computational basis.

After k iterates of G and cascades O and $H^{\otimes n}$ are applied, the state becomes

$$\begin{aligned} |\psi_{k_{H_O}}\rangle &= H^{\otimes n}O|\psi_k\rangle \\ &= A_kH^{\otimes n}|\chi_0\rangle - B_kH^{\otimes n}|\chi_1\rangle \\ &= A_kH^{\otimes n}\frac{|\psi_0\rangle - B_0|\chi_1\rangle}{A_0} - B_kH^{\otimes n}|\chi_1\rangle \\ &= \frac{A_k}{A_0}|0\rangle - \left(B_k + \frac{A_kB_0}{A_0}\right)|\chi_{1H}\rangle. \end{aligned} \quad (17)$$

Therefore, we have

$$\begin{aligned} |\rho_{k_{H_O}}\rangle = &\left(\frac{A_k}{A_0}\right)^2|0\rangle\langle 0| \\ &- \frac{A_k}{A_0}\left(B_k+\frac{A_kB_0}{A_0}\right)(|0\rangle\langle\chi_{1H}| + |\chi_{1H}\rangle\langle 0|) \\ &+ \left(B_k+\frac{A_kB_0}{A_0}\right)^2|\chi_{1H}\rangle\langle\chi_{1H}| \end{aligned} \quad (18)$$

and

$$C_{l_1}(\rho_{k_{H_O}}) = 2\frac{A_k(B_kA_0+A_kB_0)}{A_0^2}\sqrt{\frac{N}{t}} + \left(B_k+\frac{A_kB_0}{A_0}\right)^2\left(\frac{N}{t}-1\right). \quad (19)$$

In the asymptotic limits $t \ll N$, we can derive

$$C_{l_1}(\rho_{k_{H_O}}) \simeq \frac{N}{t}B_k^2 = \frac{N}{t}p_k. \quad (20)$$

∎

B. For general t

To further demonstrate the properties of the operator coherence in GSA, we first discuss the case when $t \leqslant 4$, then extend to general cases in the following.

Theorem 5. When the number of targets is small that $t \leqslant 4$ in Grover's search algorithm, the coherence of the state $|\psi_{k_{H_O}}\rangle$ will be as follows when $t \ll N$:

(1) when $t = 1$, $C_{l_1}(\rho_{k_{H_O}}) \simeq p_kN$;

(2) when $t=2$, $C_{l_1}(\rho_{k_{H_O}}) \simeq \frac{N}{2}p_k$;
(3) when $t=3$, $C_{l_1}(\rho_{k_{H_O}}) \simeq \frac{3N}{4}p_k$;
(4) when $t=4$,

$$C_{l_1}(\rho_{k_{H_O}}) \simeq \begin{cases} \frac{N}{4}p_k, & |\chi_1\rangle \textit{ is product};\\ \frac{9N}{16}p_k, & \textit{otherwise}. \end{cases}$$

Proof. (1) Let us first discuss the case of one single target state, i.e., $t=1$. For one single target state, it is always a product state. According to Theorem 4, it is easy to obtain that

$$C_{l_1}(\rho_{k_{H_O}}) \simeq p_k N \tag{21}$$

when $1 \leqslant t \ll N$.

(2) When the target number $t=2$, we denote the target states are $|x_{s_1}\rangle$ and $|x_{s_2}\rangle$ ($x_{s_1} < x_{s_2}$). Then, we have $|\chi_1\rangle = \frac{1}{\sqrt{2}}(|x_{s_1}\rangle + |x_{s_2}\rangle)$. We discuss the two cases that $|x_{s_1}\rangle = |0\rangle$ and $|x_{s_1}\rangle \neq |0\rangle$, respectively.

Case I: $|x_{s_1}\rangle = |0\rangle$. According to the properties of the matrix of $H^{\otimes n}$, its elements $H_{y,x}$ are all 1 or -1; particularly, $H_{y,0}=1$ for each y. According to Eq. (13), we have

$$t_y = \begin{cases} 1, & \text{if } H_{y,x_2} = -1\\ 2, & \text{if } H_{y,x_2} = 1. \end{cases}$$

For $y \in [0, N-1]$, the numbers of y satisfying $t_y = 1$ and 2 are both $N/2$. Therefore, we have

$$\sum_{y=0}^{N-1} |t - 2t_y| = \frac{N}{2}|2 - 2\times 2| + \frac{N}{2}|2 - 2\times 1| = N.$$

Case II: $|x_{s_1}\rangle \neq |0\rangle$. According to the properties of the matrix of $H^{\otimes n}$ and Eq. (13), we have

$$t_y = \begin{cases} 0, & \text{if } \{H_{y,x_s}\} = \{-1,-1\}\\ 1, & \text{if } \{H_{y,x_s}\} = \{1,-1\}\\ 2, & \text{if } \{H_{y,x_s}\} = \{1,1\}. \end{cases}$$

Then, we will obtain that the numbers of y satisfying $t_y = 0, 1, 2$ are $N/4, N/2, N/4$ for $y \in [0, N-1]$. Therefore, we also have $\sum_y |t - 2t_y| = N$.

As a result, according to Eq. (14), we have

$$C_{l_1}(\rho_{k_{H_O}}) \simeq 2(\sqrt{N-2}A_k - \sqrt{2}B_k)\left(\frac{A_k}{\sqrt{N-2}} + \frac{B_k}{\sqrt{2}}\right) + N\left(\frac{A_k}{\sqrt{N-2}} + \frac{B_k}{\sqrt{2}}\right)^2. \tag{22}$$

For $t \ll N$, we can derive

$$C_{l_1}(\rho_{k_{H_O}}) \simeq \frac{N}{2}p_k. \tag{23}$$

(3) When the target number $t=3$, according to Eq. (14), the coherence is always

$$C_{l_1}(\rho_{k_{H_O}}) = \frac{1}{N}\left[2\sum_{y\neq 0}|(t-2t_y)(A_k\sqrt{N-3} - B_k\sqrt{3})| \times \left(\frac{A_k}{\sqrt{N-3}} + \frac{B_k}{\sqrt{3}}\right) + \sum_{\substack{y,y'\neq 0\\ y\neq y'}} |(3-2t_y)(3-2t_{y'})| \times \left(\frac{A_k}{\sqrt{N-3}} + \frac{B_k}{\sqrt{3}}\right)^2\right]. \tag{24}$$

The numbers of y satisfying $t_y = 0, 1, 2, 3$ are either $0, \frac{3N}{8}, \frac{3N}{8}, \frac{N}{4}$ or $\frac{N}{8}, \frac{3N}{8}, \frac{3N}{8}, \frac{N}{8}$, respectively. Nevertheless, they are the same from the viewpoint of l_1 norm of coherence. In the asymptotic limits $t \ll N$, we can derive

$$C_{l_1}(\rho_{k_{H_O}}) \simeq \frac{3N}{4}p_k. \tag{25}$$

(4) For $t=4$, there are two cases that $|\chi_1\rangle$ is a product state or not.

Case 1: When $|\chi_1\rangle$ is a product state, t_y just are 0,2,4 without 1,3 and the ratio among them are $1:6:1$. In this case, according to Theorem 4, it is easy to obtain that $C_{l_1}(\rho_{k_{H_O}}) \simeq \frac{N}{4}p_k$.

Case 2: When $|\chi_1\rangle$ is not a product state, t_y are 0,1,2,3,4 with ratio $1:4:6:4:1$. For this case that $|\chi_1\rangle$ is an entangled state, and according to Eq. (14), we have $C_{l_1}(\rho_{k_{H_O}}) \simeq \frac{9N}{16}p_k$ when $4 \ll N$. ■

According to Theorems 4 and 5, we get the conjecture for general cases.

Conjecture 1. The coherence $C_{l_1}(\rho_{k_{H_O}})$ of the state $|\psi_{k_{H_O}}\rangle$ depends on the success probability p_k and the size N of the database and the target states. It may be

$$\frac{N}{t}p_k \leqslant C_{l_1}(\rho_{k_{H_O}}) \leqslant p_k N$$

when $1 \leqslant t \ll N$. The upper bound can reach when $t=1$ while the lower bound obtains when $|\chi_1\rangle$ is a product state.

V. COHERENCE PRODUCTION AND DEPLETION

In the above sections, we have discussed the coherence of the states when the operators O, H_O, P, H_P are applied. Now, we discuss how the coherence changes before and after these operators are applied.

Definition 4 (Coherence production and depletion). Let U be a unitary operator. Coherence production (depletion) when the variation of coherence after and before U applied on $|\psi\rangle$ is positive (negative), i.e., $\Delta C(\rho_U) > 0$ [$\Delta C(\rho_U) < 0$], where $\rho = |\psi\rangle\langle\psi|$.

Since O, P are incoherence operators, we can just discuss the operator coherence of two $H^{\otimes n}$. Define the variations of operator coherence between two consecutive iterations of

$H_O,\ H_P,\ G$ in Grover's search algorithm as follows:

$$\Delta C(\rho_{k_G}) \equiv C_{l_1}(\rho_{k+1}) - C_{l_1}(\rho_k), \tag{26}$$

$$\Delta C(\rho_{k_{H_O}}) \equiv C_{l_1}(\rho_{(k+1)_{H_O}}) - C_{l_1}(\rho_{k_{H_O}}), \tag{27}$$

$$\Delta C(\rho_{k_{H_P}}) \equiv C_{l_1}(\rho_{(k+1)_{H_P}}) - C_{l_1}(\rho_{k_{H_P}}). \tag{28}$$

Define the variations of suboperator coherence of each basic $H^{\otimes n}O,\ H^{\otimes n}P$ in one Grover iteration as follows:

$$\Delta C(\rho_{k_{\Delta H_O}}) \equiv C_{l_1}(\rho_{k_{H_O}}) - C_{l_1}(\rho_k), \tag{29}$$

$$\Delta C(\rho_{k_{\Delta H_P}}) \equiv C_{l_1}(\rho_{k_{H_P}}) - C_{l_1}(\rho_{k_{H_O}}). \tag{30}$$

The properties of coherence production and depletion by $H_O,\ H_P,\ G$ can be concluded by the following theorems.

Theorem 6. When $t \ll N$, the variations of operator coherence between two consecutive iterations of $H_O,\ H_P,\ G$ in Grover's search algorithm are

$$\begin{aligned}\Delta C(\rho_{k_G}) &\simeq -(p_{k+1}-p_k)N < 0,\\ \Delta C(\rho_{k_{H_O}}) &\simeq \gamma(t,t_y)(p_{k+1}-p_k)N > 0,\\ \Delta C(\rho_{k_{H_P}}) &\simeq -(p_{k+2}-p_{k+1})N < 0.\end{aligned} \tag{31}$$

Proof. By substituting Eq. (6) into (26), we get

$$\Delta C(\rho_{k_G}) \simeq -(p_{k+1}-p_k)N.$$

Since $p_k = \sin^2\theta_k$ is monotonously increasing with k as $\theta_k \in [0, \pi/2]$, we have $p_{k+1} > p_k$ and

$$\Delta C(\rho_{k_G}) \simeq -(p_{k+1}-p_k)N < 0.$$

Similarly, substitute Eq. (15) into (27), then

$$\Delta C(\rho_{k_{H_O}}) \simeq \gamma(t,t_y)(p_{k+1}-p_k)N > 0,$$

and substitute Eq. (12) into (28), then

$$\Delta C(\rho_{k_{H_P}}) \simeq -(p_{k+2}-p_{k+1})N < 0.$$

The theorem holds. ■

According to Theorem 6, H_P and G deplete coherence and their operator coherence depend only on the success probability p_k and the size N of database. However, H_O produces coherence and its operator coherence depends on p_k and the size N of database, and also depends on the target states.

Furthermore, according to Theorem 6, the relationship among the operator coherence of $H_O,\ H_P,\ G$ is clear as shown in Corollary 1.

Corollary 1. When $t \ll N$, the variations of operator coherence between two consecutive iterations of H_O, H_P, G in Grover's search algorithm satisfy

$$\Delta C(\rho_{k_G}) \simeq \Delta C(\rho_{(k-1)_{H_P}}) \simeq -\frac{1}{\gamma(t,t_y)}\Delta C(\rho_{k_{H_O}}).$$

Theorem 7. When $t \ll N$, the variations of suboperator coherence of H_O, H_P in one Grover iteration are

$$\Delta C(\rho_{k_{\Delta H_O}}) \simeq N\{[\gamma(t,t_y)+1]p_k - 1\} \begin{cases} \leqslant 0, & [\gamma(t,t_y)+1]p_k \leqslant 1\\ > 0, & [\gamma(t,t_y)+1]p_k > 1\end{cases} \tag{32}$$

and

$$\Delta C(\rho_{k_{\Delta H_P}}) \simeq N\{1 - [\gamma(t,t_y)p_k + p_{k+1}]\} \begin{cases} > 0, & [\gamma(t,t_y)p_k + p_{k+1}] \leqslant 1\\ \leqslant 0, & [\gamma(t,t_y)p_k + p_{k+1}] > 1.\end{cases} \tag{33}$$

Proof. Substitute Eqs. (6) and (15) into (29), then

$$\begin{aligned}\Delta C(\rho_{k_{\Delta H_O}}) &= [\gamma(t,t_y)+1]p_kN - (-p_kN + N)\\ &= N\{[\gamma(t,t_y)+1]p_k - 1\}.\end{aligned} \tag{34}$$

For $[\gamma(t,t_y)+1]p_k$ in Eq. (34), whether it is greater than 1 depends on $\gamma(t,t_y)$ and p_k. That is,

$$\Delta C(\rho_{k_{\Delta H_O}}) = N\{[\gamma(t,t_y)+1]p_k - 1\} \begin{cases} \leqslant 0, & [\gamma(t,t_y)+1]p_k \leqslant 1\\ > 0, & [\gamma(t,t_y)+1]p_k > 1.\end{cases}$$

Similarly, by substituting Eqs. (12) and (15) into (30), we can obtain

$$\Delta C(\rho_{k_{\Delta H_P}}) = N\{1 - [\gamma(t,t_y)p_k + p_{k+1}]\} \begin{cases} > 0, & [\gamma(t,t_y)p_k + p_{k+1}] \leqslant 1\\ \leqslant 0, & [\gamma(t,t_y)p_k + p_{k+1}] > 1.\end{cases}$$

■

According to Theorem 7, we have the following corollary.

Corollary 2. When $t \ll N$, in the curves of $C(\rho_{k_{\Delta H_P}})$ and $\Delta C(\rho_{k_{\Delta H_O}})$, there exists a turning point

$$k_T \simeq \left[\frac{\arcsin\sqrt{\frac{1}{\gamma(t,t_y)+1}}}{2\theta}\right],$$

where $[x]$ is the rounded integer of x. When $k \leqslant k_T$, H_O depletes coherence and H_P produces coherence; when $k > k_T$, H_O produces coherence and H_P depletes coherence.

Proof. According to Eq. (32), there exists a turning point in the curve of $\Delta C(\rho_{k_{\Delta H_O}})$, where $\Delta C(\rho_{k_{\Delta H_O}}) = 0$. Let k_T be the turning iteration. When $t \ll N$, we have $\gamma(t,t_y)p_{k_T} - 1 = 0$. Since $p_{k_T} = \sin^2(2k_T+1)\theta$, we can obtain

$$k_T \simeq \left[\frac{\arcsin\sqrt{\frac{1}{\gamma(t,t_y)+1}}}{2\theta}\right].$$

Similarly, there also exists a turning point in the curve of $\Delta C(\rho_{k_{\Delta H_P}})$ according to Eq. (35) which asymptotically equals to the above k_T when $t \ll N$. ■

According to Theorem 7 and Corollary 2, the coherence variations of H_O and H_P in each suboperator of one Grover iteration depend on the success probability and the target states. In addition, we can obtain that there exist turning points which depend on the target state (t, t_y) in the curves of $\Delta C(\rho_{k_{\Delta H_O}})$ and $\Delta C(\rho_{k_{\Delta H_P}})$. Before the turning point, the first $H^{\otimes n}$ with the help of O is negative that depletes the coherence; while the second $H^{\otimes n}$ with the help of P is negative that depletes the coherence after the turning point.

Remark 1. According to the above Theorems 6 and 7 and Corollaries 1 and 2, we obtain different results from different

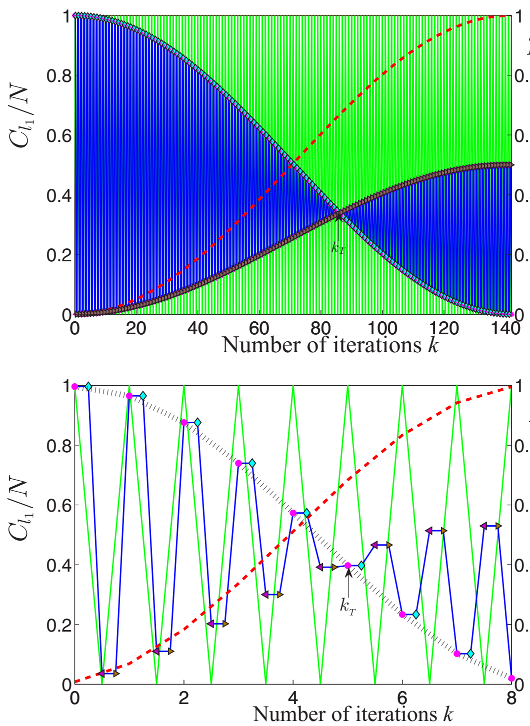

FIG. 1. The coherence dynamics with the search iteration. The red dashed line is the success probability (p_k), the black dotted line is the coherence after k iterates of G, the blue line is the coherence when O, $H^{\otimes n}$, P, $H^{\otimes n}$ (corresponding to •, ⋄, ◁, ▷) are applied in each iteration, the light-green line is the coherence just $H^{\otimes n}$, $H^{\otimes n}$ without O and P. Here, the target numbers are $t = 2$ and the qubit numbers are $n = 16$ (top) and $n = 8$ (bottom).

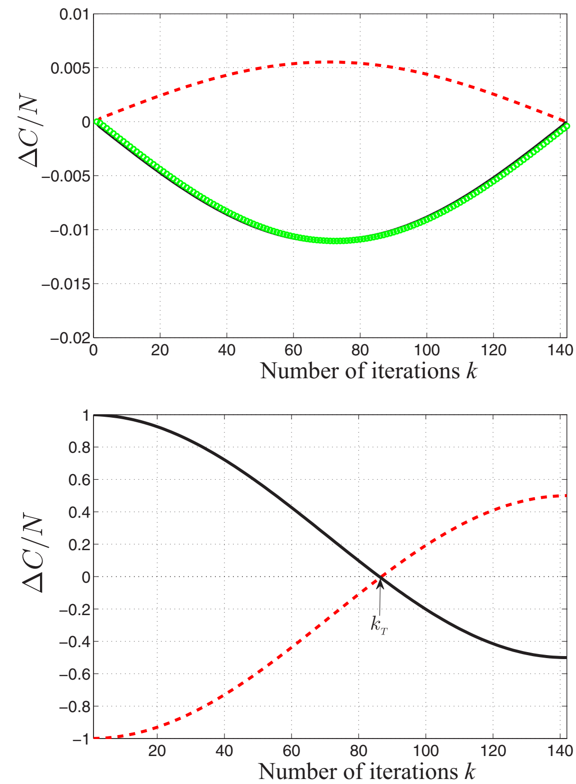

FIG. 2. (Top) The operator coherence of G, H_O, H_P (corresponding to the green circle line, red dashed line, and black line). (Bottom) The suboperator coherence of each H_O (red dashed line) and H_P (black line) in one Grover iteration. Here the qubit numbers are $n = 16$ and the target numbers are $t = 2$.

views of operator coherence in GSA. But, there is a unanimous conclusion that the two $H^{\otimes n}$ always have different effects on coherence that one produces coherence and the other depletes coherence. Therefore, during the search process, operator coherence is depleted and produced alternately instead of depleted monotonously on the essential operator level.

Examples. For clarity, we provide a diagrammatic sketch of the operators' coherence in Grover's search algorithm in Figs. 1 and 2 when $t = 2$ where $\gamma(t, t_y) = \frac{1}{2}$. More examples are presented in Appendix B. Note that we use C_{l_1}/N instead of C_{l_1} as y axis in figures for clarity. As can be seen from Fig. 1, there is a turning iteration k_T in the curve of operator coherence. When $O \to H_O \to P \to H_P$ is applied in one Grover iterator, the coherence is "unchange → decrease → unchange → increase" as$k \leqslant k_T$ while it is "unchange → increase →unchange → decrease" as $k > k_T$. In other words, the coherence does not change when O and P applied while it vibrates between $1 - p_k$ and $p_k/t = p_k/2$ when $H^{\otimes n}$ applied. Therefore, the operator coherence is oscillating during the search processing. For comparison, we also present the coherence when just two $H^{\otimes n}$ are applied, where the coherence vibrates between 1 and 0. Therefore, the incoherence operators O and P weaken the vibration of coherence.

According to Fig. 2, the operator coherences of H_O, H_P, and G begin and end with zero and $\Delta C(\rho_{k_G}) \simeq \Delta C(\rho_{k_{H_P}}) \simeq -2\Delta C(\rho_{k_{H_O}})$. The depletion plays a major role that the amount of coherence of depletion is more than that of the production. Therefore, the overall affect is that coherence depletes as the success probability improves. For suboperator coherence of each H_O and H_P in one Grover iteration, there exists a point k_T that $\Delta C(\rho_{\Delta k_{H_O}}) < 0$ and $\Delta C(\rho_{\Delta k_{H_P}}) > 0$ which means H_O depletes coherence and H_P produces coherence when $k \leqslant k_T$; on the contrary, $\Delta C(\rho_{\Delta k_{H_O}}) > 0$ and $\Delta C(\rho_{\Delta k_{H_P}}) < 0$ which means H_O produces coherence and H_P depletes coherence when $k > k_T$.

VI. COMPARISON WITH PREVIOUS WORKS

In order to clarify the contribution of this paper, we would like to compare our work with most relevant previous works.

A. Comparison with Ref. [25]

In Ref. [25], Shi *et al.* investigated the role of coherence depletion in GSA by using several typical measures of coherence and quantum correlations. They discussed the coherence of the state after each iteration of G applying on. In terms of l_1 norm of coherence measurement, the important result of Ref. [25] is presented in the Sec. II. The authors showed that the coherence of state after each G depends only on the size of the database and the success probability. In the limit case, the depletion of coherence enhances the success probability.

In this work, on essential operator level, we discuss the coherence of the states after operators O, P, $H^{\otimes n}$ applied in one G iteration. We show that the oracle operator O and the condition phase operator P are incoherent operators while $H^{\otimes n}$ is typical coherent operator. The amount of coherence of the state after $H^{\otimes n}$ with the help of O or P in one Grover iteration not only depends directly on the size of the database and the success probability, but also on the targets' number and their positions. During the search process, coherence is not always depleted, rather it is depleted and produced alternately.

B. Comparison with Ref. [36]

In [36], we showed that $H^{\otimes n}$ does not change entanglement while O increases entanglement and $R = H^{\otimes n}PH^{\otimes n}$ decreases entanglement in terms of GME. Moreover, during the process, there exists a turning point with the following properties. Before the turning point, the Oracle O increases entanglement and plays a major role, while R decreases entanglement and makes more contribution to the entanglement after the turning point.

In this paper, we show that O and P are incoherent operators while two $H^{\otimes n}$ are coherent operators. There also exists a turning point in the dynamic suboperator coherence during the process on basic operator lever. Before that turning point, the first $H^{\otimes n}$ with the help of O is negative that depletes coherence while the second $H^{\otimes n}$ with the help of P is positive that produces coherence; after the turning point, the first $H^{\otimes n}$ produces coherence while the second one depletes coherence. Meanwhile, both $H^{\otimes n}$ play important roles to coherence during the whole search process.

Furthermore, we show that the coherence values between, with, and without O and P are different. In other words, they bring a cascade effect which makes the coherence change when the oracle operator O or the condition phase operator P is applied and then $H^{\otimes n}$ is applied. When just $H^{\otimes n}$ cascading $H^{\otimes n}$ is applied on $|\psi_0\rangle$, where the coherence of the state vibrates between 0 and 1. With the help of O, P, which are incoherent but entangling operators, the coherence is vibrating that it first decreases then increases before the turning point, while it increases then decreases after the turning point. The value of coherence is larger than 0 and smaller than 1. Therefore, the entangling operators O and P weaken the vibration of the coherence. The amount of operator coherence of the first $H^{\otimes n}$ is larger when $|\chi_1\rangle$ is an entangled state than that when $|\chi_1\rangle$ is a product state. According to these works, we can learn that entanglement has an important contribution to operator coherence in GSA.

C. Discussion

According to the above comparisons, it is clear that as the success probability improves, coherence as the resource is depleted after each G application as the algorithm processes.

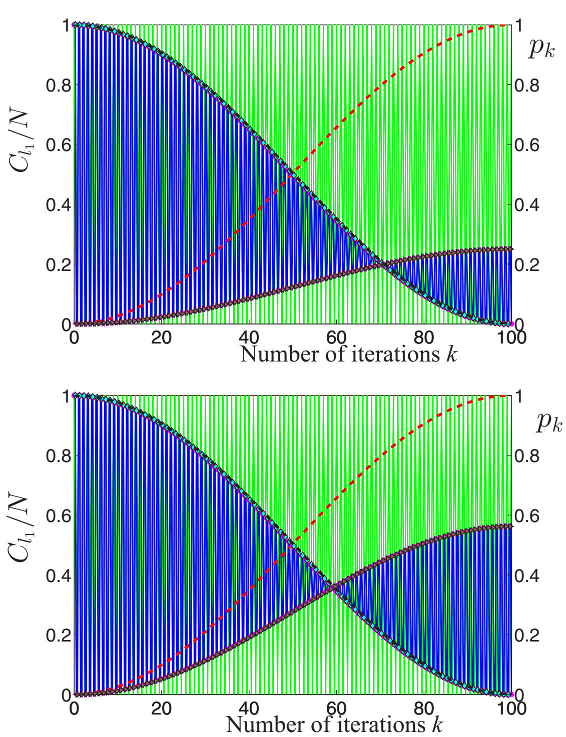

FIG. 3. The coherence dynamics with iteration number k for $n = 16$, $t = 4$. (Top) The target state $|\chi_1\rangle$ is a product state; (bottom) $|\chi_1\rangle$ is an entangled state. The red dashed line is the success probability (p_k), the blue line is the coherence when O, $H^{\otimes n}$, P, $H^{\otimes n}$ are applied in each iteration, the light-green line is the coherence just for applying $H^{\otimes n}$, $H^{\otimes n}$ without O and P.

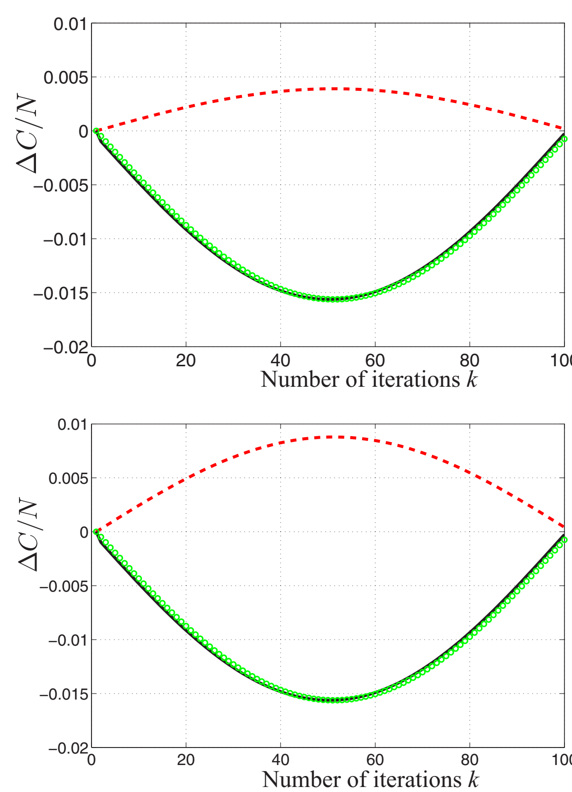

FIG. 4. The operator coherence of G, H_O, H_P with iteration number k for $n = 16$, $t = 4$. (Top) The target state $|\chi_1\rangle$ is a product state; (bottom) $|\chi_1\rangle$ is an entangled state. The green circle line, the red dashed line, and the black line correspond to the variations of the operator coherence of G, H_O, H_P, respectively.

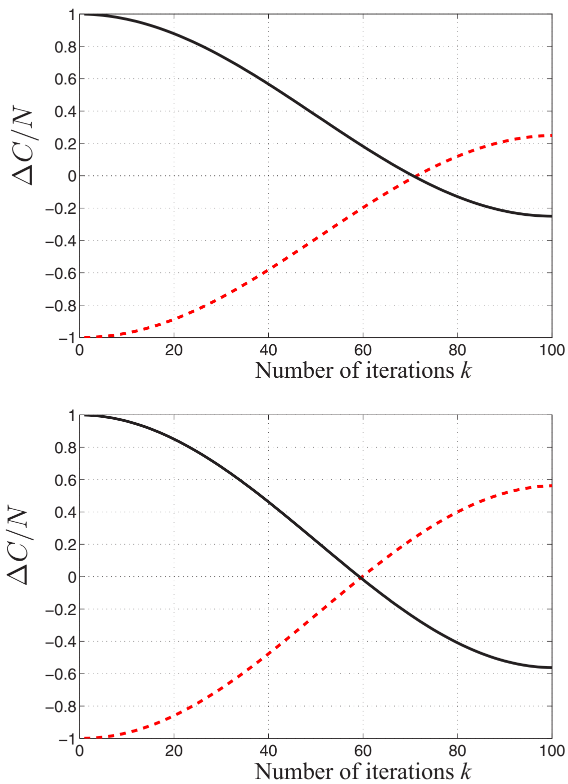

FIG. 5. The suboperator coherence of each H_O and H_P in one Grover iteration for $n = 16$, $t = 4$. (Top) The target state $|\chi_1\rangle$ is a product state; (bottom) $|\chi_1\rangle$ is an entangled state. The red dashed line and the black line correspond to the variations of suboperator coherence of each H_O and H_P, respectively.

By means of our results, the two $H^{\otimes n}$ are nonfree operations from the resource theoretical point of view for coherence. Different operators make different effects to coherence, and the coherence is changed exactly by these nonfree operations. Moreover, its coherence depends directly on the success probability. In other words, the success probability is directly affected by the application of these nonfree operations.

VII. CONCLUSION

In this paper, to clear how the essential operators contribute to the coherence and what affect the operator coherence has in Grover's search algorithm, we have studied the coherence dynamics of the state in Grover's algorithm via the l_1 norm of coherence.

We have shown that the oracle O and the phase-shift operator P are incoherent operators while two Hadamard operators $H^{\bigotimes n}$ are coherent operators. Specifically, based on the computational basis, the two $H^{\otimes n}$ have different effects on coherence that one produces coherence and another depletes coherence. In other words, the coherence is vibrating during the search process. We have also proved that the amount of the operator coherence depends not only on the size of the database and the success probability, but also on target states during the search.

According to this work, we have discovered the roles the four operators play to coherence in Grover's search algorithm and the relationship of operators' coherence with the success probability and target states. Entanglement and coherence are two important resources, and we have shown that entanglement has an impact on operator coherence in GSA. However, how entanglement and coherence are directly connected with each other is still a problem to be further clarified. Can we find suitable measurements of entanglement and coherence to obtain a conversion formula to describe their relationship? On the other hand, the hidden meanings of the oscillation of operator coherence in GSA are worthy to further study.

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APPENDIX A: PROOF OF EQ. (14)

Proof. For the sake of simplicity, denote D_y the coefficients of $|y\rangle$ of $|\psi_{k_{H_O}}\rangle$ in Eq. (10) that

$$D_y = \frac{A_k}{\sqrt{N-t}} \sum_{x\in\{x_n\}} (-1)^{xy} - \frac{B_k}{\sqrt{t}} \sum_{x\in\{x_s\}} (-1)^{xy}. \tag{A1}$$

According to the properties of Hadamard operator, we can get

$$\begin{aligned} D_{y=0} &= \frac{A_k}{\sqrt{N-t}}(N-t) - \frac{B_k}{\sqrt{t}}t \\ &= A_k\sqrt{N-t} - B_k\sqrt{t} \end{aligned} \tag{A2}$$

and

$$D_{y\neq 0} = \{(N/2 - t_y) + [N/2 - (t - t_y)](-1)\}\frac{A_k}{\sqrt{N-t}} + [t_y - (t - t_y)]\frac{-B_k}{\sqrt{t}} = (t - 2t_y)\left(\frac{A_k}{\sqrt{N-t}} + \frac{B_k}{\sqrt{t}}\right). \tag{A3}$$

Therefore, we have

$$|\psi_{k_{H_O}}\rangle = \frac{1}{\sqrt{N}}\left[(A_k\sqrt{N-t} - B_k\sqrt{t})\,|0\rangle + \sum_{y\neq 0}(t - 2t_y)\left(\frac{A_k}{\sqrt{N-t}} + \frac{B_k}{\sqrt{t}}\right)|y\rangle\right] \tag{A4}$$

and

$$\rho_{k_{H_O}} = \frac{1}{N}\Bigg[(A_k\sqrt{N-t}-B_k\sqrt{t})^2\,|0\rangle\langle 0| + \sum_{y\neq 0}(t-2t_y)(A_k\sqrt{N-t}-B_k\sqrt{t})\left(\frac{A_k}{\sqrt{N-t}}+\frac{B_k}{\sqrt{t}}\right)(|0\rangle\langle y| + |y\rangle\langle 0|)$$
$$+\sum_{y,\,y'\neq 0}(t-2t_y)(t-2t'_y)\left(\frac{A_k}{\sqrt{N-t}}+\frac{B_k}{\sqrt{t}}\right)^2|y\rangle\langle y'|\Bigg]. \quad \text{(A5)}$$

According to Eq. (3), we obtain Eq. (14) that

$$C_{l_1}(\rho_{k_{H_O}}) = \frac{1}{N}\left[2\sum_{y\neq 0}|(t-2t_y)(A_k\sqrt{N-t}-B_k\sqrt{t})|\left(\frac{A_k}{\sqrt{N-t}}+\frac{B_k}{\sqrt{t}}\right)+\sum_{\substack{y,\,y'\neq 0\\ y\neq y'}}|(t-2t_y)(t-2t'_y)|\left(\frac{A_k}{\sqrt{N-t}}+\frac{B_k}{\sqrt{t}}\right)^2\right]. \quad \text{(A6)}$$

APPENDIX B: NUMERAL EXAMPLES

For clarity, we give more examples here. Take the same qubit number $n = 16$ and the same target number $t = 4$, but different target states $|\chi_1\rangle$ that one is a product state with $\gamma(t, t_y) = 1/4$ and the other is an entangled state with $\gamma(t, t_y) = 9/16$ as shown in Theorem 5.

As is shown in Fig. 3, we can clearly see that the amount of operator coherence is larger when $|\chi_1\rangle$ is an entangled state than $|\chi_1\rangle$ is a product state. The reason may be that the operators make more contributions to coherence when the target state $|\chi_1\rangle$ is an entangled state than it is a product state as seen from Figs. 4 and 5.

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