Faithful measure of quantum non-Gaussianity via quantum relative entropy

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We introduce a measure of quantum non-Gaussianity (QNG) for those quantum states not accessible by a mixture of Gaussian states in terms of quantum relative entropy. Specifically, we employ a convex-roof extension using all possible mixed-state decompositions beyond the usual pure-state decompositions. We prove that this approach brings a QNG measure fulfilling the properties desired as a proper monotone under Gaussian channels and conditional Gaussian operations. As an illustration, we explicitly calculate QNG for the noisy single-photon states and demonstrate that QNG coincides with non-Gaussianity of the state itself when the single-photon fraction is sufficiently large.

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I. INTRODUCTION

Quantum mechanics provides a profound basis for many distinguished information processing protocols which cannot be achieved in the classical world, such as quantum computation [1], quantum teleportation [2], and quantum cryptography [3]. Those quantum protocols have been developed also using continuous variables (CVs) that can be usually described in terms of quasiprobability distributions like the Glauber-Sudarshan P function or the Wigner function in phase space [4,5]. A wide range of states like the coherent and squeezed states are categorized as the so-called Gaussian states whose quasiprobability distributions take a Gaussian form and whose statistical properties are completely characterized by their first-order moments (amplitudes) and second-order moments (covariances). Gaussian states and Gaussian operations are widely employed in many CV protocols due to their experimental feasibility in the laboratory with their compact mathematical description [6]. Nevertheless, there exist numerous no-go theorems within the Gaussian regime, which prevent Gaussian operations from performing important tasks such as universal quantum computation [7,8], quantum error correction [9], and entanglement distillation [10–12], also addressed recently in the framework of Gaussian resource theories [13]. In such tasks, non-Gaussian states and non-Gaussian operations become essential resources.

In this respect, it is of crucial importance to identify quantum non-Gaussian states that cannot be produced by Gaussian resources and their statistical mixtures. Furthermore, it may provide a valuable framework and novel insight into related studies to characterize quantum non-Gaussianity (QNG) under a proper quantitative measure. In a closely related context, several studies have investigated to quantify non-Gaussianity (NG) of quantum states [14–16], which only represents the departure of a given state from Gaussian states. In particular, it was shown that the relative entropy of NG exhibits important properties, for example, monotonicity under Gaussian channels [17]. However, the measure is not convex because the set of Gaussian states is not convex. There indeed exist non-Gaussian states which can be simply generated using Gaussian operations and classical randomness, for example, a mixture of two different coherent states $(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|)/2$. These states, a simple mixture of Gaussian states, can be generated without quantum non-Gaussian operations and they are thus not suitable to perform quantum information tasks, which require genuinely quantum non-Gaussian resources.

Recently, some works have been devoted to ruling out Gaussian mixtures and detecting genuinely quantum non-Gaussian states, i.e., $\rho \neq \sum_i p_i \rho_{G,i}$, where each component state $\rho_{G,i}$ is a Gaussian state. Though a number of criteria have been developed to assess quantum non-Gaussian states [18–27], a faithful measure of quantum non-Gaussianity has not been reported yet. Recent studies in Refs. [28,29] have remarkably adopted the Wigner negativity as a measure of QNG, which is a monotone under Gaussian protocols including classical mixing. However, it is actually not a faithful measure because it cannot detect quantum non-Gaussian states with a positive Wigner function, e.g., a highly noisy single-photon state $p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ with $p > 0.5$. A recent work by Takagi *et al.* suggests that every resource state can generally provide an operational advantage in view of subchannel discrimination even including quantum non-Gaussian states with positive Wigner functions [30]. Therefore, it seems necessary to come up with a QNG measure that can broadly and faithfully assess quantum non-Gaussian states.

In this work we propose a convex-roof measure of QNG based on quantum relative entropy. Our QNG measure is faithful because it always gives a positive value whenever a state cannot be described as a Gaussian mixture. We prove that our measure satisfies properties as a proper measure of QNG including convexity, additivity, and monotonicity under

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Gaussian channels and conditional Gaussian operations. Furthermore, we illustrate how to explicitly evaluate QNG for a noisy single-photon state. We find that its QNG coincides with its NG if the single-photon fraction is large enough.

II. QUANTUM NG MEASURE VIA RELATIVE ENTROPY AND ITS PROPERTIES

A. Non-Gaussianity

We first start with the notion of non-Gaussianity. For a given mixed state ρ, one may define its NG in terms of quantum relative entropy with reference to its Gaussified state ρ_G having the same first-order moments (average) and second-order moments (covariance) [15], that is, $\mathcal{N}[\rho] \equiv S(\rho||\rho_G)$, where $S(\rho||\sigma) \equiv -\text{Tr}\{\rho \ln \sigma\} + \text{Tr}\{\rho \ln \rho\}$ is quantum relative entropy. In particular, due to $-\text{Tr}\{\rho \ln \rho_G\} = -\text{Tr}\{\rho_G \ln \rho_G\}$, we have the relation $S(\rho||\rho_G) = S(\rho_G) - S(\rho)$, which highlights the fact that a Gaussian state among all states with the same covariance matrix possesses a maximal entropy leading to the non-negativity of the defined NG [31].

B. Quantum non-Gaussianity

We are here interested in quantum non-Gaussianity of states, which cannot be represented by a mixture of Gaussian states, namely, $\rho \neq \sum_i p_i \rho_G^i$. There can be several approaches to quantify the degree of QNG and we use the convex-roof extension of NG defined above. That is, for a given state ρ, its QNG can be measured as

$$Q[\rho] \equiv \min_{\{p_i,\rho_i\}} \sum_i p_i S(\rho_i||\rho_{i,G}), \tag{1}$$

where the minimization is taken over all possible decompositions of $\rho = \sum_i p_i \rho_i$. Note that this generalization includes the usual decomposition into pure states only, $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$, e.g., in [29]. By further allowing decompositions into mixed states, we may obtain a lower degree of QNG for a given state. We will illustrate it later by pointing out a range of noisy single-photon states whose QNG is given by a genuinely mixed-state decomposition.

We prove the following properties of the above-defined QNG.

Property 1. Quantum NG is non-negative.

This is obvious by its definition, as the relative entropies, and thus their average, are non-negative.

Property 2 (faithfulness). Quantum NG is strictly positive if and only if the state is not a mixture of Gaussian states.

This can also be readily seen. If $\rho = \sum_i p_i \rho_{G,i}$, its QNG is then zero due to the decomposition with Gaussian component states only. On the other hand, if the QNG is zero, it also means that the given state is a mixture of Gaussian states since any single non-Gaussian component state, if any, would give a strictly positive NG, leading to a positive QNG.

Property 3 (convexity). Quantum NG is convex with respect to state mixing, i.e., $Q[\lambda\rho_1 + (1-\lambda)\rho_2] \leqslant \lambda Q[\rho_1] + (1-\lambda)Q[\rho_2]$.

Proof. Let $\rho_1 = \sum_i p_i \rho_i$ and $\rho_2 = \sum_j q_j \sigma_j$ be the decompositions for their respective QNGs. Since $\sum_i \lambda p_i \rho_i + \sum_j (1-\lambda) q_j \sigma_j$ is one possible decomposition of the state $\lambda\rho_1 + (1-\lambda)\rho_2$, we have by definition

$$\begin{aligned} Q[\lambda\rho_1 + (1-\lambda)\rho_2] &\leqslant \sum_i \lambda p_i S(\rho_i||\rho_{i,G}) + \sum_j (1-\lambda) q_j S(\sigma_j||\sigma_{j,G}) \\ &= \lambda Q[\rho_1] + (1-\lambda) Q[\rho_2]. \end{aligned} \tag{2}$$

Property 4. Quantum NG is invariant under Gaussian unitary operations.

Proof. For any fixed decomposition $\rho = \sum_i p_i \rho_i$, a Gaussian unitary operation leads to $\rho' = U_G \rho U_G^\dagger = \sum_i p_i U_G \rho_i U_G^\dagger$. We also note that the relative entropy of each component NG is invariant under unitary operation, $S(\rho_i||\rho_{i,G}) = S(U\rho_i U^\dagger||U\rho_{i,G}U^\dagger)$, and that the Gaussification of state commutes with Gaussian unitary operations. The latter property means that $U_G \rho_{i,G} U_G^\dagger$ is the Gaussified state of $\rho' = U_G \rho_i U_G^\dagger$. Therefore, $\sum_i p_i S(\rho_i||\rho_{i,G})$ is invariant under Gaussian unitary operations and so is QNG.

Property 5. Quantum NG is not increasing under Gaussian channels.

Proof. We have

$$\begin{aligned} Q[\rho] &= \min \sum_i p_i S(\rho_i||\rho_{i,G}) \\ &\geqslant \sum_i p_i S[\mathcal{E}_G(\rho_i)||\mathcal{E}_G(\rho_{i,G})] \geqslant Q[\mathcal{E}_G(\rho)], \end{aligned} \tag{3}$$

where the first inequality is due to the contraction property of relative entropy under an arbitrary quantum channel. Note again that $\mathcal{E}_G(\rho_{i,G})$ is equivalent to the Gaussified state of $\mathcal{E}_G(\rho_i)$ and that $\sum_i p_i \mathcal{E}_G(\rho_i)$ is one of possible decompositions of $\mathcal{E}_G(\rho)$, which leads to the second inequality in Eq. (2).

Property 6. Quantum NG is not increasing on average under conditional Gaussian maps.

For its proof, we first introduce two preliminary tools.

Preliminary 1. Takagi and Zhuang in [28] have identified a general conditional Gaussian map as the one attaching an ancillary (multimode) vacuum to the system followed by a global unitary Gaussian operation and homodyne detection. The conditional map results from implementing a Gaussian map conditioned on the measurement outcome. That is, with $\rho_{SE} = U_G|0\rangle\langle 0| \otimes \rho_s U_G^\dagger$, we obtain $\rho' = \sum_k |k\rangle\langle k| \otimes \rho_k = \sum_k p_k |k\rangle\langle k| \otimes \tilde{\rho}_k$, where $\rho_k = \langle k|\rho_{SE}|k\rangle$ is an unnormalized state conditioned on the homodyne outcome k with $p_k = \text{Tr}\rho_k$. The final conditional map reads $\rho'' = \sum_k p_k |k\rangle\langle k| \otimes \mathcal{E}_G^k(\tilde{\rho}_k)$.

Preliminary 2. For two mixed states $\rho = \sum_j p_j^{(1)} |j\rangle\langle j| \otimes \rho_j$ and $\sigma = \sum_j p_j^{(2)} |j\rangle\langle j| \otimes \sigma_j$ where $|j\rangle$'s are orthonormal states for subsystem A, the relative entropy $S(\rho||\sigma)$ turns out to be

$$S(\rho||\sigma) = H(p^{(1)}||p^{(2)}) + \sum_j p_j^{(1)} S(\rho_j||\sigma_j), \tag{4}$$

where H is the Shannon relative entropy. Using these properties, we have the following proof.

Proof. We first note that the QNG of $\rho_{SE} = U_G|0\rangle\langle 0| \otimes \rho_s U_G^\dagger$ is the same as that of ρ_s since adding either an ancillary Gaussian state or a unitary Gaussian operation does not change QNG. Let $\rho_{SE} = \sum_i p_i \rho_i$ be the decomposition yielding its QNG, i.e., $Q[\rho_s] = Q[\rho_{SE}] = \sum_i p_i S(\rho_i||\rho_{i,G})$, where ρ_i belongs to a larger Hilbert space of $\{SE\}$.

We may introduce a further extended state of ρ_{SE} as $\rho_{SEE'} = \sum_i p_i|i\rangle\langle i| \otimes \rho_i$, where $\rho_{SE} = \mathrm{Tr}_{E'}\{\rho_{SEE'}\}$ and $|i\rangle$'s are orthonormal basis states for E'. With its Gaussified version $\sigma_{SEE'} = \sum_i p_i|i\rangle\langle i| \otimes \rho_{i,G}$, we have $Q[\rho_{SE}] = S(\rho_{SEE'}||\sigma_{SEE'})$ due to Preliminary 2, which is expressed in terms of the relative entropy of the total states without decompositions.

Let us now take a homodyne measurement with basis $|k\rangle$ on subsystem E for the two states $\rho_{SEE'}$ and $\sigma_{SEE'}$. We then obtain

$$\begin{aligned}\rho_{SEE'} \rightarrow \rho'_{SEE'} &= \sum_{i,k} p_i|i\rangle\langle i| \otimes |k\rangle\langle k| \otimes \langle k|\rho_i|k\rangle \\ &= \sum_{i,k} p_i p_{k|i}|i\rangle\langle i| \otimes |k\rangle\langle k| \otimes \tilde{\rho}_{k|i},\end{aligned} \tag{5}$$

where $\tilde{\rho}_{k|i}$ is the normalized state obtained on the measurement outcome k starting with the state ρ_i and $p_{k|i} = \mathrm{Tr}\langle k|\rho_i|k\rangle$ is the corresponding conditional probability. The product $p_i p_{k|i} \equiv p_{ik}$ defines a joint probability as such. Similarly, we obtain the state after measurement for $\sigma_{SEE'}$; however, the conditional probability $p^G_{k|i} = \mathrm{Tr}\langle k|\rho_{i,G}|k\rangle$ is not necessarily the same as $p_{k|i} = \mathrm{Tr}\langle k|\rho_i|k\rangle$. Nevertheless, with

$$\begin{aligned}\sigma_{SEE'} \rightarrow \sigma'_{SEE'} &= \sum_{i,k} p_i|i\rangle\langle i| \otimes |k\rangle\langle k| \otimes \langle k|\rho_{i,G}|k\rangle \\ &= \sum_{i,k} p_i p^G_{k|i}|i\rangle\langle i| \otimes |k\rangle\langle k| \otimes \tilde{\rho}_{k|i,G},\end{aligned} \tag{6}$$

and since a measurement on a partial system is a completely positive map (its action is actually to eliminate all off-diagonal elements in the subsystem), we have $S(\rho_{SEE'}||\sigma_{SEE'}) \geqslant S(\rho'_{SEE'}||\sigma'_{SEE'})$. Using Preliminary 2 again, the latter quantity is given by $H(p_{ik}||p^G_{ik}) + \sum_{i,k} p_{ik} S(\tilde{\rho}_{k|i}||\tilde{\rho}_{k|i,G}) \geqslant \sum_k p_k S_k$, where $H \geqslant 0$ is used. We have here defined the marginal probability $p_k = \sum_i p_{ik}$ and $S_k = \frac{1}{p_k}\sum_i p_{ik} S(\tilde{\rho}_{k|i}||\tilde{\rho}_{k|i,G})$.

Noting that $\rho_k = \frac{1}{p_k}\sum_i p_{ik}\tilde{\rho}_{k|i}$ is the state of system conditioned on the measurement outcome k on E, we have therefore proved $Q[\rho] = Q[\rho_{SE}] = S(\rho_{SEE'}||\sigma_{SEE'}) \geqslant S(\rho'_{SEE'}||\sigma'_{SEE'}) \geqslant \sum_k p_k S_k \geqslant \sum_k p_k Q[\rho_k] \geqslant \sum_k p_k Q[\mathcal{E}^k_G(\rho_k)]$.

III. CASE OF NOISY SINGLE-PHOTON STATES

In the preceding section we have demonstrated that our entropic QNG measure fulfills desirable properties as a proper measure of quantum non-Gaussianity. Operationally, we may interpret our measure as quantifying the minimum required non-Gaussian resources to prepare a given quantum non-Gaussian state. We have specifically introduced the convex-roof extension adopting mixed-state decompositions beyond the usual pure-state decompositions to define the degree of QNG. One may then be interested in knowing if there exist quantum non-Gaussian states whose QNG is given strictly by a mixed-state decomposition and not by a pure-state decomposition. We illustrate it by an example of noisy single-photon states with the explicit calculation of their QNG based on our approach. Before that, we remark on the case of pure non-Gaussian states.

A. Pure states

If the state is pure, $\rho = |\Psi\rangle\langle\Psi|$, the state itself is the only possible decomposition of it. Therefore, its QNG coincides with its NG, $Q[\rho] = \mathcal{N}[\rho]$.

B. Noisy single-photon state

We now consider the case of mixed states. Specifically, we obtain the QNG of a noisy single-photon state, i.e., $p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0|$, as follows.

(i) To begin with, we obtain the non-Gaussianity, not QNG yet, of a noisy single-photon state in a general form of $\rho = p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0| + re^{i\theta}|0\rangle\langle 1| + re^{-i\theta}|1\rangle\langle 0|$. For this state, we have $\langle\hat{a}\rangle = \langle\hat{a}^\dagger\rangle^* = re^{-i\theta}$, $\langle\hat{a}^2\rangle = \langle(\hat{a}^\dagger)^2\rangle = 0$, and $\langle\hat{a}\hat{a}^\dagger\rangle = \langle\hat{a}^\dagger\hat{a}\rangle + 1 = p + 1$, which yield $\langle\hat{q}\rangle = \sqrt{2}r\cos\theta$, $\langle\hat{p}\rangle = -\sqrt{2}r\sin\theta$, $\langle\hat{q}^2\rangle = \langle\hat{p}^2\rangle = \frac{1}{2} + p$, and $\langle\hat{q}\hat{p} + \hat{p}\hat{q}\rangle = 0$, where $\hat{q} = \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}$ and $\hat{p} = \frac{\hat{a}-\hat{a}^\dagger}{\sqrt{2}i}$ are two orthogonal quadrature amplitudes. The covariance matrix of ρ is then given by

$$\Gamma = \begin{pmatrix} \frac{1}{2} + p - 2r^2\cos^2\theta & 2r^2\sin\theta\cos\theta \\ 2r^2\sin\theta\cos\theta & \frac{1}{2} + p - 2r^2\sin^2\theta \end{pmatrix}, \tag{7}$$

where the covariance matrix elements are defined as $\Gamma_{ij} = \frac{1}{2}\langle\hat{x}_i\hat{x}_j + \hat{x}_j\hat{x}_i\rangle - \langle\hat{x}_i\rangle\langle\hat{x}_j\rangle$, with $\hat{x}_1 = \hat{q}$ and $\hat{x}_2 = \hat{p}$. It determines the quantum entropy of the reference Gaussian state ρ_G as

$$S(\rho_G) = (\bar{n}_{\mathrm{th}} + 1)\ln(\bar{n}_{\mathrm{th}} + 1) - \bar{n}_{\mathrm{th}}\ln\bar{n}_{\mathrm{th}}, \tag{8}$$

where

$$\bar{n}_{\mathrm{th}} = \sqrt{\det\Gamma} - \tfrac{1}{2} = \sqrt{\left(\tfrac{1}{2} + p\right)\left(\tfrac{1}{2} + p - 2r^2\right)} - \tfrac{1}{2}. \tag{9}$$

The non-Gaussianity ρ is thus given by

$$\begin{aligned}\mathcal{N}[\rho] &= S(\rho_G) - S(\rho) \\ &= (\bar{n}_{\mathrm{th}} + 1)\ln(\bar{n}_{\mathrm{th}} + 1) - \bar{n}_{\mathrm{th}}\ln\bar{n}_{\mathrm{th}} \\ &\quad + \lambda_+\ln\lambda_+ + \lambda_-\ln\lambda_-,\end{aligned} \tag{10}$$

where $\lambda_\pm = \frac{1}{2} \pm \sqrt{(\frac{1}{2} - p)^2 + r^2}$ are the eigenvalues of ρ. Note that the NG of the state ρ is independent of the phase θ, which is indeed due to the invariance property under Gaussian unitary operations, particularly phase rotation in this case, i.e., $\mathcal{N}[\rho] = \mathcal{N}[e^{i\hat{n}\theta}\rho e^{-i\hat{n}\theta}]$.

(ii) From the non-Gaussianity in Eq. (10), we may find the minimum of NG$_\rho$ among all states for a fixed p as

$$\mathcal{M}(p) \equiv \min_r \mathcal{N}[\rho], \tag{11}$$

which can be obtained by solving

$$\begin{aligned}\frac{d}{dr}\mathcal{N}[\rho] &= 4r\left\{\frac{\tanh^{-1}(2\lambda_+ - 1)}{2\lambda_+ - 1} - \frac{1+2p}{2\bar{n}_{\mathrm{th}}}\tanh^{-1}\frac{1}{2\bar{n}_{\mathrm{th}}}\right\} \\ &= 0\end{aligned} \tag{12}$$

and comparing the extremal values. We plot the minimum $\mathcal{M}(p)$ and the corresponding optimal parameter r_{opt} as a function of p in Fig. 1. The minimum NG is given by a partially mixed state $[0 < r_{\mathrm{opt}} < \sqrt{p(1-p)}]$ and a maximally mixed state ($r_{\mathrm{opt}} = 0$) for $p \lesssim 0.062$ and $p \gtrsim 0.062$, respectively.

(iii) Using the above result, we obtain the QNG of $\rho_p = p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0|$ as follows. Given a

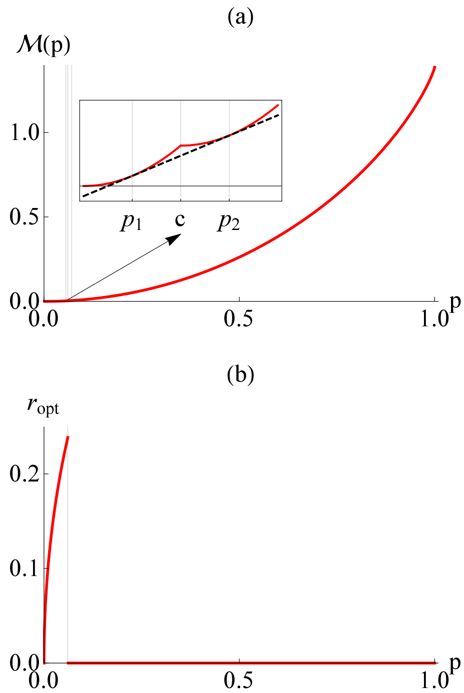

FIG. 1. (a) Minimum $\mathcal{M}(p)$ defined in Eq. (11) as a function of the single-photon fraction p and (b) corresponding optimal parameter r_{opt} for minimum $\mathcal{M}(p)$.

state ρ_p, our task is to find a decomposition yielding $Q[\rho_p] = \min_{\{f_k,\rho_k\}} \sum_k f_k \mathcal{N}[\rho_k]$ among all decompositions $\rho_p = \sum_k f_k \rho_k$. In particular, we let ρ_k be the state with single-photon fraction p_k thus satisfying the constraint $p = \sum_k f_k p_k$. The idea of optimization here is to find values p_k with the constraint $p = \sum_k f_k p_k$ to have a minimum $\sum_k f_k \mathcal{M}[p_k]$, where $\mathcal{M}[p]$ is the function whose values are shown in Fig. 1.

This optimization actually corresponds to the lower convex envelope of $\mathcal{M}(p)$ defined by

$$\breve{\mathcal{M}}(p) \equiv \sup\{f(p) \mid f \text{ is convex and } f \leqslant \mathcal{M} \text{ in } [0,1]\}, \quad (13)$$

which is obtained as follows. Investigating $\mathcal{M}''(p)$, we find that $\mathcal{M}(p)$ itself is convex on the two intervals $[0, c]$ and $[c, 1]$ individually with $c \simeq 0.062$, but not in the whole interval [red solid curve in the inset of Fig. 1(a)]. Then we may construct the lower convex envelope by finding a line tangent to $\mathcal{M}(p)$ in both intervals [black dashed line in the inset of Fig. 1(a)]. If there exists a solution to the equation

$$\mathcal{M}'(p_1)(p_2 - p_1) + \mathcal{M}(p_1) = \mathcal{M}(p_2), \quad (14)$$

together with the condition $\mathcal{M}'(p_1) = \mathcal{M}'(p_2)$, the line is tangent to $\mathcal{M}(p)$ in both intervals. Indeed, we find the solution $p_1 \simeq 0.0559$ and $p_2 \simeq 0.0701$, respectively. Therefore, we obtain the QNG of $\rho_p = p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0|$ as

$$Q[\rho_p] = \begin{cases} \mathcal{M}(p) & \text{for } 0 \leqslant p \leqslant p_1 \\ \frac{p-p_1}{p_2-p_1}\mathcal{M}(p_2) + \frac{p_2-p}{p_2-p_1}\mathcal{M}(p_1) & \text{for } p_1 \leqslant p \leqslant p_2 \\ \mathcal{M}(p) & \text{for } p_2 \leqslant p \leqslant 1, \end{cases} \quad (15)$$

where $p_1 \simeq 0.0559$ and $p_2 \simeq 0.0701$.

From the above analysis, we can also readily identify an optimal decomposition of ρ_p. For $p \geqslant p_2$ we have $\breve{\mathcal{M}}(p) = \mathcal{N}[\rho]$, which means that the state ρ_p itself is the optimal decomposition attaining minimum convex-roof QNG. This is a clear example for which the mixed-state decomposition becomes optimal rather than the pure-state decomposition. For $p \leqslant p_1$, the equal mixture of two optimal states $\rho_\pm^p = p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0| \pm r_{opt}(|0\rangle\langle 1| + |1\rangle\langle 0|)$ achieves the bound. For the remaining case, i.e., $p_1 \leqslant p \leqslant p_2$, the optimal decomposition becomes $\{\rho_+^{p_1}, \rho_-^{p_1}, p_2|1\rangle\langle 1| + (1-p_2)|0\rangle\langle 0|\}$ with the probability distribution $\left\{\frac{1}{2}\frac{p_2-p}{p_2-p_1}, \frac{1}{2}\frac{p_2-p}{p_2-p_1}, \frac{p-p_1}{p_2-p_1}\right\}$.

IV. DISCUSSION

We have proposed a faithful measure of quantum non-Gaussianity adopting quantum relative entropy. Specifically, we have introduced a convex-roof extension of non-Gaussianity using all possible mixed-state decompositions beyond the typical pure-state decompositions. This enabled us to come up with properties desired as a proper measure of QNG including convexity and monotonicity under Gaussian channels and conditional Gaussian operations. Our measure is faithful in that it strictly gives a positive value for an arbitrary quantum non-Gaussian state that cannot be represented as a mixture of Gaussian states.

As an illustration, we studied the case of a noisy single-photon state, which is a practically important QNG resource for many applications like linear-optical quantum computation [32]. We have shown the procedures to identify its QNG rigorously, which may be extended to quantum non-Gaussian states with higher photon numbers. By doing so, we have clearly illustrated that there exists a range of quantum states for which QNG is given by a mixed-state decomposition, not a pure-state one. Moreover, it turns out that the QNG actually coincides with NG if the single-photon fraction is sufficiently large.

Our measure of QNG may be interpreted as quantifying the minimum required non-Gaussian resource to produce a given quantum non-Gaussian state. Namely, it addresses a way of preparing different non-Gaussian states with a proper probability distribution such that the average of non-Gaussianity of each state becomes minimal to constitute the quantum non-Gaussian state under investigation. While this measure has its own merit, a more comprehensive study is still needed concerning the characterization of QNG in a full variety of physical contexts. There have been several investigations demonstrating the usefulness of non-Gaussian states and operations, e.g., the improvement of quantum entanglement [33–39] and enhancement of performance in quantum teleportation and dense coding [40–43]. However, there were only a few studies to comprehensively and critically identify the role of QNG in CV quantum information processing beyond the level of case studies [44]. For instance, it is an interesting question whether an arbitrary quantum non-Gaussian state, even though it possesses a positive-definite Wigner function, can be a critically useful resource to provide an advantage for practical quantum tasks. If so, what sort of QNG measure would appropriately address such criticality in a rigorous way? These and other related issues are left for future investigation.

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