Entropy, purity, and fidelity in Majorana phase space

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Majorana phase-space representations for fermions map fermionic many-body physics into a distribution over one of the Cartan symmetric spaces of Lie group theory. The representation is in terms of $2M \times 2M$ complex antisymmetric matrices, which generate the Gaussian Majorana operators. Here we show how this expansion can be utilized to calculate quantities arising in quantum thermodynamics and quantum information. Purity and the linear entropy are calculated, as well as the quantum fidelity between two general fermionic states, with numerical examples for pure states. We describe the geometrical properties of the phase space, and show that the overlap between two Gaussian Majorana states depends on the product of their antisymmetric matrices. Fermionic phase space is divided up into two orthogonal subspaces of different number parity, whose matrix representations differ by an orthogonal reflection in the phase space.

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I. INTRODUCTION

Much research in recent years has focused on Majorana fermions and quasiparticles [1–9]. While studying particles that are their own antiparticles is of intrinsic interest, the corresponding operators have an important significance as quadrature components of any fermionic field. One tool to study these particles is the use of quantum phase-space methods. This approach has grown in popularity since Wigner's paper [10] on quantum corrections to thermodynamic equilibrium. Such methods are widely used to study interacting Bose systems [11–14], and this approach can also be used in fermionic cases [15–18]. Recently, fermionic phase-space methods have been extended to Majorana space [19,20], leading to fermionic analogs of both the Q function [18,21] and the P function [22,23] expansions.

Majorana phase-space methods are used in this paper to calculate the fundamental information-related quantities of entropy, purity, and fidelity in Fermi systems. The Majorana representation has been recently used to develop fidelity witnesses [24]. This formalism has also been applied to Ising chains [25]. Therefore, it is important to obtain an expression for the Renyi entropy, purity, and fidelity in these representations. Entropy in quantum information is used to quantify disorder and entanglement. As the entropy of a system increases, information about the system decreases, making this a fundamental concept in quantum thermodynamics. The relation between entropy and information of a quantum system is due to Shannon [26]. The Renyi or linear entropy [27] of a density matrix $\hat{\rho}$, generalizes the Shannon entropy so that it is simpler to treat finite systems. Utilizing these results allows one to study the thermodynamical properties of quantum many-body Fermi systems [28] and their entanglement [29,30]. Some of these have been performed by using Monte Carlo methods [31–33]. The Renyi entropy has been used to study velocity distributions [34], threshold selection [35], and cryptography [36].

In calculating or measuring a quantum many-body state, purity signifies whether a pure state has undergone decoherence. For a given quantum density matrix $\widehat{\rho}$, purity is defined as $\Pi(\widehat{\rho}) = \mathrm{Tr}[\rho^2]$. Methods of studying purity include quantum homodyne tomography (QHT) [37], discrimination methods [38], and many others, although we note that these techniques are mostly developed for bosonic cases. Purity calculations are essential in the studies of quantum information processing [39], as these applications are uniquely sensitive to decoherence.

Fidelity is another fundamental quantity in quantum information, which has been used to investigate quantum phase transitions [40–42], quantum teleportation [43,44], quantum metrology [45], evolution of open quantum systems [46], nonadiabaticity measurements [47], quantum chemistry [48], and quantum chaos [49]. In the literature there are investigations on the Renyi entropy based on Monte Carlo [50,51] and holographic methods [52].

Previous calculations of Renyi entropy utilizing phase-space methods [53] employed the positive-P distribution [23,54,55] to calculate the linear entropy, for both bosons and fermions. A recent review [56] analyzes the fidelity of mixed states in a detailed way. The present paper quantifies orthogonality properties and extends these earlier calculations of purity, linear entropy, and fidelity to Majorana phase space by utilizing the Majorana P representation [19,20].

In order to carry out these calculations, we give a detailed analysis of the geometrical properties and dimensionality of the Gaussian phase space. We prove a theorem on the representations of Gaussian state parity for the case of pure states, showing how the Gaussian pure states can be classified into two orthogonal subspaces.

This paper is organized as follows: Sec. II describes the properties of the Majorana P representation. Next, in Sec. III we describe the fidelity measures and the derivation of the inner product of two Gaussian Majorana operators. There we also show the relation between the parity operator and fidelity.

Section IV gives results for entropy and purity calculations. Fidelity calculations for pure states are discussed in Sec. V. Finally our conclusions are given in Sec. VI.

II. GAUSSIAN MAJORANA P REPRESENTATION

The Gaussian basis of fermions used here leads to either fermionic P functions [16,17,57] or Q functions [18,19]. Either approach gives rise to distributions on a space of $2M \times 2M$ matrices that grow only quadratically with the mode number M. Another interesting and useful approach to phase-space representations for fermions was first investigated using Grassmann variables [58]. This method is also utilized here for analytic proofs. Grassmann variables have an exponentially large $2^{2M} \times 2^{2M}$ matrix representation [59], which is infeasible for direct computational representation at large M.

Thus, there is a practical advantage in using a Gaussian basis, as one can directly represent and visualize the phase space. In this section we review the basic properties of the Majorana P function, which is a fermionic P function expressed using Majorana fermion operators.

A. Gaussian operators

The fermionic Gaussian operators are exponentials of quadratic forms in the fermionic raising and lowering operators. There is extensive literature on this approach using un-ordered exponentials [60,61]. Here we use an alternative approach: the normally ordered Gaussian basis [16,17,57]. This is equivalent to the traditional approach, and has a number of advantages. In particular, one can explicitly write the Gaussian basis in terms of a $2M \times 2M$ correlation matrix. This allows the calculation of differential identities, as well as existence theorems for expansions of the identity operator.

Using unitary transformations, it is possible to transform fermionic phase-space representations from the annihilation and creation operator approach to an expansion using Majorana operators [19]. This turns out to have many appealing properties. The fundamental phase-space variable is simply one of these antisymmetric matrices. As a result, there are many useful properties from Lie group theory and the theory of homogeneous spaces and matrix polar coordinates [62] that are directly applicable.

Throughout the paper, we use the following notation for an M-mode Hilbert space: M vectors are denoted in bold type as $\hat{\boldsymbol{a}}$, $2M$ vectors with a single underline as $\underline{\hat{a}}$, $M \times M$ matrices are in bold type, for example, $\mathbf{I}$ for the M-dimensional identity matrix, and $2M \times 2M$ matrices are denoted with a double underline as in $\underline{\underline{I}}$, which is the $2M$-dimensional identity.

The most general Majorana Gaussian operator is then defined in a normally ordered form, as [20]

$$\hat{\Lambda}(\underline{\underline{x}}) = N(\underline{\underline{x}}) : \exp[-i\underline{\hat{\gamma}}^T[\underline{\underline{i}} + (\underline{\underline{i}} + \underline{\underline{i}}\,\underline{\underline{x}}\,\underline{\underline{i}})^{-1}]\underline{\hat{\gamma}}/2] :, \quad (2.1)$$

where $\underline{\underline{i}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$ is a matrix square root of $-\underline{\underline{I}}$, while $\underline{\underline{x}}$ is a complex antisymmetric matrix and $\underline{\hat{\gamma}}$ is a vector of Majorana operators such that

$$\{\hat{\gamma}_i, \hat{\gamma}_j\} = 2\delta_{ij}. \quad (2.2)$$

The normalization factor N is defined as

$$N(\underline{\underline{x}}) = \frac{1}{2^M}\sqrt{\det[\underline{\underline{i}} - \underline{\underline{x}}]}, \quad (2.3)$$

which gives a unit trace for the Gaussian operator.

The Majorana operators themselves are obtained as a result of the action of a matrix [60], $\underline{\underline{U}} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ -i\mathbf{I} & i\mathbf{I} \end{bmatrix}$ on the extended fermionic creation and annihilation operator, $\underline{\hat{a}} = (\hat{\boldsymbol{a}}^T, \hat{\boldsymbol{a}}^\dagger)^T$ and $\underline{\hat{a}}^\dagger = (\hat{\boldsymbol{a}}^\dagger, \hat{\boldsymbol{a}}^T)$, so that $\underline{\hat{\gamma}} = \underline{\underline{U}}\,\underline{\hat{a}}$.

The choice of a Gaussian operator with normal ordering, with all creation operators ordered to the left, and signs chosen in the standard way, allows us to define the antisymmetric matrix $\underline{\underline{x}}$ so that there are well-defined differential identities. In the case of a real matrix $\underline{\underline{x}}$, $\hat{\Lambda}(\underline{\underline{x}})$ is Hermitian and positive definite and corresponds to a definite quantum state. If we introduce the basic Majorana variance commutator,

$$\hat{X}_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu], \quad (2.4)$$

then the matrix $\underline{\underline{x}}$ is simply the operator variance of the Gaussian quantum state $\hat{\Lambda}(\underline{\underline{x}})$, so that

$$\underline{\underline{x}} = \mathrm{Tr}[\underline{\underline{\hat{X}}}\hat{\Lambda}(\underline{\underline{x}})]. \quad (2.5)$$

B. Fermionic P representations

It is possible to express any fermionic quantum density matrix as an expansion of the Gaussian basis and the Majorana P function as follows [16,19,57]:

$$\hat{\rho}(t) = \int_{\mathcal{D}} P(\underline{\underline{x}}, t)\hat{\Lambda}(\underline{\underline{x}})d\underline{\underline{x}}, \quad (2.6)$$

where t is the time, $\hat{\Lambda}(\underline{\underline{x}})$ is a normalized Gaussian operator corresponding to the complex phase-space variable $\underline{\underline{x}}$, and $P(\underline{\underline{x}}, t)$ is the P function. This can be real or complex. The general integration measure is $d\underline{\underline{x}} = \prod_{i<j} d^2x_{ij}$ over a complex integration domain $\mathcal{D}$, which we treat in more detail below.

One of the properties of the $P(\underline{\underline{x}}, t)$ distribution is that the normalization of $\hat{\rho}$ and $\hat{\Lambda}$ implies that this distribution is also normalized such that

$$\int_{\mathcal{D}} P(\underline{\underline{x}}, t)d\underline{\underline{x}} = 1. \quad (2.7)$$

The expectation value of any Hermitian quantum operator $\hat{A}$ can therefore be calculated at time t given $A(\underline{\underline{x}}) = \mathrm{Tr}[\hat{A}\hat{\Lambda}(\underline{\underline{x}})]$, so that

$$\langle\hat{A}\rangle = \int_{\mathcal{D}} P(\underline{\underline{x}}, t)A(\underline{\underline{x}})d\underline{\underline{x}}. \quad (2.8)$$

This is a complete representation like the positive P distribution for bosons [23], provided complex antisymmetric matrices are included in the phase-space domain $\mathcal{D}$. One can also define a more restricted representation similar to the Glauber-Sudarshan P representation for bosons, if only real antisymmetric matrices are included. In this restricted case the basis of $\hat{\Lambda}$ operators is Hermitian and positive definite.

An important property of $\hat{\Lambda}(\underline{\underline{x}})$ is that an orthogonal matrix transformation such that $\underline{\underline{x}} \to \underline{\underline{O}}^T\underline{\underline{x}}\,\underline{\underline{O}}$ has the same effect as a transformation on the Majorana operators such that $\underline{\hat{\gamma}} \to \underline{\underline{O}}\,\underline{\hat{\gamma}}$.

Since this transformation leaves the fundamental Majorana anticommutators invariant, it also gives a unitary transformation in the quantum Hilbert space, which is equivalent to a change of basis. However, this is a larger group of transformations than a simple mode transformation of the fermion operators. It can swap the roles of annihilation and creation operators, thus turning particles into antiparticles, as well as other more general transformations.

C. Dimensionality of manifolds and pure states

The Hermitian Majorana Gaussian density matrices $\hat{\Lambda}(\underline{\underline{x}})$ are defined in terms of an antisymmetric real matrix $\underline{\underline{x}}$ of size $2M \times 2M$. Since this matrix is required to be antisymmetric, the diagonal elements are zero, and the transposed off-diagonal elements are equal and opposite. This defines a homogeneous real vector space of real dimension $D = M(2M-1)$. When bounded by the additional condition that

$$\underline{\underline{I}} - \underline{\underline{x}}\underline{\underline{x}}^{\dagger} > 0, \tag{2.9}$$

the space is known as one of the six group theoretic irreducible homogeneous symmetric domains [62,63]. The inequality means that all eigenvalues of the matrix $\underline{\underline{I}} - \underline{\underline{x}}\underline{\underline{x}}^{\dagger}$ are positive. We extend this to include the boundary, so our integration domain $\mathcal{D}$ is for $\underline{\underline{I}} - \underline{\underline{x}}\underline{\underline{x}}^{\dagger} \geqslant 0$.

In the case of Hermitian Gaussian states, the physical states are characterized by real antisymmetric matrices such that $\underline{\underline{I}} + \underline{\underline{x}}^2 \geqslant 0$. This is the real subspace of the complex symmetric space, with the addition of the real boundary. These correspond to physical density matrices that are Hermitian and positive definite. We can introduce $\underline{\underline{y}} = i\underline{\underline{x}}$, which is a Hermitian matrix. By the spectral theorem, it is diagonalizable, and has $2M$ real eigenvalues, λ_m. Eigenvalues of an even dimensional antisymmetric matrix are paired, and hence the antisymmetric imaginary matrix $\underline{\underline{y}}$ has at most M independent real eigenvalues $\pm\lambda_m$. The same is true for $\underline{\underline{x}}^2 + \underline{\underline{I}} = \underline{\underline{I}} - \underline{\underline{y}}^2$, except that by applying the same diagonalization, this has at most M distinct real eigenvalues $1 - \lambda_m^2$.

The condition that $\underline{\underline{x}}^2 + \underline{\underline{I}} \geqslant 0$ means that this inequality holds for all the eigenvalues. Therefore, there are M independent conditions that $1 - \lambda_m^2 \geqslant 0$. This defines an interval for each eigenvalue such that at the end points, $\lambda_m = \pm 1$, which is the boundary of the real homogeneous space. This boundary condition is not sufficient to characterize a Gaussian Majorana pure state, for which $\underline{\underline{x}}^2 = -\underline{\underline{I}}$. In the pure state case, *all* the M eigenvalues of y must therefore satisfy $\lambda_m = \pm 1$, which defines M additional constraints on the coefficients of $\underline{\underline{x}}$. Hence the real dimensionality of the pure states must be $d = M(2M-1) - M = 2M(M-1)$.

As a result, almost all of the boundary of the homogeneous space consists of mixed states, in which only one eigenvalue has reached its maximal value. The dimensionality of the pure state manifold is much smaller than this, as indicated schematically in Fig. 1.

To give an example, in the single fermion case of a pure state with $M = 1$, one has $d = 0$. There are only two possible antisymmetric $\underline{\underline{x}}$ matrices that satisfy these conditions, without any continuous vector space of coefficients. These are the two pure Fock state density matrices, so that $\hat{\Lambda}(\underline{\underline{x}}) = |n\rangle\langle n|$ with $n = 0, 1$, and there are no other pure states.

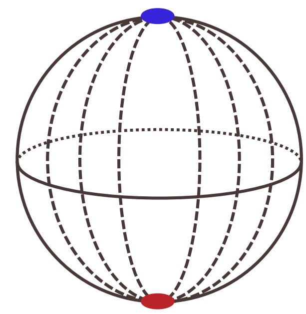

FIG. 1. Schematic diagram of the real homogeneous space as a hypersphere with polar caps at the North and South Poles, representing pure states with positive and negative parity. These are indicated by the red and blue colors, respectively. Mixed states occur in the interior of the sphere, but can also exist at the boundaries, away from the poles.

In the $M = 2$ Hilbert space, it follows that for the pure states, $d = 4$, while the total space of Gaussian density matrices has dimension $D = 6$. In this case, the pure state manifold has two less dimensions than the mixed state manifold. For this and for all higher dimensional cases, the pure state manifold cannot be a boundary of the homogeneous space, because this is necessarily a manifold of dimension $\delta D = D - 1 = M(2M-1) - 1$ [64]. As a result, the domain boundary is generically composed of partially mixed and pure states, for which at least one of the eigenvalues has reached its bounding value of $\lambda_m = \pm 1$.

In general the manifold of pure states is a subspace of the real manifold boundary space. For $M = 2$, it has dimension $d = 4$, in which case the manifold boundary has dimension $\delta D = 5$, and the homogeneous space dimension is $D = 6$.

D. Majorana identities and observables

In an earlier paper [19], we have explained how one may calculate observables from these representations using operator identities. The main identities are given below, for (left) mixed order and (right) products, respectively:

$$\begin{aligned}
\underline{\hat{\gamma}\hat{\gamma}^T}\,\hat{\Lambda} &= i\left[\underline{\underline{x}}^{-}\frac{d\hat{\Lambda}}{d\underline{\underline{x}}}\underline{\underline{x}}^{+} - \hat{\Lambda}\underline{\underline{x}}^{+}\right],\\
\underline{\hat{\gamma}}\hat{\Lambda}\underline{\hat{\gamma}}^T &= i\left[-\underline{\underline{x}}^{-}\frac{d\hat{\Lambda}}{d\underline{\underline{x}}}\underline{\underline{x}}^{-} + \hat{\Lambda}\underline{\underline{x}}^{-}\right],\\
\hat{\Lambda}\underline{\hat{\gamma}\hat{\gamma}^T} &= i\left[\underline{\underline{x}}^{+}\frac{d\hat{\Lambda}}{d\underline{\underline{x}}}\underline{\underline{x}}^{-} - \hat{\Lambda}\underline{\underline{x}}^{+}\right].
\end{aligned} \tag{2.10}$$

Here $\underline{\underline{x}}^{\pm} \equiv \underline{\underline{x}} \pm i\underline{\underline{I}}$, so that if we define the Majorana correlation function as $\hat{X}_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, then its expectation value is given by

$$\langle \hat{X}_{\mu\nu} \rangle = \mathrm{Tr}[\hat{\rho}\widehat{X}_{\mu\nu}] = \int_{\mathcal{D}} P(\underline{\underline{x}}) x_{\mu\nu} d\underline{\underline{x}}. \tag{2.11}$$

Since $\underline{\underline{x}}$ is simply the Majorana correlation matrix for a Majorana Gaussian state, we can utilize Wick's theorem [65,66] to write all higher order correlations. If we define $\hat{C}^{(p)}_{j_1,\dots j_{2p}} = i^p\widehat{\gamma}_{j_1}\dots\widehat{\gamma}_{j_{2p}}$, these correlations can therefore be obtained from $\underline{\underline{x}}$ as

$$C^{(p)}_{j_1\dots j_{2p}}(\underline{\underline{x}}) = \mathrm{Tr}\big(\hat{\Lambda}(\underline{\underline{x}})i^p\widehat{\gamma}_{j_1}\dots\widehat{\gamma}_{j_{2p}}\big) = Pf(\underline{\underline{x}}'), \tag{2.12}$$

where Pf denotes the Pfaffian, $1 \leqslant j_1 < \dots < j_{2p} \leqslant 2M$, and $x'_{ik} = x_{j_i,j_k}$ is the corresponding $2p \times 2p$ submatrix obtained through deletion of all rows and columns of $\underline{\underline{x}}$ that do not occur in the correlation function. Thus, one can calculate any correlation function defined as a polynomial in the Majorana operators. This is possible for any density matrix, whether pure, mixed, or non-Gaussian, via a phase-space average as in Eq. (2.8) with $A(\underline{\underline{x}}) = Pf(\underline{\underline{x}}')$.

III. PARITY AND FIDELITY

In this section we explain how number parity can be calculated from the antisymmetric covariance matrix of the Majorana Gaussian operator. We also explain how parity and orthogonal transformations relate to the inner product of the Gaussian operators, and therefore to the fidelity and entropy measures. These quantities are closely related to the Renyi entropy, which is defined by using the Euclidean norm, $\|\hat{\rho}\|_2 = \sqrt{\mathrm{Tr}(\hat{\rho}\hat{\rho}^\dagger)}$, with $S_2 = -\ln \|\hat{\rho}\|_2^2$.

A. Fidelity measures

The Euclidean fidelity measure between two mixed quantum states $\widehat{\rho}$ and $\widehat{\sigma}$ is defined by the relation [56],

$$\mathcal{F}_2 = \frac{\mathrm{Tr}(\hat{\rho}\hat{\sigma})}{\max(\mathrm{Tr}(\hat{\rho}^2), \mathrm{Tr}(\hat{\sigma}^2))}. \tag{3.1}$$

For Hermitian Gaussian pure states, $\hat{\Lambda}(\underline{\underline{x}})$, $\hat{\Lambda}(\underline{\underline{y}})$ the corresponding $\underline{\underline{x}}$, $\underline{\underline{y}}$ matrices are real, and

$$\mathcal{F}_2(\hat{\Lambda}(\underline{\underline{x}}), \hat{\Lambda}(\underline{\underline{y}})) = \mathrm{Tr}(\hat{\Lambda}(\underline{\underline{x}})\hat{\Lambda}(\underline{\underline{y}})). \tag{3.2}$$

The first necessary step for the evaluation of the entropy, purity, and fidelity using Gaussian representations is therefore to calculate the trace of the product of two unit trace Majorana Gaussian operators. This must include the non-Hermitian Gaussian operators with complex $\underline{\underline{x}}$ matrices. Following the calculations given in Appendix, we obtain that the inner product of two Majorana Gaussian operators is

$$F(\underline{\underline{x}}, \underline{\underline{y}}) = \mathrm{Tr}[\hat{\Lambda}(\underline{\underline{x}})\hat{\Lambda}^\dagger(\underline{\underline{y}})] = 2^{-M}\sqrt{\det[\underline{\underline{I}} - \underline{\underline{x}}\underline{\underline{y}}^*]}. \tag{3.3}$$

B. Number parity

The number parity eigenvalue is an essential property of fermion states. The corresponding parity operator depends on the choice of basis used to define the original number states. If one interchanges a single mode creation and annihilation operator, which does not change the commutation relations, the parity changes sign. However, in any basis the parity operator is a conserved quantity under a Hamiltonian that is quadratic in the Majorana operators, or equivalently quadratic in the raising and lowering operators. This is because such a Hamiltonian can only change the particle number in even steps of 0, $\pm 2n$. Of course, a lossy reservoir can change the parity.

We wish to show that this can be calculated directly as the Pfaffian function of the phase-space matrix $\underline{\underline{x}}$. The Majorana parity operator is the maximal Mth-order Majorana correlation function [67]:

$$\mathcal{P}(\widehat{\boldsymbol{\gamma}}) = (-1)^{\hat{N}} = i^M \prod_{i=1}^{2M} \widehat{\gamma}_i, \tag{3.4}$$

where $\hat{N} = \sum_i \hat{n}_i$ is the total fermionic number operator. This can be written as

$$\mathcal{P}(\widehat{\boldsymbol{\gamma}}) = i^M \widehat{\gamma}_1\widehat{\gamma}_2\widehat{\gamma}_3\dots\widehat{\gamma}_{2M}. \tag{3.5}$$

Using the expression in the above equation we evaluate the higher order correlations given in Eq. (2.12) as

$$\mathrm{Tr}(\hat{\Lambda}\mathcal{P}(\widehat{\boldsymbol{\gamma}})) = Pf(\underline{\underline{x}}). \tag{3.6}$$

Hence the mean parity of any Gaussian is simply the Pfaffian function of the Majorana correlation matrix $\underline{\underline{x}}$ so that $\mathcal{P}(\underline{\underline{x}}) = Pf(\underline{\underline{x}})$.

For pure state Gaussian density matrices, $\mathrm{Tr}[\hat{\Lambda}(\underline{\underline{x}})^2] = 1$. The inner-product result (3.3) shows that in this case, $\det[\underline{\underline{I}} - \underline{\underline{x}}^2] = 2^{2M}$. Pure state Gaussian density matrices are diagonalizable as number states in some orthogonally transformed number state basis [18,68]. Number states with an ith mode occupation of $n_i = 0, 1$, from which one can deduce that $\det(\underline{\underline{x}}) = 1$ and $\underline{\underline{x}}^2 = -\underline{\underline{I}}$. This is consistent with having $\det[\underline{\underline{I}} - \underline{\underline{x}}^2] = 2^{2M}$. For antisymmetric matrices $Pf(\underline{\underline{x}})^2 = \det(\underline{\underline{x}})$, so this also means that the corresponding parity is $\mathcal{P}(\underline{\underline{x}}) = \pm 1$.

Since these are the extremal eigenvalues of the quantum parity operator, it follows that all Gaussian pure states are number parity eigenstates. However, they are generally not number eigenstates, and hence are similar to the coherent states of bosons, which are not number states either. While physical pure states are number states, due to superselection rules, both coherent and Gaussian states allow one to obtain a complete mathematical basis for the physical states, which is an extremely useful property, even though unphysical superpositions may be included in the basis.

C. Fundamental orthogonality theorem

We now show that, unlike the coherent states of bosons, the Gaussian pure states of fermions are fundamentally divided into two orthogonal groups that depend on their number parity. The fidelity result given above can then be interpreted as showing that when two states are related by a rotation, there is a finite relative fidelity. We will also show that if they are Gaussian pure states related by a reflection, they are orthogonal to each other, since there is zero relative fidelity for pure states of opposite parity.

Theorem. If $\underline{\underline{x}}$ is real, antisymmetric and orthogonal with $\det(\underline{\underline{x}}) = 1$, and $\underline{\underline{y}} = \underline{\underline{O}}^T\underline{\underline{x}}\underline{\underline{O}}$ where $\underline{\underline{O}}$ is a real orthogonal matrix with $\det(\underline{\underline{O}}) = -1$, then

$$F(\underline{\underline{x}}, \underline{\underline{y}}) = 0. \tag{3.7}$$

Proof. A standard identity connecting Pfaffians and determinants is as follows:

$$Pf[\underline{\underline{OxO^T}}] = \det[\underline{\underline{O}}]Pf[\underline{\underline{x}}]. \tag{3.8}$$

Using the relation $\mathcal{P}(\underline{\underline{x}}) = Pf(\underline{\underline{x}})$, the above relation can be written as

$$Pf[\underline{\underline{OxO^T}}] = \det[\underline{\underline{O}}]\mathcal{P}(\underline{\underline{x}}). \tag{3.9}$$

In the case that $\det(\underline{\underline{x}}) = 1$ with $\underline{\underline{x}}$ real, antisymmetric and orthogonal, the corresponding $\hat{\Lambda}(\underline{\underline{x}})$ is a pure Fermi state with $\mathcal{P}(\underline{\underline{x}}) = \pm 1$, and hence is an eigenstate of the parity operator $\hat{\mathcal{P}}$. Since $\det(\underline{\underline{O}}) = -1$ the parity of $\underline{\underline{y}}$ must change sign, i.e., $\mathcal{P}(\underline{\underline{y}}) = \mp 1$, as evident from Eq. (3.9). As a consequence, $\hat{\Lambda}(\underline{\underline{x}})$ and $\hat{\Lambda}(\underline{\underline{y}})$ are eigenstates of a Hermitian operator with different eigenvalues, and are orthogonal, so $F(\underline{\underline{x}}, \underline{\underline{y}}) = 0$. ■

In summary, an orthogonal rotation with $\det(\underline{\underline{O}}) = 1$ does not change the parity of the Gaussian Fermi state, but a reflection with $\det(\underline{\underline{O}}) = -1$ does. Two Gaussian pure states that have different parity are orthogonal, since they are distinct eigenstates of a Hermitian observable. Physically, the rotations are caused by unitary evolution generated by a Bogoliubov-de Gennes Hamiltonian which at most creates and destroys fermions in pairs, and does not change parity. Reflections, which change parity, are caused by decoherence processes in which a single particle is lost or gained from coupling to a reservoir.

To give a more direct proof in the simplest case of a single reflection, suppose that only one reflection is involved, so that $\underline{\underline{O}} = \underline{\underline{R}}$ and $\underline{\underline{y}} = \underline{\underline{RxR}}$, where $\underline{\underline{R}}^2 = \underline{\underline{I}}$ and $\det(\underline{\underline{R}}) = -1$. Then,

$$\begin{aligned} F(\underline{\underline{x}}, \underline{\underline{y}}) &= \det[\underline{\underline{I}} - \underline{\underline{xRxR}}] \\ &= \det[\underline{\underline{I}} - \underline{\underline{xR}}]\det[\underline{\underline{I}} + \underline{\underline{xR}}]. \end{aligned} \tag{3.10}$$

However, $\det[\underline{\underline{Rx^T}}] = \det[\underline{\underline{R}}]\det[\underline{\underline{x}}] = -1$, so

$$\begin{aligned} \det[\underline{\underline{I}} + \underline{\underline{xR}}] &= -\det[\underline{\underline{Rx^T}}]\det[\underline{\underline{I}} + \underline{\underline{xR}}] \\ &= -\det[\underline{\underline{Rx^T}} + \underline{\underline{Rx^T}}\,\underline{\underline{xR}}] \\ &= -\det[\underline{\underline{I}} + \underline{\underline{xR}}]. \end{aligned} \tag{3.11}$$

This means that $\det[\underline{\underline{I}} + \underline{\underline{xR}}] = F(\underline{\underline{x}}, \underline{\underline{y}}) = 0$, as required by the general theorem above. This will be investigated numerically and shown to be valid in later sections.

D. Visualization

It is difficult to visualize the geometry of the homogeneous space of the rotations and reflections of the real antisymmetric matrices. They have a high dimensionality. One simple way to envisage them is shown in Figs. 1 and 2. In Fig. 1, the real homogeneous space is depicted as a sphere, although this is a simplified picture of the actual high dimensional geometry, since the homogeneous space exists in $M(2M-1)$ dimensions. The surface of the sphere represents the boundary, which has $M(2M-1) - 1$ dimensions.

The North Pole of the sphere represents the positive parity pure states, and the South Pole the negative parity pure states. These poles are in fact two distinct lower dimensional spaces,

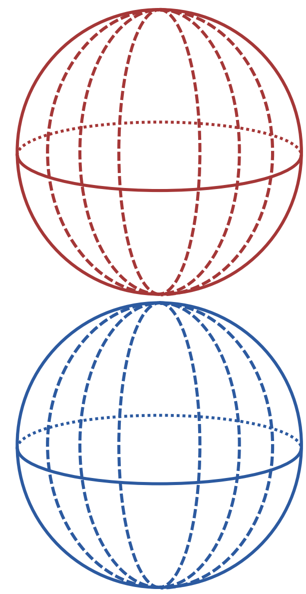

FIG. 2. Diagram of the pure states in the homogeneous space as the two polar caps unfolded into hyperspherical surfaces of positive and negative parity, indicated by the red and blue colors, respectively. These represent spaces that are of M dimensions lower than the original homogeneous space.

each of dimension $2M(M-1)$. The pure states themselves are two distinct, compact spaces represented as the hyperspherical surfaces shown in Fig. 2, although of M lower dimensions than the original space. The red hypersphere corresponds to $P = 1$ and the blue to $P = -1$. Orthogonal rotations can lead to any point on a sphere. Orthogonal reflections take one to a point to the other sphere. Once there, more rotations will lead to the point staying on this sphere, as previously. Another reflection is needed in order to return to the starting sphere.

IV. ENTROPY AND PURITY

Entropy is a quantity that defines a system's uncertainty and randomness [69]. Experimentally entropy has been measured as a function of energy [70]. These results are in agreement with the theoretical results [71] by considering strongly interacting fermions in a harmonic trap. Finite temperature Monte Carlo simulations [72] of the entropy of a unitary Fermi gas have also been verified by experimental results. In some cases it is difficult to measure temperature and entropy is measured instead.

Quantum entropies are usually defined through the von Neumann entropy $S = -\mathrm{Tr}\widehat{\rho}\ln\widehat{\rho}$. Here we focus on the linear or Renyi entropy [27], which is defined as

$$S_2 = -\ln \mathrm{Tr}(\widehat{\rho}^2). \tag{4.1}$$

This is important, as it is related to both purity and fidelity.

We wish to express the Renyi entropy in terms of the Majorana P representation. This can be done by using the expression of Eq. (2.6) in Eq. (4.1), obtaining

$$S_2 = -\ln \iint P(\underline{\underline{x}})P^*(\underline{\underline{y}})\mathrm{Tr}[\hat{\Lambda}(\underline{\underline{x}})\hat{\Lambda}^\dagger(\underline{\underline{y}})]d\underline{\underline{x}}d\underline{\underline{y}}. \tag{4.2}$$

This result is expressed in terms of the inner product of two Gaussian operators, which is given in Eq. (3.3).

A. Sampled entropy

In order to sample the Renyi entropy, one can use a sampling technique which is described below. As an example, we consider the case where $P(\underline{\underline{x}})$ is a distribution over the pure states. The antisymmetric matrix is expressed as an orthogonal transformation of pure states. Thus, it is possible to evaluate the double integral in Eq. (4.2) as a summation of samples of orthogonal transformation of pure states. It is given by

$$S_2 \approx -\ln\left\{\frac{1}{N^2}\sum_{i,j=1}^{N} F\left(\underline{\underline{x}}^{(i)}, \underline{\underline{y}}^{(j)}\right)\right\},$$

provided $\underline{\underline{x}}^{(i)}$ and $\underline{\underline{y}}^{(j)}$ are the i.i.d samples from real positive distributions $P(\underline{\underline{x}})$ and $P(\underline{\underline{y}})$, respectively. Using Eq. (3.3), it is possible to rewrite the above equation as

$$S_2 \approx -\ln\left\{\frac{2^{-M}}{N^2}\sum_{i,j=1}^{N}\sqrt{\det\left(\underline{\underline{x}}^{(i)}\underline{\underline{y}}^{*(j)} - \underline{\underline{I}}\right)}\right\}.$$

We now consider a special case where $P(\underline{\underline{x}})$ is a uniform measure over the surface of pure states with $\underline{\underline{x}}^2 = -\underline{\underline{I}}$. From known operator identities [18,68], this corresponds to the identity operator, which is an infinite temperature mixed state, $\hat{\rho}_\infty = 2^{-M}\hat{I}$. The Renyi entropy in this case is

$$S_2 = -\ln\frac{1}{2^{2M}}\mathrm{Tr}(\widehat{I}) = M\ln 2. \tag{4.3}$$

To sample over the surface, we start with a reference Gaussian pure state, and use randomly chosen orthogonal transformations to generate the sampled pure states $\hat{\Lambda}(\underline{\underline{x}})$. The initial pure state utilized for orthogonal transformations to generate these real antisymmetric matrices is given by $\underline{\underline{B}} = \oplus_{i=1}^{M}\mathbf{B}_i$, where $\mathbf{B}_i = \left[\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right]$ and $\underline{\underline{B}} = -\underline{\underline{B}}^T$. This has the explicit form of

$$\underline{\underline{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0\\ -1 & 0 & 0 & 0 & . & . & . & 0\\ 0 & 0 & 0 & 1 & & . & . & .\\ 0 & 0 & -1 & 0 & . & . & . & .\\ . & 0 & 0 & . & . & . & . & .\\ . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & 0 & 1\\ 0 & 0 & 0 & . & . & . & -1 & 0 \end{bmatrix}. \tag{4.4}$$

After applying an orthogonal transformation to $\underline{\underline{B}}$, one can generate i.i.d samples as $\underline{\underline{x}} = \underline{\underline{O}}^T\underline{\underline{BO}}$, where $\underline{\underline{O}}$ is an orthogonal matrix. These pure $\underline{\underline{x}}$ matrices, as well as any products of $\underline{\underline{x}}$ matrices, are isomorphic to the $O(2M)$ group. As a result, sampling the Gaussian pure states is equivalent to sampling members of the $O(2M)$ group located on the surface of the Cartan symmetric space.

In Fig. 3, the Renyi entropy for infinite temperature states corresponding to different mode numbers is plotted. Here a sample size of $N = 400$ randomly chosen pure states is used. The result agrees with the expected analytic result for the entropy of the infinite temperature state given in Eq. (4.3).

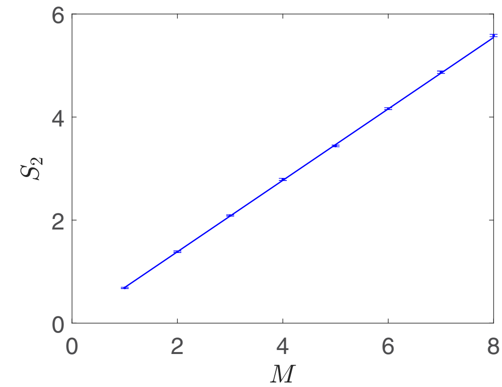

FIG. 3. Renyi entropy S_2 of an infinite temperature maximally mixed state, obtained from samples of random orthogonal transformations of Gaussian pure states with $\underline{\underline{x}}^2 = -\underline{\underline{I}}$, and different mode numbers M for a sample size of $N = 400$. The sampled results are the error bars. They agree with the solid line giving analytic values within sampling error.

B. Purity

The purity of a quantum state $\widehat{\rho}$ is given by $\mathrm{Tr}(\widehat{\rho}^2)$ [39], which is the standard expression. Since an expression for the inner product of two Gaussian operators in Majorana representation has already been evaluated, one can write the purity of a quantum state using the expansion of the density operator provided by the Majorana P representation in Eq. (2.6), in the form,

$$\begin{aligned}\Pi(\widehat{\rho}) &= \iint P(\underline{\underline{x}})P^*(\underline{\underline{y}})\mathrm{Tr}[\hat{\Lambda}(\underline{\underline{x}})\hat{\Lambda}^\dagger(\underline{\underline{y}})]d\underline{\underline{x}}d\underline{\underline{y}}\\ &= \iint P(\underline{\underline{x}})P^*(\underline{\underline{y}})F(\underline{\underline{x}}, \underline{\underline{y}})d\underline{\underline{x}}d\underline{\underline{y}}.\end{aligned} \tag{4.5}$$

The purity of a state can be determined from the entropy results as well, since $S_2 = 0$ corresponds to a pure state, while $S_2 > 0$ corresponds to a mixed state. This is clearly very similar to the Renyi entropy, as shown in Fig. 4.

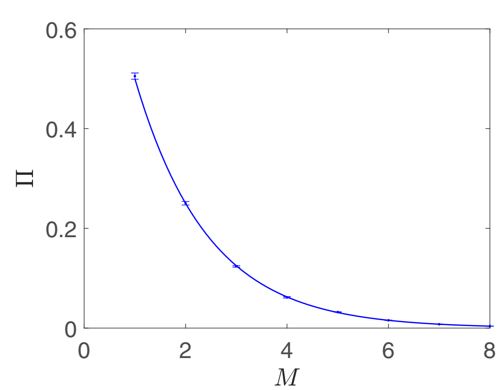

FIG. 4. Purity, Π, of an infinite temperature maximally mixed state, obtained using samples of random orthogonal transformations of Gaussian pure states each with $\underline{\underline{x}}^2 = -\underline{\underline{I}}$, and different mode numbers, M for a sample size of $N = 400$. Just as in Fig. 3, the sampled results are the error bars, which agree with the solid line giving analytic values.

V. FIDELITY

The general expression for the fidelity of two pure states $\widehat{\rho}$ and $\widehat{\sigma}$ is given by

$$\mathcal{F}(\widehat{\rho},\widehat{\sigma}) = \mathrm{Tr}(\widehat{\rho}\widehat{\sigma}). \tag{5.1}$$

In recent work on fidelity measures for two mixed states $\widehat{\rho}$ and $\widehat{\sigma}$ [56], different types of fidelities were defined. Here we focus on the norm-based fidelity $\mathcal{F}_2$ which can be evaluated as

$$\mathcal{F}_2\,(\widehat{\rho},\widehat{\sigma}) = \frac{\mathrm{Tr}(\widehat{\rho}\widehat{\sigma})}{\max[\mathrm{Tr}(\widehat{\rho}^2),\mathrm{Tr}(\widehat{\sigma}^2)]}. \tag{5.2}$$

This is valid in the pure state limit because, if $\widehat{\rho}$ and $\widehat{\sigma}$ are pure states then $\mathrm{Tr}(\widehat{\rho}^2)=\mathrm{Tr}(\widehat{\sigma}^2)=1$. This leads to

$$\max[\mathrm{Tr}(\widehat{\rho}^2),\mathrm{Tr}(\widehat{\sigma}^2)] = 1, \tag{5.3}$$

which in turn converts Eq. (5.2) as in Eq. (5.1).

Noting that $\widehat{\sigma}=\widehat{\sigma}^\dagger$, and utilizing the Majorana P function, $\mathrm{Tr}(\widehat{\rho}\widehat{\sigma})$ can be evaluated for any mixed state as

$$\mathrm{Tr}(\widehat{\rho}\widehat{\sigma}) = \iint P_\rho(\underline{\underline{x}})P_\sigma^*(\underline{\underline{y}})\mathrm{Tr}[\hat{\Lambda}(\underline{\underline{x}})\hat{\Lambda}^\dagger(\underline{\underline{y}})]d\underline{\underline{x}}d\underline{\underline{y}}. \tag{5.4}$$

As the inner product of two Gaussian operators has already been evaluated in Eq. (3.3), it is possible to write the above expression in terms of it, as:

$$\mathrm{Tr}(\widehat{\rho}\widehat{\sigma}) = \iint P_{\widehat{\rho}}(\underline{\underline{x}})P_{\widehat{\sigma}}^*(\underline{\underline{y}})F(\underline{\underline{x}},\underline{\underline{y}})d\underline{\underline{x}}d\underline{\underline{y}}. \tag{5.5}$$

If this is combined with the purity measure given above, we obtain a complete expression for the fidelity of two mixed states. In general, since we are sampling the distribution N times, then

$$\mathcal{F}_2(\widehat{\rho},\widehat{\sigma}) \approx \mathcal{F}_2^S(\widehat{\rho},\widehat{\sigma}) = \frac{\sum_{i,j=1}^N \mathrm{Tr}(\hat{\Lambda}(\underline{\underline{x}}^{(i)})\hat{\Lambda}(\underline{\underline{y}}^{(j)}))}{\max[\Pi(\widehat{\rho}),\Pi(\widehat{\sigma})]N^2}. \tag{5.6}$$

Here we can make use of the purity sampling result from the previous section as well.

A. Fidelity distributions

Fidelities can be calculated using sampling techniques, as given above. Here we illustrate the type of results obtained for pure states, when the corresponding density operators are real Majorana Gaussian operators, so that $\widehat{\rho}=\hat{\Lambda}(\underline{\underline{x}})$, $\widehat{\sigma}=\hat{\Lambda}(\underline{\underline{y}})$. This leads to

$$\mathcal{F}_2(\widehat{\rho},\widehat{\sigma}) = \mathrm{Tr}(\hat{\Lambda}(\underline{\underline{x}})\hat{\Lambda}(\underline{\underline{y}})). \tag{5.7}$$

We now ask this question: What is the distribution of relative fidelities $P(\mathcal{F}_2)$, given two random Gaussian pure states? Applying the expression for the inner product of two Gaussian operators, so that only one sample is needed, it is possible to

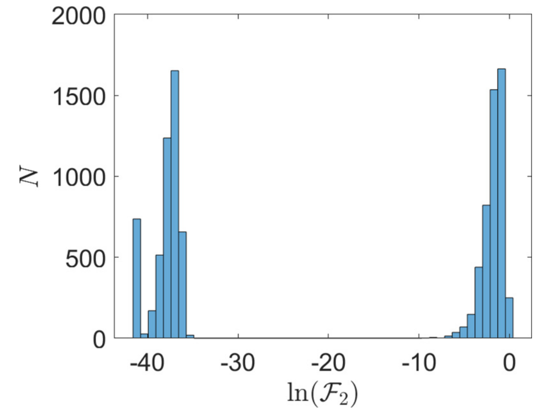

FIG. 5. Numbers of log fidelities, $\ln\mathcal{F}_2$, in a given bin from two orthogonal transformations $O(2M)$ of a pure state. Here we have used $M=3$, $N=10\,000$ random samples of matrices and the number of bins is 50. The bins at $\ln\mathcal{F}_2\approx-40$ correspond to states that are orthogonal up to numerical rounding errors, as they have the opposite parity.

rewrite the above equation as

$$\mathcal{F}_2(\widehat{\rho},\widehat{\sigma}) = \frac{1}{2^M}\sqrt{\det(\underline{\underline{x}}^{(i)}\underline{\underline{y}}^{(j)}-\underline{\underline{I}})}, \tag{5.8}$$

where $\underline{\underline{x}}$, $\underline{\underline{y}}$ correspond to pure states. In the numerical results given below we consider random choices of the two Gaussian pure state correlation matrices, and numerically investigate the distributions of resulting fidelity. These choices are given as $\underline{\underline{x}}=\underline{\underline{O}}^T\underline{\underline{B}}\underline{\underline{O}}$ and $\underline{\underline{y}}=\underline{\underline{\tilde{O}}}^T\underline{\underline{B}}\underline{\underline{\tilde{O}}}$, where $\underline{\underline{O}}$ and $\underline{\underline{\tilde{O}}}$ are random orthogonal matrices chosen with a Haar measure.

Sampling of fidelity between two random pure states generated by means of orthogonal transformation on $\underline{\underline{B}}$ leads to two possibilities, depending on the group transformations that are sampled. In the first two figures, Figs. 5 and 6, the orthogonal matrices are chosen randomly with a Haar measure [73,74]. The plots are of the number of random pairs with

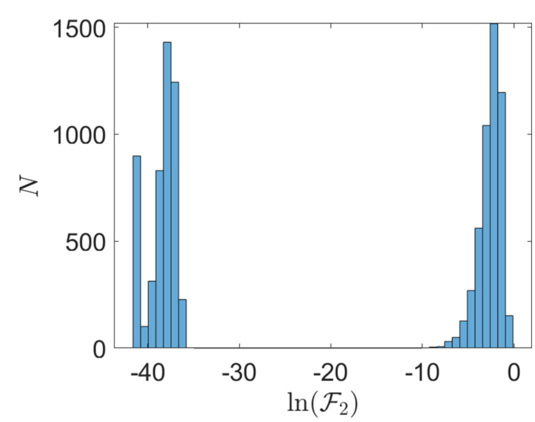

FIG. 6. Numbers of log fidelities, $\ln\mathcal{F}_2$, in a given bin from two orthogonal transformations $O(2M)$ of a pure state. Here we have used $M=4$, $N=10\,000$ random samples of matrices and the number of bins is 50. The bins at $\ln\mathcal{F}_2\approx-40$ correspond to states that are orthogonal up to numerical rounding errors, as they have the opposite parity.

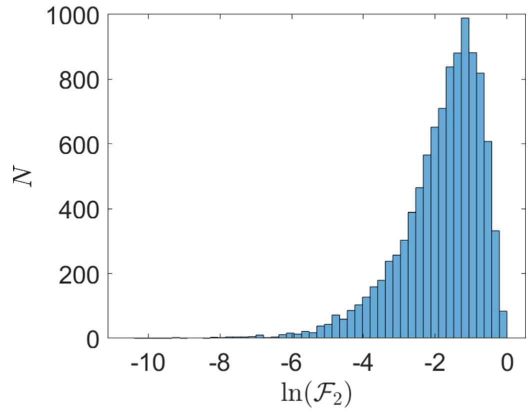

FIG. 7. Numbers of log fidelities, $\ln \mathcal{F}_2$, in a given bin from two orthogonal transformations $SO(2M)$ of a pure state. Here we have used $M = 3$, $N = 10\,000$ random samples of matrices and the number of bins is 50.

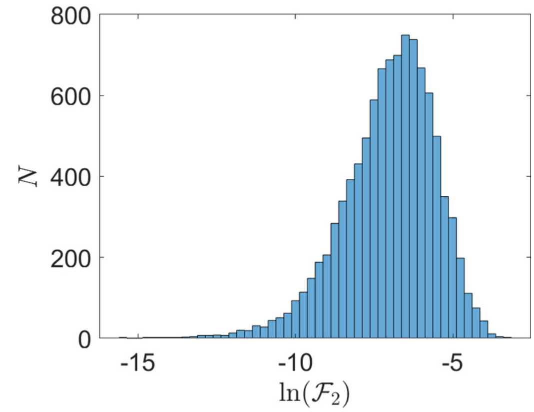

FIG. 9. Numbers of log fidelities, $\ln \mathcal{F}_2$, in a given bin from two orthogonal transformations $SO(2M)$ of a pure state. Here we have used $M = 10$ and $N = 10\,000$ random samples of matrices, and the number of bins is 50.

inner products given in the plotted range of binned fidelities. The logarithm ($\ln \mathcal{F}_2$) is binned to allow better visualization of results where the inner products have small values.

To explain this, we see that if the two states are generated using an $O(2M)$ transformation with $\det O = \pm 1$, then the results corresponding to different modes are as in Figs. 5 and 6, with half of the fidelities being very small (zero apart from numerical errors). If the matrices are restricted to the $SO(2M)$ group with $\det O = 1$, the results are as in Figs. 7–9.

The difference is that in the larger $O(2M)$ group, half of the transformations are parity changing, and lead to inner products that are zero, as expected from our analytic results. Numerically, this leads to a fidelity of order e^{-40} due to rounding errors, as shown in the graphs. In the case of two Gaussian states generated within the same subspace, the average fidelities do not vanish, but they are reduced as the space dimension is increased. As the Hilbert space is exponentially large, the probability that two randomly chosen Gaussian states will overlap is exponentially small.

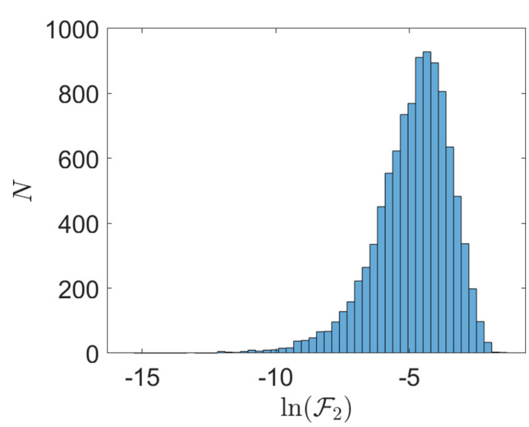

FIG. 8. Numbers of log fidelities, $\ln \mathcal{F}_2$, in a given bin from two orthogonal transformations $SO(2M)$ of a pure state. Here we have used $M = 7$, $N = 10\,000$ random samples of matrices and the number of bins is 50.

VI. SUMMARY

We have given expressions for the information-related quantities of the Renyi entropy, fidelity, and purity for Majorana P representations. The essential component in the calculation of these quantities is the trace of the inner product of two Majorana Gaussian operators. This is given in terms of two antisymmetric matrices. These quantities are useful in quantum information theory, and the Majorana representation has been recently used to develop fidelity witnesses [24]. We have given a relation between the number parity operator and the inner product of the Gaussian operator. This result has been used to explain the fidelity results when considering orthogonal transformations that relate the covariance matrices of the Gaussian operators. We also give numerical results for Renyi entropy and fidelity when considering pure states.

ACKNOWLEDGMENTS

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APPENDIX: INNER PRODUCT OF TWO MAJORANA GAUSSIAN OPERATORS

In this Appendix we give detailed calculations of trace of two Majorana Gaussian operators. Some of these identities have been proved in the Appendix of Ref. [18], for the case of real matrices. To demonstrate that these can also be used for complex matrices, we include the proofs here.

An unnormalized Majorana Gaussian operator has the form,

$$\hat{\Lambda}^u(\underline{\underline{X}}) =: \exp[i\underline{\hat{\gamma}}^T \underline{\underline{X}}\,\underline{\hat{\gamma}}/2] :, \tag{A1}$$

where $\underline{\underline{X}}$ is a $2M$ X $2M$ complex antisymmetric matrix and the relationship connecting $\underline{\underline{x}}$ and $\underline{\underline{X}}$ is given by

$$\underline{\underline{x}} = \underline{\underline{i}} - \underline{\underline{i}}(\underline{\underline{X}} + \underline{\underline{i}})^{-1}\underline{\underline{i}}. \tag{A2}$$

The inner product of two unnormalized Gaussian operators in the Majorana basis is

$$F^u(\underline{\underline{X}}, \underline{\underline{Y}}) = \mathrm{Tr}[\hat{\Lambda}^u(\underline{\underline{X}})\hat{\Lambda}^{u\dagger}(\underline{\underline{Y}})]. \tag{A3}$$

For an M-mode case, $F^u(\underline{\underline{X}}, \underline{\underline{Y}})$ can be expressed as

$$F^u(\underline{\underline{X}}, \underline{\underline{Y}}) = \mathrm{Tr}[: e^{i\underline{\hat{\gamma}}^T \underline{\underline{X}}\underline{\hat{\gamma}}/2} :: e^{i\underline{\hat{\gamma}}^T \underline{\underline{Y}}^*\underline{\hat{\gamma}}/2} :]. \tag{A4}$$

The trace of any fermion operator can be expanded [58] via Grassmann integration utilizing Grassmann numbers $\boldsymbol{\alpha}$ and Grassmann coherent states $|\boldsymbol{\alpha}\rangle$:

$$\mathrm{Tr}[\widehat{O}] = \int d^{2M}\boldsymbol{\alpha}\langle -\boldsymbol{\alpha}|\widehat{O}|\boldsymbol{\alpha}\rangle. \tag{A5}$$

The resolution of the identity has the expression,

$$\int d^{2M}\boldsymbol{\alpha}|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}| = \hat{I}. \tag{A6}$$

Utilizing these relations, $F(\underline{\underline{X}}, \underline{\underline{Y}})$ can be evaluated as

$$F^u(\underline{\underline{X}}, \underline{\underline{Y}}) = \int \langle -\boldsymbol{\alpha}| : e^{i\underline{\hat{\gamma}}^T \underline{\underline{X}}\underline{\hat{\gamma}}/2} : |\boldsymbol{\beta}\rangle \\ *\langle\boldsymbol{\beta}| : e^{i\underline{\hat{\gamma}}^T \underline{\underline{Y}}^*\underline{\hat{\gamma}}/2} : |\boldsymbol{\alpha}\rangle d^{2M}\boldsymbol{\alpha} d^{2M}\boldsymbol{\beta}. \tag{A7}$$

Evaluating the action of Grassmann variables $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ on the Majorana variables, the inner product expression can be rewritten as

$$F^u(\underline{\underline{X}}, \underline{\underline{Y}}) = \int \langle -\boldsymbol{\alpha}| e^{i[-\boldsymbol{\alpha}^\dagger \; \boldsymbol{\beta}]\underline{\underline{U_0}}^{-1}\underline{\underline{X}}\,\underline{\underline{U_0}}[{}^{\boldsymbol{\beta}}_{-\boldsymbol{\alpha}^\dagger}]} |\boldsymbol{\beta}\rangle \\ *\langle\boldsymbol{\beta}| e^{i[\boldsymbol{\beta}^\dagger \; \boldsymbol{\alpha}]\underline{\underline{U_0}}^{-1}\underline{\underline{Y}}^*\underline{\underline{U_0}}[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}^\dagger}]} |\boldsymbol{\alpha}\rangle d^{2M}\boldsymbol{\alpha} d^{2M}\boldsymbol{\beta}. \tag{A8}$$

It is easier to find the inner products in terms of a $2M$ X $2M$ complex antisymmetric matrix $\underline{\underline{\mu}}$, and then transform it to the required form. The relation connecting $\underline{\underline{\mu}}, \underline{\underline{\mu}}'$ and $\underline{\underline{X}}, \underline{\underline{Y}}$ is

$$\underline{\underline{\mu}} = -2i\underline{\underline{U_0}}^{-1}\underline{\underline{X}}\,\underline{\underline{U_0}}, \quad \underline{\underline{\mu}}' = -2i\underline{\underline{U_0}}^{-1}\underline{\underline{Y}}^*\underline{\underline{U_0}}. \tag{A9}$$

In this way, the inner product of two Gaussian operators in terms of the complex matrix $\underline{\underline{\mu}}$ is

$$F^u = \int d^{2M}\boldsymbol{\alpha} d^{2M}\boldsymbol{\beta} e^{-(s_1+s_2+s_3)}, \tag{A10}$$

where $s_1 = \frac{1}{2}[-\boldsymbol{\alpha}^\dagger \quad \boldsymbol{\beta}]\underline{\underline{\mu}}[{}^{\boldsymbol{\beta}}_{-\boldsymbol{\alpha}^\dagger}]$, $s_2 = \frac{1}{2}[\boldsymbol{\beta}^\dagger \quad \boldsymbol{\alpha}]\underline{\underline{\mu}}'[{}^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}^\dagger}]$, and $s_3 = \boldsymbol{\alpha}^\dagger\boldsymbol{\beta} - \boldsymbol{\beta}^\dagger\boldsymbol{\alpha} + \boldsymbol{\beta}^\dagger\boldsymbol{\beta} + \boldsymbol{\alpha}^\dagger\boldsymbol{\alpha}$.

As a next step, to get a simplified expression, we introduce a double dimension complex matrix $\underline{\underline{\Gamma}}$ as

$$\underline{\underline{\Gamma}} = \begin{bmatrix} \underline{\underline{I}} & \underline{\underline{J}} + \underline{\underline{\mu}} \\ \underline{\underline{J}} + \underline{\underline{\mu}}' & -\underline{\underline{I}} \end{bmatrix}. \tag{A11}$$

Utilizing $\underline{\underline{\Gamma}}$, we can write the inner product expression as

$$F^u = \int d^{2M}\underline{\boldsymbol{\eta}} d^{2M}\underline{\boldsymbol{\eta}}' e^{-\frac{1}{2}\underline{\boldsymbol{\eta}}'^\dagger \underline{\underline{\Gamma}}\underline{\boldsymbol{\eta}}}, \tag{A12}$$

where two new Grassmann variables $\underline{\boldsymbol{\eta}}'^\dagger$ and $\underline{\boldsymbol{\eta}}$ are given as

$$\underline{\boldsymbol{\eta}}'^\dagger = [\underline{\boldsymbol{\alpha}}^\dagger \quad \underline{\boldsymbol{\beta}}'^\dagger], \tag{A13}$$

and

$$\underline{\boldsymbol{\eta}} = \begin{bmatrix} \underline{\boldsymbol{\alpha}}' \\ \underline{\boldsymbol{\beta}} \end{bmatrix}. \tag{A14}$$

Also we have $\underline{\boldsymbol{\alpha}} = [{}^{-\boldsymbol{\alpha}}_{\boldsymbol{\beta}^\dagger}]$, $\underline{\boldsymbol{\alpha}}'^\dagger = [-\boldsymbol{\alpha}'^\dagger \quad \boldsymbol{\beta}']$, $\underline{\boldsymbol{\beta}}' = [{}^{\boldsymbol{\beta}'}_{-\boldsymbol{\alpha}'^\dagger}]$, and $\underline{\boldsymbol{\beta}}^\dagger = [\boldsymbol{\beta}^\dagger \quad -\boldsymbol{\alpha}]$. After applying the standard identity [75], it is possible to get a simple expression for F^u as

$$F^u = \sqrt{\det(\underline{\underline{\Gamma}})}. \tag{A15}$$

Expansion of the determinant gives

$$F^u_\mu(\underline{\underline{\mu}}, \underline{\underline{\mu}}') = \sqrt{\det[\underline{\underline{I}}^2 + (\underline{\underline{J}} + \underline{\underline{\mu}})(\underline{\underline{J}} + \underline{\underline{\mu}}')]}, \tag{A16}$$

provided

$$\underline{\underline{J}} = \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{A17}$$

The covariance matrix $\underline{\underline{\sigma}}$ is more general since it is used in the definition of the normalized fermionic Gaussian operator. It will be easier to derive the required trace for normalized Majorana Gaussian operator when a relation for the trace of the products of the normalized fermionic Gaussian operator is obtained. The complex antisymmetric matrix $\underline{\underline{\sigma}}$ is related to $\underline{\underline{\mu}}$ as

$$\underline{\underline{\sigma}} = [\underline{\underline{\mu}} + 2\underline{\underline{J}}]^{-1}, \quad \underline{\underline{\sigma}}' = [\underline{\underline{\mu}}' + 2\underline{\underline{J}}]^{-1}. \tag{A18}$$

Modifying the expression in Eq. (A16) after incorporating the normalization factor corresponding to the fermionic Gaussian operator $\sqrt{\det[i\underline{\underline{\sigma}}]}$ and utilizing Eq. (A18), it is possible to write Eq. (A16) as

$$F = \sqrt{\det(i\underline{\underline{\sigma}})(i\underline{\underline{\sigma}}')[\underline{\underline{I}}^2 + (\underline{\underline{\sigma}}^{-1} - \underline{\underline{J}})(\underline{\underline{\sigma}}'^{-1} - \underline{\underline{J}})]}. \tag{A19}$$

Expanding the terms and simplifying further using $\underline{\underline{\sigma}}\,\underline{\underline{\sigma}}^{-1} = \underline{\underline{J}}^2 = \underline{\underline{I}}$, we obtain a simplified relation,

$$F_\sigma(\underline{\underline{\sigma}}, \underline{\underline{\sigma}}') = \sqrt{\det[\underline{\underline{\sigma}}\,\underline{\underline{\sigma}}' + (\underline{\underline{J}} - \underline{\underline{\sigma}})(\underline{\underline{J}} - \underline{\underline{\sigma}}')]}. \tag{A20}$$

At this stage we utilize another transformation that converts the $\underline{\underline{\sigma}}$ matrices to the $\underline{\underline{x}}$ matrices [19]. This is given by

$$\underline{\underline{\sigma}} = \frac{-i\underline{\underline{U_0}}^{-1}(\underline{\underline{i}}\underline{\underline{x}}\underline{\underline{i}} + \underline{\underline{i}})\underline{\underline{U_0}}}{2}, \quad \underline{\underline{\sigma}}' = \frac{-i\underline{\underline{U_0}}^{-1}(\underline{\underline{i}}\underline{\underline{y}}^*\underline{\underline{i}} + \underline{\underline{i}})\underline{\underline{U_0}}}{2}. \tag{A21}$$

Applying the above relations to Eq. (A20) and making use of the property, $\underline{\underline{U_0}}^{-1}\underline{\underline{U_0}} = \underline{\underline{I}}$, the expansion $\underline{\underline{J}} = -i\underline{\underline{U_0}}^{-1}\underline{\underline{i}}\,\underline{\underline{U_0}}$, and the relation $\underline{\underline{i}}\,\underline{\underline{i}} = -\underline{\underline{I}}$, it is possible to write the inner product in terms of the antisymmetric matrices $\underline{\underline{x}}, \underline{\underline{y}}$ as

$$F(\underline{\underline{x}}, \underline{\underline{y}}) = \mathrm{Tr}[\hat{\Lambda}(\underline{\underline{x}})\hat{\Lambda}^\dagger(\underline{\underline{y}})] \\ = 2^{-M}\sqrt{\det[\underline{\underline{I}} - \underline{\underline{x}}\underline{\underline{y}}^*]}. \tag{A22}$$

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