Dynamical model for positive-operator-valued measures

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We tackle the dynamical description of the quantum measurement process by explicitly addressing the interaction between the system under investigation and the measurement apparatus, the latter ultimately considered as a macroscopic quantum object. We consider arbitrary positive-operator-valued measures (POVMs) such that the orthogonality constraint on the measurement operators is relaxed. We show that, as with the well-known von Neumann scheme for projective measurements, it is possible to build up a dynamical model holding a unitary propagator characterized by a single time-independent Hamiltonian. This is achieved by modifying the standard model so as to compensate for the possible lack of orthogonality among the measurement operators of arbitrary POVMs.

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I. INTRODUCTION

A distinctive trademark of quantum mechanics is represented by the quantum measurements and by the randomness of their outcomes. The postulates of the theory dictate how to compute the associated statistics for quantum observables through projective measures, while no mechanism is provided to predict how the actual finally observed result comes about. In this respect, the measurement process still represents an open field of research [1–8]. Actually, from the dawn of quantum theory, two main steps toward complementary directions have been performed. On the one hand, a clear description of the measurement process entailing the definition of a time-independent interaction Hamiltonian between the system and an ultimately macroscopic apparatus has been provided by von Neumann [9], and fully characterized by Ozawa [10] several years later. On the other hand, the statistical description of quantum measurements has been extended to non-necessarily orthonormal measurement operators by the introduction of the so-called positive-operator-valued measures (POVMs) [1–3].

In this paper, we unify these two approaches, introducing a dynamical description of arbitrary quantum measurements. We show that, in order to achieve a well-defined—i.e., completely positive trace preserving (CPT) [1,11,12]—dynamical map, the lack of orthogonality of arbitrary measurement operators needs to be compensated by properly modifying the von Neumann–Ozawa (vN-O), time-independent Hamiltonian representation. In our analysis, we rely on the Naimark extension theorem [13–16], which allows one to describe an arbitrary POVM performed on the system of interest, in terms of a projective measurement performed on an external probing system that was properly coupled with the latter. This provides a proper generalization of the von Neumann model to arbitrary measurements. We recall that addressing the actual dynamics behind the formal description of a quantum measurement not only helps us to understand fundamental aspects of the process, but it also gives a relevant indication about the actual design of quantum-measurement experiments (see, e.g., Refs. [17–19]).

The paper is structured as follows: as a premise in Sec. II we introduce the notation and review a few basic notions regarding POVMs and the vN-O construction for projective measurements. Section III contains the original part of the work. Here we rigorously define the problem we wish to address and present a solution for it; the fundamental element of our analysis is the explicit construction of a Naimark Hamiltonian discussed in Secs. III A and III B. The conclusion and final remarks are given in Sec. IV, while technical considerations are presented in the Appendixes.

II. QUANTUM MEASUREMENTS

The minimal description of a quantum measurement requires two elements: a set of n_Γ distinguishable outcomes, $\{\mu_\gamma; \gamma = 1, \dots, n_\Gamma\}$, and the corresponding probability distribution $\{p_\gamma\}$. Herewith, without loss of generality, we will exclusively consider countable sets of outputs and hence discrete distributions. This process involves at least two players interacting with each other: the system S, upon which the measurement is performed, and the apparatus Ξ, from which one actually obtains the outcomes. Let $\mathcal{H}_s$ be the Hilbert space of S. Formally, a quantum measure on a state $\rho_s^{\rm in}$ is defined by a bijection from $\{\mu_\gamma\}$ into the set of positive operators $\{F_s^{(\gamma)}\}$ on $\mathcal{H}_s$, called *elements of the measure* or *effects*, such that $p_\gamma = {\rm Tr}[\rho_s^{\rm in} F_s^{(\gamma)}]$, $\forall\gamma$. For $\sum_\gamma p_\gamma = 1$ to hold, it must be $\sum_\gamma F_s^{(\gamma)} = \mathbb{I}_s$. As a process on S, a single measurement acts on an input state $\rho_s^{\rm in}$, upon which we want to gain some information, and produces one output $\mu_{\bar\gamma}$ with probability $p_{\bar\gamma}$, as defined above. After the interaction with the apparatus Ξ and before the production of the outcomes, the system is described by the so-called *postmeasurement* state $\rho_S^{\rm out}$ of S, defined as

$$\rho_S^{\rm out} = \sum_{\gamma=1}^{n_\Gamma} M_S^{(\gamma)} \rho_S^{\rm in} M_S^{(\gamma)\dagger} = \sum_{\gamma=1}^{n_\Gamma} p_\gamma \rho_S^{(\gamma)}, \tag{1}$$

$$\rho_S^{(\gamma)} = \frac{1}{p_\gamma} M_S^{(\gamma)} \rho_S^{\rm in} M_S^{(\gamma)\dagger}, \quad p_\gamma = {\rm Tr}\big[M_S^{(\gamma)} \rho_S^{\rm in} M_S^{(\gamma)\dagger}\big]. \tag{2}$$

Herewith we focus only on data acquisition operations, without considering any additional dynamical evolution that could be involved during the whole measurement process. In the expressions above, the operators $M_S^{(\gamma)}$, dubbed as *measurement* or *detection* operators, are defined by $F_S^{(\gamma)} = M_S^{(\gamma)\dagger} M_S^{(\gamma)}$, a decomposition always allowed due to the positiveness of $F_S^{(\gamma)}$. Actually, the relation between the elements of the POVM and the measurement operators is not univocal. Indeed, the former equation admits an infinite amount of solutions that can be obtained multiplying $M_S^{(\gamma)}$ by an arbitrary unitary transformation $W_S^{(\gamma)}$, i.e., for all γ's we can define $\tilde{M}_S^{(\gamma)} = W_S^{(\gamma)} M_S^{(\gamma)}$ such that $F_S^{(\gamma)} = \tilde{M}_S^{(\gamma)\dagger} \tilde{M}_S^{(\gamma)}$. We will refer to the states $\rho_S^{(\gamma)}$ as γ*-detected states*, meaning that a precise choice for the detection operator $M_S^{(\gamma)}$ has been made. In general, n_Γ is not constrained by the dimension of the Hilbert space $n_S = \dim \mathcal{H}_S$. This is because neither the elements $F_S^{(\gamma)}$ of the POVM nor the measurement operators $M_S^{(\gamma)}$ are required to satisfy any orthogonality constraint. This is actually the case for a more specific type of quantum measurement defined by a projective-valued measure (PVM) or a projective measure. The latter is characterized by a set of operators $\Pi_S^{(\gamma)}$ being orthonormal projectors on S, which implies that $n_\Gamma \leqslant n_S$. A PVM $\{\Pi_S^{(\gamma)}\}$ defines self-adjoint operators $O_S = \sum_\gamma o_\gamma \Pi_S^{(\gamma)}$, with o_γ real $\forall\gamma$ and in one-to-one relation with μ_γ via an invertible *calibration* function $f(o_\gamma) = \mu_\gamma$ [4]. In fact, the usual formulation of the quantum-measurement postulate refers to the above operators as "observables" and assigns the probability $p_\gamma = {\rm Tr}[\rho_S^{\rm in} \Pi_S^{(\gamma)}]$ to the eigenvalue o_γ. As for the γ-detected states, their definition as $\{\rho_S^{(\gamma)} = \Pi_S^{(\gamma)} \rho_S^{\rm in} \Pi_S^{(\gamma)}/p_\gamma\}$ is an integral part of the postulate for PVM in its standard form, asserting that, after one single measurement with output $\mu_{\bar\gamma}$, the system is in the state $\rho_S^{\bar\gamma}$ with absolute certainty (as for the case of nonorthogonal POVMs, the mapping between the output probabilities p_γ and the γ-detected state is preserved even multiplying $\Pi_S^{(\gamma)}$ by a unitary transformation). This gives the state $\rho_S^{\rm out}$ the consistent meaning of a statistical mixture of the detected states produced in a series of many identical repetitions of the measurement. When rank $[\Pi_S^{(\gamma)}] = 1, \forall\gamma$, i.e., $\Pi_S^{(\gamma)} = |\gamma\rangle_S\langle\gamma|$, the PVM is called *ideal*, and $n_\Gamma = \dim\mathcal{H}_S$.

A. The Naimark extension theorem

The Naimark extension theorem [13–16] establishes a formal connection between POVMs and PVMs. Specifically, it states that any given POVM $\{F_S^{(\gamma)}\}$ for S can be represented as a PVM $\{\Pi_A^{(\gamma)}\}$ for an ancillary system A that has unitarily interacted with S prior to being tested. Let n_A be the dimension of the Hilbert space $\mathcal{H}_A$ associated with the ancilla. Formally, the Naimark theorem requires that $n_A \geqslant n_\Gamma$, allowing the choice $n_A = n_\Gamma$ that entails an ideal PVM on A, $\Pi_A^{(\gamma)} = |\gamma\rangle_A\langle\gamma|$. It then states that there exist the following: (i) a state $|\psi_0\rangle_A\langle\psi_0| \in \mathcal{L}(\mathcal{H}_A)$, (ii) a unitary operator $V_{SA} \in \mathcal{L}(\mathcal{H}_{SA})$, and (iii) an ideal PVM $\{|\gamma\rangle_A\langle\gamma|\}$ for A (see Fig. 1), such that

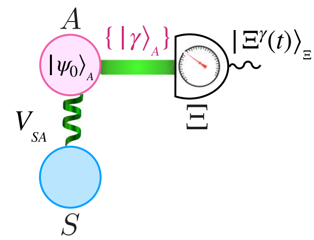

FIG. 1. Scheme of the Naimark representation for a POVM $\{F_S^{(\gamma)}\}$ on S, $F_S^{(\gamma)} = {\rm Tr}_A[\mathbb{I}_S \otimes |0\rangle_A\langle 0|\, V_{SA}^\dagger(\mathbb{I}_S \otimes |\gamma\rangle_A\langle\gamma|)V_{SA}]$.

$$F_S^{(\gamma)} = {\rm Tr}_A[(\mathbb{I}_S \otimes |\psi_0\rangle_A\langle\psi_0|)V_{SA}^\dagger(\mathbb{I}_S \otimes |\gamma\rangle_A\langle\gamma|)V_{SA}] \tag{3}$$

and

$$p_\gamma = {\rm Tr}\big[\big(\rho_S^{\rm in} \otimes |\psi_0\rangle_A\langle\psi_0|\big)\big(V_{SA}^\dagger(\mathbb{I}_S \otimes |\gamma\rangle_A\langle\gamma|)V_{SA}\big)\big], \tag{4}$$

which allows us to consistently write

$$M_S^{(\gamma)} = {}_A\langle\gamma|V_{SA}|\psi_0\rangle_A. \tag{5}$$

An explicit example of the above construction is presented in Sec. III A: it should be stressed, however, that this is by no means the only possibility, as different choices for the Naimark operator V_{SA} are typically available for each given POVM $\{F_S^{(\gamma)}\}$ and the associated choice of measurement operators $\{M_S^{(\gamma)}\}$. It should also be noticed that, conversely, a unitary transformation of the state $\rho_S^{\rm in} \otimes |\psi_0\rangle_A\langle\psi_0|$ into $V_{SA}(\rho_S^{\rm in} \otimes |\psi_0\rangle_A\langle\psi_0|)V_{SA}^\dagger$, followed by an ideal PVM $\{|\gamma\rangle_A\langle\gamma|\}$ on A, defines a proper POVM on S. In this respect, the entrance of the ancilla is extremely valuable, as it provides the theoretical scheme with the versatility needed to describe diverse experimental situations, such as those in which a physical mediator actually exists, and is ultimately responsible for the information transfer from S to Ξ [20,21].

B. Dynamical models for PVM

Dynamical models for quantum measurements are meant to describe how a measurement process takes place in time, in terms of a (time-independent) Hamiltonian coupling between the system S and an external environment Ξ playing the role of the apparatus, which, at the end of the process, will store the measurement outcomes. More specifically, in its simplest yet completely general version, the von Neumann–Ozawa (vN-O) dynamical model for PVMs [9,10,22–25] assumes that the interaction between S and Ξ reads

$$H_{S\Xi} := O_S \otimes O_\Xi, \tag{6}$$

with $O_S = \sum_\gamma o_\gamma \Pi_S^{(\gamma)}$ an observable on S, and O_Ξ a self-adjoint operator on Ξ, which is canonically conjugated to what is typically referred to as "the pointer" observable [7]. Hence, the associated unitary evolution reads

$$U_{S\Xi}(t) := e^{-itO_S \otimes O_\Xi} = \sum_{\gamma=1}^{n_\Gamma} \Pi_S^{(\gamma)} \otimes U_\Xi^\gamma(t), \tag{7}$$

where $U_\Xi^\gamma(t) = e^{-ito_\gamma O_\Xi}$ in units $\hbar = 1$. The model also assumes that Ξ is initially prepared in a pure state $|D\rangle$ that is not

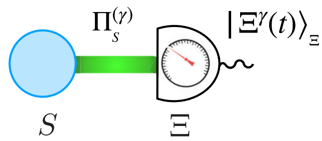

FIG. 2. Schematic representation of the von Neumann–Ozawa dynamical scheme for projective measures: an observable $O_S = \sum_\gamma o_\gamma \Pi_S^{(\gamma)}$ is measured on S by letting it interact with the measurement apparatus Ξ, so as to encode the information on the states of the apparatus $|\Xi^\gamma(t)\rangle_\Xi$.

an eigenstate of O_Ξ. If the system S is initialized at $t = 0$ in the state $\rho_S^{\rm in}$, the unitary (7) makes the system $S + \Xi$ evolve into the joint density matrix

$$\rho_{S\Xi}(t) := \sum_{\gamma,\gamma'=1}^{n_\Gamma} \Pi_S^{(\gamma)} \rho_S^{\rm in} \Pi_S^{(\gamma')} \otimes |\Xi^\gamma(t)\rangle_\Xi \langle \Xi^{\gamma'}(t)|, \quad (8)$$

which, upon partial trace with respect to Ξ, corresponds to the following local mapping:

$$\rho_S(t) = \sum_{\gamma,\gamma'=1}^{n_\Gamma} \Pi_S^{(\gamma)} \rho_S^{\rm in} \Pi_S^{(\gamma')} \, {}_\Xi\langle \Xi^{\gamma'}(t)|\Xi^\gamma(t)\rangle_\Xi \quad (9)$$

for S. In the above expressions, $|\Xi^\gamma(t)\rangle_\Xi := U_\Xi^\gamma(t)|D\rangle_\Xi$ are pure states of Ξ, which encode the measurement outcomes γ (see Fig. 2 for a schematic representation of the process). The more distinguishable such states are, the larger is the information stored in Ξ that allows one to distinguish between the different outcomes. In fact, the most favorable situation in terms of information transfer from S to Ξ corresponds to having the $|\Xi^\gamma(t)\rangle_\Xi$'s be orthonormal. It is easily seen that when this condition holds, it follows from Eq. (9) that the matrix representation of $\rho_S(t)$ on the basis of the O_S eigenstates is block-diagonal, and vice versa, i.e., $\rho_S(t) = \sum_\gamma \Pi_S^{(\gamma)} \rho_S^{\rm in} \Pi_S^{(\gamma)}$, as required by (1) if $M_S^{(\gamma)} = \Pi_S^{(\gamma)} = F_S^{(\gamma)}$. This clarifies why decoherence plays such an important role in the quantum measurement process [26–30]. Therefore, we say that the PVM $\{\Pi_S^{(\gamma)}\}$ can be successfully realized on S only if, in the limit of a macroscopic apparatus [29,30], there exists a time $t_{\rm d}$, typically referred to as *decoherence time* [6,7], such that for $t > t_{\rm d}$ one has

$${}_\Xi\langle \Xi^{\gamma'}(t)|\Xi^\gamma(t)\rangle_\Xi = \delta_{\gamma\gamma'}, \quad (10)$$

or at least such that the above condition is approximately verified over some nontrivial time interval preceding the data acquisition event (notice that although these scalar products are in principle periodic functions of time, in the limit of a macroscopic measuring device Ξ one can safely take the time during which they stay approximately null much longer than the time necessary to perform the measurement).

III. DYNAMICAL MODEL FOR ARBITRARY POVM

In this section, we discuss how to generalize the vN-O construction for PVMs to the case of arbitrary POVMs, removing the constraint on the orthonormality of the measurement operators. More precisely, we show how to modify Eqs. (6) and (7) in such a way that for times t larger than a certain characteristic threshold time $t_{\rm d}$, the interaction between S and Ξ will yield a joint density matrix of the form similar to Eq. (8), i.e.,

$$\rho_{S\Xi}(t) = \sum_{\gamma,\gamma'=1}^{n_\Gamma} M_S^{(\gamma)} \rho_S^{\rm in} M_S^{(\gamma')\dagger} \otimes |\Xi^\gamma(t)\rangle_\Xi \langle \Xi^{\gamma'}(t)|, \quad (11)$$

where $\{M_S^{(\gamma)};\, \gamma = 1, \ldots, n_\Gamma\}$ are the selected measurement operators for the POVM and where the vectors $\{|\Xi^\gamma(t)\rangle_\Xi;\, \gamma = 1, \ldots, n_\Gamma\}$ form a mutually orthonormal set as in Eq. (10).

Let us start by observing that at variance with the PVM scenario discussed in the previous section, we cannot expect Eq. (11) to apply at those times $t < t_{\rm d}$ for which Eq. (10) does not hold. Indeed, due to the lack of orthogonality of the operators $M_S^{(\gamma)}$ in this regime, the resulting transformation would not be CPT in general, hence it would be nonphysically implementable—see Appendix A. This of course does not imply that dynamical models cannot be found that describe a generic POVM: simply speaking, we need to replace the vN-O Hamiltonian coupling (6) with something else. The key ingredient for this construction is clearly provided by the Naimark extension theorem [13–16] we reviewed in Sec. II A, which could be pictorially summarized as in Fig. 1. A tentative idea would be to work in an $S + A + \Xi$ scenario with a conventional vN-O couplings linking the apparatus Ξ to A or to $S + A$ (A being the Naimark ancillary system). However, this approach, which we briefly review in Appendix B, does not conclusively work because, although it can reproduce the correct outcome probability distribution, it cannot yield a solution capable of approaching Eq. (11). On the contrary, a simpler and more effective way to construct a dynamical model for an arbitrary POVM is found by identifying the system environment Ξ directly with A. Under this assumption, we then look for a proper Hamiltonian coupling H_{SA} generating a unitary evolution $U_{SA}(t) := e^{-itH_{SA}}$, which for all times t larger than a certain critical time $t_{\rm d}$ fulfills, at least approximately, the constraint

$$U_{SA}(t) = V_{SA}, \quad (12)$$

V_{SA} being the unitary entering Eq. (5). Clearly due to the Stone theorem [31,32] such a Naimark Hamiltonian can always be identified. However, our goal is to produce an explicit construction for such a term, as we show in the following.

To construct our candidate for H_{SA}, we start with a first example that utilizes a small ancilla A, hence inducing an $S + A$ dynamics, which is explicitly periodic: accordingly, this model can produce the same correlations as in Eq. (11) only for specific values of t, with cyclic recurrence that prohibits the possibility of maintaining such a configuration indefinitely or at least for some nonzero time intervals. The second model, which is actually the central result of this paper, corrects this drawback adopting a much larger ancilla. A pictorial representation of the model is presented in Fig. 3, while the complete analytical derivation is presented in Sec. III B. We introduce a degeneracy parameter $\ell = 1, \ldots, n_L$ for the ancilla Hilbert space $\mathcal{H}_A$ and define a coupling between S and A formally equivalent to first neighboring hopping terms, characterizing models for perfect state transfer [33,34]. Therefore, by increasing n_L it is possible to extend the condition Eq. (11) over arbitrarily large (ideally infinitely long) time

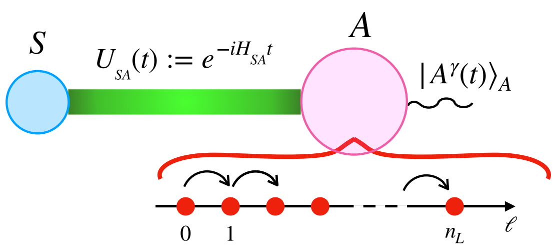

FIG. 3. Schematic representation of our dynamical model for POVMs. The principal system S interacts with an ultimately macroscopic ancilla A. The $S+A$ coupling is ruled by a time-independent generator H_{SA}. In fact, the unitary transformation $U_{SA}(t)$ induces the transition from an arbitrary initial state $\rho_S^{\rm in} \otimes |\psi_0\rangle_A\langle\psi_0|$ to the final state (13). The information about the possible outputs μ_γ is encoded in the orthonormal states of ancilla $|A^\gamma(t)\rangle_A$.

intervals, as shown in Fig. 4. Actually, our model allows us to translate (11) into

$$\rho_{SA}(t) = \sum_{\gamma,\gamma'=1}^{n_\Gamma} M_S^{(\gamma)} \rho_S^{\rm in} M_S^{(\gamma')^\dagger} \otimes |A^\gamma(t)\rangle_A\langle A^{\gamma'}(t)|, \tag{13}$$

where we have explicitly identified the state $|\Xi^\gamma(t)\rangle_\Xi$ with the state of the enlarged ancilla $|A^\gamma(t)\rangle_A$. A crucial difference between $\{|\Xi^\gamma(t)\rangle_\Xi\}$ and $\{|A^\gamma(t)\rangle_A\}$ is that the latter are orthogonal to each other for all times t. This compensates for the possible lack of orthogonality of the measurement operators $M_S^{(\gamma)}$, guaranteeing *a posteriori* the complete positivity of the unitary transformation $U_{SA}(t)$.

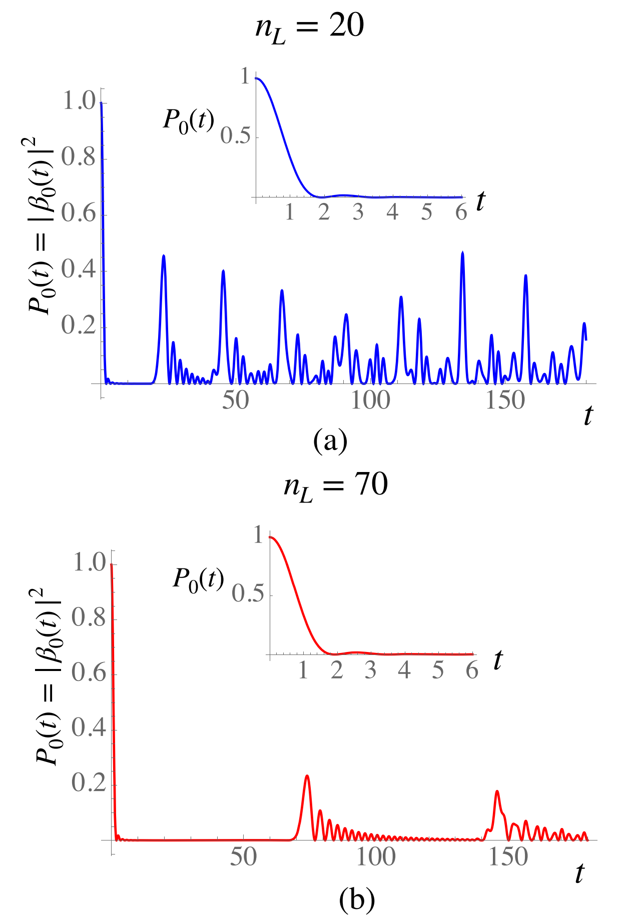

FIG. 4. Plot of the probability function $P_0(t) := |\beta_0(t)|^2$ entering Eq. (46) obtained by solving Eq. (41) for $n_L = 20$ [panel (a)] and $n_L = 70$ [panel (b)] for the case in which the frequency parameters ω_ℓ of Eq. (35) are taken to be uniform and equal to ω_0. In both cases, $P_0(t)$ drops from 1 to almost zero around $t \sim 2/\omega_0$. Then small revivals appear quite periodically after a time interval approximately given by n_L. Therefore, in the limit of $n_L \to \infty$, and after a given lapse of time (see the insets), the state of the $S+A$ can be safely approximated by (48).

A. First implementation: Periodic dynamics

Our first step to tackle the problem is to explicitly write down a suitable candidate for the Naimark unitary V_{SA}. We observe that Eq. (5) can be satisfied, e.g., by requiring that for all $|\psi\rangle_S$ of S the following condition holds:

$$V_{SA}|\psi\rangle_S \otimes |\psi_0\rangle_A = e^{i\alpha} \sum_{\gamma=1}^{n_\Gamma} M_S^{(\gamma)}|\psi\rangle_S \otimes |\gamma\rangle_A, \tag{14}$$

with $\{|\gamma\rangle_A;\ \gamma = 1, 2, \ldots, n_\Gamma\}$ being the orthonormal set of vectors of A entering Eq. (5), the phase α being absolutely irrelevant but being inserted for future reference (notice that the above requirement is fully consistent with the dimension n_A of A being larger than the total number of measurements outcomes n_Γ). This transformation does not completely characterize V_{SA} on the full Hilbert space of $S+A$, but does it only on a proper subspace of the latter—specifically the subspace associated with vectors having A into the input state $|\psi_0\rangle$. By construction, at least on these vectors, it preserves the scalar product: hence it can be generalized to a global unitary acting on the full space of the system and of the ancilla. What we are going to do next is to explicitly construct such an extension using a simplifying trick. Specifically, we assume the input vector $|\psi_0\rangle_A$ of A to be orthogonal to all the elements of the orthonormal set $\{|\gamma\rangle_A;\ \gamma = 1, \ldots, n_\Gamma\}$, i.e.,

$${}_A\langle\psi_0|\gamma\rangle_A = 0, \quad \forall \gamma = 1, 2, \ldots, n_\Gamma. \tag{15}$$

This, of course, automatically implies that the dimension of A we are considering has to be at least larger than or equal to $n_\Gamma + 1$, i.e., slightly larger than the minimum value required by the Naimark theorem (i.e., n_Γ). Such small overhead turns out to be extremely useful as we now can decompose the matrix V_{SA} of (12) into a collection of 2×2 independent blocks. Indeed, let us introduce an orthonormal basis $\{|j\rangle_S;\ j = 1, \ldots, n_S\}$ for $\mathcal{H}_S$. Expanding $|\psi\rangle_S$ in such a basis, we can then observe that the identity (14) gets replaced by

$$V_{SA}\big|\xi_j^{(0)}\big\rangle_{SA} = e^{i\alpha}\big|\xi_j^{(1)}\big\rangle_{SA}, \tag{16}$$

where for all $j = 1, \ldots, n_S$ we defined the pure states

$$\big|\xi_j^{(0)}\big\rangle_{SA} := |j\rangle_S \otimes |\psi_0\rangle_A, \tag{17}$$

$$\big|\xi_j^{(1)}\big\rangle_{SA} := \sum_{\gamma=1}^{n_\Gamma} M_S^{(\gamma)}|j\rangle_S \otimes |\gamma\rangle_A, \tag{18}$$

which by construction are all mutually orthonormal, i.e.,

$${}_{SA}\big\langle\xi_j^{(\ell)}\big|\xi_{j'}^{(\ell')}\big\rangle_{SA} = \delta_{j,j'}\delta_{\ell,\ell'}, \tag{19}$$

with $\ell, \ell' = 0, 1$. They can be grouped in a collection of n_S mutually orthogonal, two-dimensional subspaces

$$\mathcal{H}^{(j)}_{SA} := \text{Span}\{|\xi^{(0)}_j\rangle_{SA}, |\xi^{(1)}_j\rangle_{SA}\}, \tag{20}$$

labeled by j and spanned by the couple $|\xi^{(0)}_j\rangle_{SA}$ and $|\xi^{(1)}_j\rangle_{SA}$. According to (16), the unitary V_{SA} operates separately on each one of the $\mathcal{H}^{(j)}_{SA}$ where, up to the global phase factor $e^{i\alpha}$, it acts as the following effective Pauli transformations:

$$\sigma^{(j)}_{SA} = [\sigma^{(j)}_{SA}]^\dagger := |\xi^{(0)}_j\rangle_{SA}\langle\xi^{(1)}_j| + |\xi^{(1)}_j\rangle_{SA}\langle\xi^{(0)}_j|, \tag{21}$$

leading to the identification

$$V_{SA} = e^{i\alpha} \oplus_j \sigma^{(j)}_{SA}, \tag{22}$$

the direct sum being performed over all $j = 1, \dots, n_S$. Our first choice for the Naimark Hamiltonian is hence provided by the self-adjoint operator

$$H_{SA} := \omega \sum_{j=1}^{n_S} \sigma^{(j)}_{SA}, \tag{23}$$

with $\omega > 0$ an arbitrary positive constant, which, using (17) and (18), can be equivalently expressed as

$$H_{SA} = \omega \sum_{\gamma=1}^{n_\Gamma} \left(M^{(\gamma)}_S \otimes |\gamma\rangle_A\langle\psi_0| + \text{H.c.}\right). \tag{24}$$

Its associated unitary evolution is periodic of period $2\pi/\omega$ and reads

$$\begin{aligned} U_{SA}(t) &:= e^{-iH_{SA}t} = \oplus_j e^{-i\omega t \sigma^{(j)}_{SA}} \\ &= \oplus_j [\mathbb{1}^{(j)}_{SA}\cos(\omega t) - i\sigma^{(j)}_{SA}\sin(\omega t)], \end{aligned} \tag{25}$$

where we used the property

$$\sigma^{(j)}_{SA}\sigma^{(j')}_{SA} = \delta_{j,j'}\mathbb{1}^{(j)}_{SA}, \tag{26}$$

with

$$\mathbb{1}^{(j)}_{SA} := |\xi^{(0)}_j\rangle_{SA}\langle\xi^{(0)}_j| + |\xi^{(1)}_j\rangle_{SA}\langle\xi^{(1)}_j| \tag{27}$$

being the projection operator on $\mathcal{H}^{(j)}_{SA}$. From Eq. (25) it then follows that

$$\begin{aligned} U_{SA}(t)|\psi\rangle_S \otimes |\psi_0\rangle_A &= \cos(\omega t)|\psi\rangle_S \otimes |\psi_0\rangle_A \\ &\quad - i\sin(\omega t)\sum_{\gamma=1}^{n_\Gamma} M^{(\gamma)}_S|\psi\rangle_S \otimes |\gamma\rangle_A, \end{aligned} \tag{28}$$

which yields Eq. (14) for $t = t_d = \pi/(2\omega)$ upon identifying the phase term α with $-\pi/2$. We remark that $U_{SA}(t)$ is explicitly tailored for the selected detection operators $M^{(\gamma)}_S$, and in this sense it can be actually dubbed as a dynamical model for instruments.

B. Second implementation: Nonperiodic dynamics

The main drawback of the previous example is that it exhibits a definite period $2\pi/\omega$, so that Eq. (28) reproduces Eq. (14) only at the precise instants $t_n = (2n+1)t_d$, where n is an integer number. Hence it does not exactly fit into our requirement to enforce Eq. (11) for an extended time interval after a given premeasurement time t_d. Here we show, however, how one can easily modify the construction to explicitly fulfill this requirement, too. The idea is to increase the dimension of the subspaces $\mathcal{H}^{(j)}_{SA}$ of Eq. (20) and to equip the associated Hamiltonian with a reacher frequency spectrum. For this purpose, we replace the orthonormal set $\{|\gamma\rangle_A; \gamma = 1, \dots, n_\Gamma\}$ entering the previous construction with a larger set of orthonormal vectors $\{|\gamma, \ell\rangle_A; \gamma = 1, \dots, n_\Gamma; \ell = 1, \dots, n_L\}$, where ℓ is a degeneracy parameter that can take up to n_L different values, i.e.,

$${}_A\langle\gamma', \ell'|\gamma, \ell\rangle_A = \delta_{\gamma,\gamma'}\delta_{\ell,\ell'}, \quad {}_A\langle\psi_0|\gamma, \ell\rangle_A = 0 \quad \forall \gamma, \ell, \tag{29}$$

which implicitly dictates that now A must have a dimension n_A that is larger than or equal to $n_\Gamma n_L + 1$. With that in mind, we then replace Eq. (20) with the $(n_L + 1)$-dimensional spaces

$$\mathcal{H}^{(j)}_{SA} := \text{Span}\{|\xi^{(0)}_j\rangle_{SA}, |\xi^{(1)}_j\rangle_{SA}, \dots, |\xi^{(n_L)}_j\rangle_{SA}\}, \tag{30}$$

with $|\xi^{(0)}_j\rangle_{SA}$ still defined as in Eq. (17) and where, for $\ell = 1, \dots, n_L$, $|\xi^{(\ell)}_j\rangle_{SA}$ are instead given by

$$|\xi^{(\ell)}_j\rangle_{SA} := \sum_{\gamma=1}^{n_\Gamma} M^{(\gamma)}_S|j\rangle_S \otimes |\gamma, \ell\rangle_A, \tag{31}$$

which still fulfills the orthogonality conditions (19). Define hence the new self-adjoint operators

$$H^{(j)}_{SA} := \sum_{\ell=0}^{n_L-1} \omega_\ell \sigma^{(j,\ell)}_{SA}, \tag{32}$$

with $\omega_\ell > 0$ being frequency terms that play the role of free parameters in the model, and where, for $\ell = 0, \dots, n_L - 1$, the new Pauli operators $\sigma^{(j,\ell)}_{SA}$ are given by

$$\sigma^{(j,\ell)}_{SA} = [\sigma^{(j,\ell)}_{SA}]^\dagger := |\xi^{(\ell)}_j\rangle_{SA}\langle\xi^{(\ell+1)}_j| + |\xi^{(\ell+1)}_j\rangle_{SA}\langle\xi^{(\ell)}_j|. \tag{33}$$

Notice that from the orthonormality conditions (19) it follows that, irrespective of the values of ℓ and ℓ', the product of any two operators $\sigma^{(j,\ell)}_{SA}$ and $\sigma^{(j',\ell')}_{SA}$ with $j \neq j'$ vanishes, i.e.,

$$\sigma^{(j,\ell)}_{SA}\sigma^{(j',\ell')}_{SA} = 0. \tag{34}$$

Furthermore, the various $H^{(j)}_{SA}$ terms have exactly the same matrix form with respect to the associated canonical basis of the associated spaces $\mathcal{H}^{(j)}_{SA}$, i.e.,

$${}_{SA}\langle\xi^{(\ell')}_j|H^{(j)}_{SA}|\xi^{(\ell)}_j\rangle_{SA} = \omega_\ell(\delta_{\ell,\ell'+1} + \delta_{\ell+1,\ell'}). \tag{35}$$

Finally, we observe that $H^{(j)}_{SA}$ formally corresponds to the 1-excitation sector of a spin-1/2 chain Hamiltonian, with open boundary conditions, characterized by first-neighboring hopping terms, whose coupling terms are gauged by the frequencies ω_ℓ's.

We hence introduce as the new Hamiltonian of the $S + A$ system the operator

$$H_{SA} := \sum_{j=1}^{n_S} H^{(j)}_{SA}, \tag{36}$$

which, making use of Eqs. (31) and (17), can be equivalently recast in the following compact form:

$$H_{SA} = \sum_{\gamma=1}^{n_\Gamma} M^{(\gamma)}_S \otimes \Theta^{(\gamma)}_A + \sum_{\gamma,\gamma'=1}^{n_\Gamma} M^{(\gamma)}_S M^{(\gamma')\dagger}_S \otimes \Theta^{(\gamma,\gamma')}_A + \text{H.c.} \tag{37}$$

after defining the operators

$$\Theta_A^{(\gamma)} := \omega_0|\gamma, 1\rangle_A\langle\psi_0|,$$
$$\Theta_A^{(\gamma,\gamma')} := \sum_{\ell=1}^{n_L-1}\omega_\ell|\gamma,\ell\rangle_A\langle\gamma',\ell+1|. \tag{38}$$

From Eqs. (34) and (35) we notice that, as in the case of Sec. III A, H_{SA} is block-diagonal, with respect to the extended subspaces $\mathcal{H}_{SA}^{(j)}$, with isospectral blocks. Hence it acts independently on each one of such subspaces, inducing on each of them the same local unitary rotation, i.e.,

$$U_{SA}(t) = e^{-iH_{SA}t} = \oplus_j e^{-itH_{SA}^{(j)}}. \tag{39}$$

If we now consider the evolution it induces on an input state of the form $|\psi\rangle_S \otimes |\psi_0\rangle_A$, where $|\psi\rangle_S$ is a generic vector of S, expanding the input state as a linear combination of the vectors $|\xi_j^{(0)}\rangle_{SA}$, we can write

$$U_{SA}(t)|\psi\rangle_S \otimes |\psi_0\rangle_A = \sum_{j=1}^{n_S}\alpha_j\big|\xi_j^{(0)}(t)\big\rangle_{SA}, \tag{40}$$

α_j being the expansion coefficients of $|\psi\rangle_S$ with respect to the basis $\{|j\rangle_S;\ j = 1,\dots,n_S\}$ and where the vector

$$\big|\xi_j^{(0)}(t)\big\rangle_{SA} := e^{-itH_{SA}^{(j)}}\big|\xi_j^{(0)}\big\rangle_{SA} \tag{41}$$

is the evolution of $|\xi_j^{(0)}(t)\rangle_{SA}$ induced by the Hamiltonian component $H_{SA}^{(j)}$ that is active on the subspace $\mathcal{H}_{S,A}^{(j)}$. By construction $|\xi_j^{(0)}(t)\rangle_{SA} \in \mathcal{H}_{S,A}^{(j)}$ so that we can write it as

$$\big|\xi_j^{(0)}(t)\big\rangle_{SA} = \sum_{\ell=0}^{n_L}\beta_\ell(t)\big|\xi_j^{(\ell)}\big\rangle_{SA}. \tag{42}$$

In this expression, the quantities $\beta_\ell(t)$ are (properly normalized) amplitude probabilities associated with the canonical orthonormal basis $|\xi_j^{(0)}\rangle_{SA}, |\xi_j^{(1)}\rangle_{SA},\dots,|\xi_j^{(n_L)}\rangle_{SA}$, whose explicit functional dependence on t can be freely tailored by properly choosing the frequencies $\omega_1,\omega_2,\dots,\omega_{n_L}$ of the model. The relevant observation here is the fact that due to the isospectral property (35), such coefficients do not bear any functional dependence upon the index j. Exploiting this fact and replacing Eq. (42) into (40), we can hence write

$$\begin{aligned}U_{SA}(t)|\psi\rangle_S &\otimes |\psi_0\rangle_A\\ &= \beta_0(t)|\psi\rangle_S \otimes |\psi_0\rangle_A + \sqrt{1-|\beta_0(t)|^2}\sum_{\gamma=1}^{n_\Gamma}M_S^{(\gamma)}|\psi\rangle_S\\ &\quad\otimes|A^\gamma(t)\rangle_A,\end{aligned} \tag{43}$$

where for $\gamma = 1,\dots,n_\Gamma$,

$$|A^\gamma(t)\rangle_A := \frac{1}{\sqrt{1-|\beta_0(t)|^2}}\sum_{\ell=1}^{n_L}\beta_\ell(t)|\gamma,\ell\rangle_A \tag{44}$$

form an orthonormal set of vectors of A, which are also orthogonal to $|\psi_0\rangle_A$, i.e., they fulfill the conditions

$$_A\langle A^{\gamma'}(t)|A^\gamma(t)\rangle_A = \delta_{\gamma,\gamma'}, \quad _A\langle\psi_0|A^\gamma(t)\rangle_A = 0. \tag{45}$$

As a consequence of Eq. (40), it follows that the evolved density matrix $\rho_{SA}(t) := U_{SA}(t)(\rho_S^{\text{in}} \otimes |\psi_0\rangle_A\langle\psi_0|)U_{SA}^\dagger(t)$ of $S + A$ at time t can be written as

$$\begin{aligned}\rho_{SA}(t) &= |\beta_0(t)|^2\rho_S^{\text{in}} \otimes |\psi_0\rangle_A\langle\psi_0| + |\beta_0(t)|\Delta_{SA}(t)\\ &\quad+[1-|\beta_0(t)|^2]\sum_{\gamma,\gamma'=1}^{n_\Gamma}M_S^{(\gamma)}\rho_S^{\text{in}}M_S^{(\gamma')\dagger}\\ &\quad\otimes|A^\gamma(t)\rangle_A\langle A^{\gamma'}(t)|,\end{aligned} \tag{46}$$

where we have define the bounded operator on $S + A$,

$$\begin{aligned}\Delta_{SA}(t) &= e^{-i\xi_0(t)}\sqrt{1-|\beta_0(t)|^2}\sum_{\gamma=1}^{n_\Gamma}M_S^{(\gamma)}\rho_S^{\text{in}} \otimes |A^\gamma(t)\rangle_A\langle\psi_0|\\ &\quad+\text{H.c.},\end{aligned} \tag{47}$$

$e^{i\xi_0(t)}$ being the phase of $\beta_0(t)$.

The relevant quantity in Eq. (46) is the probability amplitude function $\beta_0(t)$: for $t = 0$ it is equal to 1, in agreement with the requirement that $\rho_{SA}(0) = \rho_S^{\text{in}} \otimes |\psi_0\rangle_A\langle\psi_0|$, but $\beta_0(t) \to 0$ in an extended time interval for large enough n_L, as shown in Fig. 4. Accordingly, in such a time interval the above expression reduces to

$$\rho_{SA}(t) \simeq \sum_{\gamma,\gamma'=1}^{n_\Gamma}M_S^{(\gamma)}\rho_S^{\text{in}}M_S^{(\gamma')\dagger} \otimes |A^\gamma(t)\rangle_A\langle A^{\gamma'}(t)|, \tag{48}$$

which effectively achieves our target (11) by identifying $|A^\gamma(t)\rangle_A$ with $|\Xi^\gamma(t)\rangle_\Xi$.

IV. CONCLUSIONS

In this paper, we discussed how to provide a comprehensive dynamical description for the quantum measurement process. For the case of projective measures, an exhaustive well-established answer is provided by the von Neumann–Ozawa model hinging upon the decoherence induced by an ultimately macroscopic apparatus on the system under investigation. As the decoherence process takes place, the states of the apparatus, on which the information about measurement outputs is encoded, progressively become orthogonal to each other. Once the decoherence process has taken place, such states turn out to be perfectly distinguishable, thus allowing for an optimal encoding of the measurement results. We proved that this model cannot be directly applied to tackle the case of nonorthogonal measurements, as it could induce a violation of the complete positivity requirement for such dynamical process before the decoherence is completed. We showed different strategies in order to overcome this hindrance. On the one hand, it turns out that it is possible to retrieve the correct probability distribution prescribed by an arbitrary POVM by extending the von Neumann description to an ancillary system and performing a joint projective measure on the system and the ancilla (Appendix B). However, this solution does not return the expression for the postmeasurement state of the system prescribed by the definition of POVMs. In Sec. III we show that a possible solution to this problem can be realized by getting rid of such a net separation between the ancilla and the apparatus, and finally identifying the latter with a macroscopic ancilla. The key mechanism underlying our model consists in engineering a coupling between the system and the ancilla in terms of state transfer Hamiltonians acting on orthogonal eigenspaces of the global Hilbert space. By construction, this allows us to encode the information about the output results of an arbitrary POVM into the states of the ancilla that, contrary to the standard decoherence model, constitute an orthonormal set at all times. This allows us to retrieve not only the correct probability distribution for the output results, but also the correct expression for the post-measurement state of POVMs for an assigned set of detector operators. In this sense, we have determined a dynamical model for instruments.

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APPENDIX A: CPT CONDITIONS FOR THE MAPPING (8)

If we force the mapping (8) to apply also when the projectors $\Pi_S^{(\gamma)}$'s are replaced by the element $M_S^{(\gamma)}$ associated with a generic POVM, at the local level on S this would induce the following transformation:

$$\rho_S^{\rm in} \longrightarrow \sum_{\gamma,\gamma'=1}^{n_\Gamma} M_S^{(\gamma)} \rho_S^{\rm in} M_S^{(\gamma')\dagger}\, {}_{\Xi}\langle \Xi^{\gamma'}(t)|\Xi^{\gamma}(t)\rangle_{\Xi}. \tag{A1}$$

Notice that the scalar product ${}_{\Xi}\langle \Xi^{\gamma'}(t)|\Xi^{\gamma}(t)\rangle_{\Xi}$ can be seen as the element γ,γ' of a positive-semidefinite matrix $Q(t)$ in a given orthonormal basis $\{|\phi_\gamma\rangle\}_{\gamma=1,\dots,n_\Gamma}$ of an n_Γ-dimensional Hilbert space. Let then $Q(t)=\sum_{j=1}^{n_\Gamma} q_j(t)|u_j(t)\rangle\langle u_j(t)|$ be the spectral decomposition of $Q(t)$, with $q_j(t)\geqslant 0$ and

$$\sum_{j=1}^{n_\Gamma} q_j(t) = {\rm Tr}[Q] = \sum_{\gamma=1}^{n_\Gamma} {}_{\Xi}\langle \Xi^{\gamma}(t)|\Xi^{\gamma}(t)\rangle_{\Xi} = n_\Gamma. \tag{A2}$$

Writing ${}_{\Xi}\langle \Xi^{\gamma'}(t)|\Xi^{\gamma}(t)\rangle_{\Xi}$ in the eigenbasis of $Q(t)$, we can then recast the mapping (A1) as

$$\rho_S^{\rm in} \longrightarrow \sum_{j=1}^{n_\Gamma} q_j(t) L_S^{(j)}(t)\rho_S^{\rm in} L_S^{(j)}(t)^{\dagger}, \tag{A3}$$

where $L_S^{(j)}(t)=\sum_{\gamma=1}^{n_\Gamma}\langle u_j(t)|\phi_\gamma\rangle M_S^{(\gamma)}$ are operators fulfilling the constraint

$$\sum_{j=1}^{n_\Gamma} L_S^{(j)}(t)^{\dagger} L_S^{(j)}(t) = \sum_{\gamma=1}^{n_\Gamma} {M^{(\gamma)}}^{\dagger}_S M^{(\gamma)}{}_S = \mathbb{1}_S. \tag{A4}$$

It is then easy to verify that Eq. (A3) is CPT if and only if the following condition holds:

$$\sum_{j=1}^{n_\Gamma} q_j(t) L_S^{(j)}(t)^{\dagger} L_S^{(j)}(t) = \sum_{\gamma,\gamma'=1}^{n_\Gamma} {}_{\Xi}\langle \Xi^{\gamma'}(t)|\Xi^{\gamma}(t)\rangle_{\Xi} M_S^{(\gamma')\dagger} M_S^{(\gamma)} = \mathbb{1}_S. \tag{A5}$$

The identity is trivially attained when the $M_S^{(\gamma)}$ form a complete set of orthogonal projectors, as in the case of PVMs. On the contrary, if this condition is not met, then Eq. (A5) is in general in conflict with (A4) with the exception of the case when the $q_j(t)$ are all equal to 1, forcing $Q(t)$ to be the identity operator, and forcing the vectors $|\Xi^{\gamma}(t)\rangle_{\Xi}$ to be orthonormal, i.e., ${}_{\Xi}\langle \Xi^{\gamma'}(t)|\Xi^{\gamma}(t)\rangle_{\Xi}=\delta_{\gamma,\gamma'}$.

APPENDIX B: $S+A+\Xi$ APPROACH TO DYNAMICAL MAPPING

A reasonable, yet not completely satisfying, approach to produce a generic dynamical model for describing an arbitrary POVM follows by considering the extended $S+A$ system of the Naimark representation as the system of interest, and introducing an external environment Ξ that performs a PVM on it. First, we notice that any PVM $\{\Pi_{SA}^{(\gamma)}\}$ on $S+A$ together with an arbitrary state $|\psi_0\rangle_A\langle\psi_0|$ defines a POVM on S with measurement operators $\{F_S^{(\gamma)}={}_A\langle\psi_0|\Pi_{SA}^{(\gamma)}|\psi_0\rangle_A\}$. Actually, thanks to the Naimark theorem, the reverse statement is also true. Indeed, if we take an arbitrary POVM $\{F_S^{(\gamma)}\}$ on S, from Eqs. (3) and (4) we can define the projectors

$$P_{SA}^{(\gamma)} := V_{SA}^{\dagger}\mathbb{I}_S\otimes|\gamma\rangle_A\langle\gamma|V_{SA}, \tag{B1}$$

which form a complete orthonormal set in the space $\mathcal{L}(\mathcal{H}_{SA})$ of linear operators of $S+A$. Let us now construct the vN-O dynamical model for such PVM introducing the interaction $O_{SA}\otimes O_{\Xi}$, with $O_{SA}=\sum_\gamma o_\gamma P_{SA}^{(\gamma)}$, $o_\gamma\in\mathbb{R}$, and O_Ξ self-adjoint; the corresponding propagator reads

$$U_{SA\Xi}(t) := e^{-itO_{SA}\otimes O_\Xi} = \sum_{\gamma=1}^{n_\Gamma} P_{SA}^{(\gamma)}\otimes U_{\Xi}^{(\gamma)}(t), \tag{B2}$$

with $U_\Xi^{(\gamma)}(t)=e^{-ito_\gamma O_\Xi}$. Subject to such a unitary, an initial state $\rho_S^{\rm in}\otimes|\psi_0\rangle_A\langle\psi_0|\otimes|D\rangle_\Xi\langle D|$ evolves into

$$\rho_{SA\Xi}(t) = \sum_{\gamma,\gamma'=1}^{n_\Gamma} P_{SA}^{(\gamma)}\rho_S^{\rm in}\otimes|\psi_0\rangle_A\langle\psi_0|P_{SA}^{(\gamma')}\otimes|\Xi^{\gamma}(t)\rangle_\Xi\langle\Xi^{\gamma'}(t)|$$

at a later time t, and for $t>t_{\rm d}$ the density operator of the joint system $S+A$ will have a block-diagonal form with respect to the basis of the PVM $\{P_{SA}^{(\gamma)}\}$. From the viewpoint of the principal system S, the composite system $A+\Xi$ is, however, seen as a single measurement apparatus. In this perspective, if we expand $\rho_{SA\Xi}(t)$ into an arbitrary basis $\{|e_k\rangle_A\}$ of $\mathcal{H}_A$, we have

$$\rho_{SA\Xi}(t) = \sum_{k,k'=1}^{n_A}\sum_{\gamma,\gamma'=1}^{n_\Gamma} {}_A\langle e_k|P_{SA}^{(\gamma)}|\psi_0\rangle_A\ \rho_S^{\rm in}\ {}_A\langle\psi_0|P_{SA}^{(\gamma')}|e_{k'}\rangle_A \times|e_k\rangle_A\langle e_{k'}|\otimes|\Xi^{\gamma}(t)\rangle_\Xi\langle\Xi^{\gamma'}(t)| \tag{B3}$$

and

$$\rho_{SA}(t) = \sum_{\gamma=1}^{n_\Gamma} P_{SA}^{(\gamma)}\rho_S^{\rm in}\otimes|\psi_0\rangle_A\langle\psi_0|P_{SA}^{(\gamma)} = \sum_{\gamma=1}^{n_\Gamma} p_\gamma\rho_{SA}^{(\gamma)} = \rho_{SA}^{\rm out}. \tag{B4}$$

Therefore, the system experiences a decoherence process that takes place in n_Γ (n_A-dimensional) subspaces spanned by $\{|e_k\rangle_A|\Xi^{\gamma}(t)\rangle_\Xi\}_{k=1,\dots,n_A}$ of $\mathcal{H}_{A\Xi}$. Indeed, since just the vectors $\{|\Xi^{\gamma}(t)\rangle_\Xi\}$ evolve in time, where Ξ is the actual macroscopic part of the apparatus, the effective decoherence process will emerge only with respect to the label γ. As we are aiming at

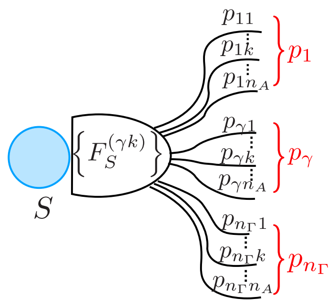

FIG. 5. Correspondence between the statistics $\{p_\gamma\}$ yielded by the POVM $\{F_S^{(\gamma)}\} = \{_A\langle\psi_0|P_{SA}^{(\gamma)}|\psi_0\rangle_A\}$ and the one resulting from $\{F_S^{(\gamma k)}\}$.

a dynamical model for the original POVM on S, a relevant question is as follows: what happens at the level of the principal system S? Clearly, no matter which subsystem we are going to identify as the apparatus (say Ξ or $\Xi + A$), Eq. (B3) does not have the form we are aiming to, not yielding to something like (11) even after the orthogonalization of the $|\Xi^\gamma(t)\rangle_\Xi$'s. As for the probability distribution $\{p_\gamma\}$, the outcomes statistics generated by $\{F_S^{(\gamma)}\}$ via $p_\gamma = \mathrm{Tr}[\rho_S^{\text{in}} F_S^{(\gamma)}]$ is nevertheless the same as that entering Eq. (B4), as can be easily seen by Eqs. (3) and (B1). As for the state of S, by inserting the explicit expression for $P_{SA}^{(\gamma)}$ into (B1) and tracing over the ancilla, we get

$$\rho_S(t) = \sum_{\gamma=1}^{n_\Gamma}\sum_{k=1}^{n_A} N_S^{(k)\gamma}\big(M_S^{(\gamma)}\rho_S^{\text{in}} M_S^{(\gamma)\dagger}\big)N_S^{(k)\gamma\dagger}$$
$$= \sum_{\gamma=1}^{n_\Gamma} p_\gamma \sum_{k=1}^{n_A} N_S^{(k)\gamma}\rho_S{}^{(\gamma)} N_S^{(k)\gamma\dagger}, \tag{B5}$$

where $N_S^{(k)\gamma} := {}_A\langle e_k|V_{SA}^\dagger|\gamma\rangle_A$. Therefore, $\rho_S(t)$ does not coincide with the postmeasurement state ρ_S^{out} defined in Eq. (1). The only exception is represented by the case in which $\dim \mathcal{H}_S = n_\Gamma = n_A$ and V_{SA} coincides with the *swap* operator $\mathbb{S}_{SA} := \sum_{\gamma,\gamma'} |\gamma\rangle_S\langle\gamma'| \otimes |\gamma'\rangle_A\langle\gamma|$: in this case, it results in $P_{SA}^{(\gamma)} = |\gamma\rangle_S\langle\gamma| \otimes \mathbb{I}_A$, which pulls back to the vN-O model for the ideal PVM $\{|\gamma\rangle_S\langle\gamma|\}$ on S. [The operator $\mathbb{S}_{SA}$ is a unitary self-adjoint transformation such that for all operators, $\Theta_S \in \mathcal{L}(\mathcal{H}_S)$ and $\Upsilon_A \in \mathcal{L}(\mathcal{H}_A)$ gives $\mathbb{S}_{SA}(\Theta_S \otimes \Upsilon_A)\mathbb{S}_{SA} = \Theta_A \otimes \Upsilon_S$.]

However, we can push forward. Let us observe that for any fixed γ the set of operators $\{N_S^{(k)\gamma\dagger}N_S^{(k)\gamma}\}$ returns a resolution of the identity, i.e., $\sum_k N_S^{(k)\gamma\dagger}N_S^{(k)\gamma} = \mathbb{I}_S$. From this, two facts follow: The first is that $\rho_S(t)$ reads as the postmeasurement state of a double-labeled POVM $\{F_S^{(\gamma,k)}\}$, with measurement operators $\{M_S^{(\gamma k)} := N_S^{(k)\gamma} M_S^{(\gamma)}\}$. Such a POVM accounts for $n_\Gamma n_A$ possible outcomes, and the associated probability distribution $p_{\gamma k} = \mathrm{Tr}[\rho_S^{\text{in}} F_S^{(\gamma k)}]$ is related to that of the original POVM $\{F_S^{(\gamma)}\}$ via

$$\sum_{k=1}^{n_A} p_{\gamma k} = p_\gamma. \tag{B6}$$

This means that, if we gather the $n_\Gamma n_A$ outcomes $\mu_{\gamma k}$ of the POVM $\{F_S^{(\gamma k)}\}$ in n_Γ sets O_γ, each bearing n_A elements, $O_\gamma = \{\mu_{\gamma 1}, \mu_{\gamma 2}, \dots, \mu_{\gamma n_A}\}$ (see Fig. 5), the probability for each set is the sum of the probabilities for the outcomes it collects. This is consistent with the fact that, as observed through Eq. (B3), from the viewpoint of the principal system, the decoherence process emerges in the form of n_Γ subspaces (one for each γ) in $\mathcal{H}_{A\Xi}$.

The second fact following from the condition $\sum_k N_S^{(k)\gamma\dagger}N_S^{(k)\gamma} = \mathbb{I}_S$ is that for all γ's, the set $\{N_S^{(k)\gamma\dagger}N_S^{(k)\gamma}\}$ itself defines a POVM on $\mathcal{H}_S$. This represents a meaningful result, as it tells us that the state (B5) prior to the output production is the statistical mixture, with the original POVM's probability distribution $\{p_\gamma\}$ of the n_Γ output states of a set of nonselective measurements, each labeled by γ and defined by the set of measurement operators $\{N_S^{(k)\gamma}\}$, performed upon the respective γ-detected state $\rho_S^{(\gamma)}$ resulting from the action of the original POVM on ρ_S^{in}.

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