Ruling out the class of statistical processes involving two noninteracting identical particles in two modes

S. Aravinda*
Department of Physics, Indian Institute of Technology Madras, Chennai 600036, India

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In the framework of generalized probabilistic theories (GPT), we illustrate a class of statistical processes in the case of two noninteracting identical particles in two modes that satisfies a well-motivated notion of physicality conditions, namely the double stochasticity and the no-interaction condition proposed by Karczewski *et al.* [Phys. Rev. Lett. **120**, 080401 (2018)], which cannot be realized through a quantum mechanical process. This class of statistical process is ruled out by an additional requirement, called the evolution condition, imposed on two-particle evolution. We also show that any statistical process of two noninteracting identical particles in two modes that satisfies all three physicality conditions can be realized within quantum mechanics using the beam-splitter operation.

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I. INTRODUCTION

Of the many ways in which quantum mechanics (QM) deviates from classical probability theory, the indistinguishable nature of identical particles and nonlocal nature [1] of quantum correlations are prominent. The statistical nature of indistinguishable particles plays a central role in our everyday understanding, from molecules, atoms, and solids to many astronomical events. The nonlocal nature exhibited by quantum theory via violation of Bell-type inequality [2] has revolutionized the fundamental understanding of nature and has contributed to development of quantum information theory and computation.

The generalized probabilistic theories (GPT) framework begins with adopting the formalism of operational quantum mechanics, which was advocated by the quantum logic community [3–5] in the pre-quantum-information era. Later, as a result of advancement in quantum information theory, GPT was formulated using Euclidean real probabilistic vector spaces to describe the state space [6–8]. It is considered one of the fundamental frameworks in which to study the quantum correlations in any probabilistic theory [6–14] and to find the operationally motivated information theoretic axioms for quantum theory [15–20].

The quest for physical axioms to characterize QM began with an attempt by Popescu and Rohrlich in which they constructed a probabilistic model (later called the PR box) for maximally nonlocal correlations that violate the Clauser, Horne, Shimoney and Halt (CHSH) inequality to its algebraic maximum. The impossibility of realizing PR-box correlations in QM was found by Tsirelson [21], even though the correlations are nonlocal as well as nonsignaling. This prompted the search for different physical principles to reproduce the quantum mechanical bound in the case of CHSH inequality [22–26].

The GPT framework, which can accommodate any probabilistic physical theory, facilitated such constructions and thus provided a deeper understanding of the nature of QM and in turn of nature. Motivated by these developments, the GPT framework has been extended to accommodate general relativity [27] and indefinite causal structure [28]. Hardy has extended the GPT framework to field theories and has provided a manifesto for constructing quantum gravity [29].

Recently, Karczewski *et al.* [30] proposed a general probabilistic framework to deal with noninteracting identical particles. In this work, the authors formulated a well-motivated physical principle called the consistency condition, which is similar to the no-signaling condition that any theory with noninteracting identical particles should satisfy, and provided an example of a statistical process with three identical particles in three modes that satisfies the consistency condition but cannot be realized in QM. Following the framework of Karczewski *et al.* [30], in this work we show that there exists a much simpler configuration, i.e., two noninteracting identical particles in two modes, which satisfies the consistency condition yet fails to produce such a process in QM. We also show that an extra physicality condition proposed by Karczewski *et al.*, which recovers quantum mechanical statistics in the case of three particles in three modes, can also be used to rule out a proposed impossible process involving two identical particles in the two-mode case. We call this principle the "evolution principle" and show that the class of physical theories involving generalized statistics of identical particles, interpolated between fermions and bosons [31–33], also satisfy this principle. We show that any process which satisfies all of the three physicality conditions can be realized by a quantum mechanical beam-splitter configuration.

II. GENERALIZED PROBABILITY THEORY FRAMEWORK

The basic ingredients of GPTs are states, transformations, and measurements and are formulated in the language

*aravinda@physics.iitm.ac.in

independent of Hilbert space formalism so that GPTs can accommodate any probabilistic physical theory. In the case of distinguishable particles, the framework is very well developed and has been used as a cornerstone for both foundational understanding and applications. In the case of foundations, the framework is used to provide the physicality conditions that characterizes QM [22–26], to understand the limits and advantages of general correlations [9,10,34–38], and to find an axiomatic formulation of QM [6,18,39–41]. On the application side, it has provided a methodology [42] of device-independent certification of many quantum information theoretic and quantum computational tasks.

Recently, Karczewski *et al.* [30] developed a GPT framework for noninteracting identical particles, which we briefly review here.

Consider N particles in M modes. The states of identical particles are determined by the particle occupation number in each mode, $\phi = \{n_1, n_2, \dots n_M\}$, where $n_1, n_2, \dots n_M$ are occupation number in mode $\{1, 2, \dots M\}$ with $\sum_{i=1}^{M} n_i = N$. The state probability vector Φ is a d-dimensional vector representing the probability distribution of particles over all modes. The GPT framework consists of an initial state Φ_i, an evolution (or transformation) $\mathcal{T}$ given by a stochastic matrix, and a final state Φ_f.

For example, consider a single boson in two mode with a symmetric beam-splitter (BS) transformation. The set of occupation number states are $\{1, 0\}$ and $\{0, 1\}$. The state probability vector $\Phi = [P(\{1, 0\}), P(\{0, 1\})]^T$, where $P(\{1, 0\})$ is the probability distribution of particles. The transformation matrix $\mathcal{T}^{(1)}_{\text{BS}}$ for a BS for single particle in two modes is given by

$$\mathcal{T}^{(1)}_{\text{BS}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{1}$$

Similarly, a GPT framework for two bosons in a symmetric BS is given by representing the state $\Phi = [P(\{2, 0\}), P(\{1, 1\}), P(\{0, 2\})]^T$ and the transformation $\mathcal{T}^{(2)}_{\text{BS}}$ by

$$\mathcal{T}^{(2)}_{\text{BS}} = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}. \tag{2}$$

III. PHYSICALITY CONDITIONS

In the case of characterizing nonlocal correlations in a Bell-type scenario, the natural physicality condition is to satisfy relativistic causality, i.e., the impossibility of instantaneous communication. The no-signaling condition states that the marginal probability distribution of one party should not be affected by the choice of observables by any another spatially separated party. Similarly, in the case of contextuality [43], it is Gleason's no signaling (also called "no disturbance" in the literature) which acts as the physicality or consistency condition.

The consistency condition for noninteracting identical particles has to consider the noninteracting character. This requirement as formulated in Ref. [30], which we call here the no-interaction criteria:

No interaction. The transformation of a single-particle distribution should not be affected by the presence of any other particle.

The precise mathematical characterization requires defining a transition matrix $\mathcal{R}^{(N)}$ with elements $R^{(N)}_{ij}$ which specify the transition of an N-particle state Φ_j to an $(N-1)$-particle state Φ_i by randomly removing one particle from the N-particle state:

$$\Phi^{(N-1)} = \mathcal{R}^{(N)}\Phi^{(N)}. \tag{3}$$

Any K-particle state can be obtained from an N-particle state by sequentially removing a single particle. For example, a transition from three-particle state to single-particle state can be obtained as $\Phi^{(1)} = \mathcal{R}^{(2)}\mathcal{R}^{(3)}\Phi^{(3)}$. Thus, a transition matrix $\mathcal{R}^{(N\to K)}$ for an N-particle state to a K-particle state is given by

$$\mathcal{R}^{(N\to K)} = \mathcal{R}^{(K+1)}\dots\mathcal{R}^{(N-1)}\mathcal{R}^{(N)}. \tag{4}$$

With these mathematical devices, the no-interaction condition constrains the allowed transformations $\mathcal{T}$ as

$$\mathcal{R}^{(N\to K)}\mathcal{T}^{(N)}\Phi_i = \mathcal{T}^{(K)}\mathcal{R}^{(N\to K)}\Phi_i, \quad \forall i. \tag{5}$$

This means that the state probability vector obtained by first reducing an N-particle state to a K-particle state and then transferring a K-particle state must be the same as first transferring an N-particle state and then reducing it to a K-particle state.

This will be evident by considering an elementary system with $N = 2$ and $M = 2$, for which $\mathcal{R}^{(2)}$ is given by

$$\mathcal{R}^{(2)} = \frac{1}{2}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}. \tag{6}$$

The no-interaction condition constraints any $\mathcal{T}^{(2)}$ that satisfies

$$\mathcal{R}^{(2)}\mathcal{T}^{(2)}\Phi^{(2)}_i = \mathcal{T}^{(1)}\mathcal{R}^{(2)}\Phi^{(2)}_i \tag{7}$$

for all states Φ_i. It is very clear from Eqs. (1) and (6) that the symmetric BS transformation given in Eq. (2) satisfies the no-interaction condition (7).

No increase of entropy after transformation demands the *double-stochasticity* condition, which is stated as follows:

Double stochasticity. The transformation matrix $\mathcal{T}$ must be doubly stochastic.

The final condition on the transformation $\mathcal{T}$ to be physical is the *evolution principle,* which is satisfied within QM [44,45], and it can be stated as follows:

Evolution principle. The evolution of the states in which all the particles are in the same mode should be equal to the evolution generated by its single-particle counterpart.

In the two-particle case, this principle is written as

$$\mathcal{T}^{(2)}\Phi^{(2)} = \mathcal{T}^{(1)}\Phi^{(1)} \times \mathcal{T}^{(1)}\Phi^{(1)}. \tag{8}$$

IV. CHARACTERIZATION OF TWO NONINTERACTING IDENTICAL PARTICLES

In this section, we characterize the transformation of two noninteracting identical particles in two modes satisfying the conditions of double stochasticity, no interaction and the evolution principle. It is shown that for two noninteracting identical particles in two modes, any transformation that satisfies all the three physicality conditions can be realized by quantum mechanical identical particles in a general beam-splitter transformation.

A. GPT characterization

Consider a general transformation $\mathcal{T}_g^{(2)}$ of two identical particles in two modes that satisfies the doubly stochastic condition:

$$\mathcal{T}_g^{(2)} = \begin{pmatrix} \alpha_1 & \alpha_2 & 1-\alpha_1-\alpha_2 \\ \alpha_3 & \alpha_4 & 1-\alpha_3-\alpha_4 \\ 1-\alpha_1-\alpha_3 & 1-\alpha_2-\alpha_4 & -1+\sum_{i=1}^4 \alpha_i \end{pmatrix}. \tag{9}$$

Similarly, the general transformation $\mathcal{T}_g^{(1)}$ of a single particle in two modes is

$$\mathcal{T}_g^{(1)} = \begin{pmatrix} \beta & 1-\beta \\ 1-\beta & \beta \end{pmatrix}. \tag{10}$$

Now, applying the no-interaction condition (7) on the general transformations (9) and (10) constrains the values of α_i as

$$\alpha_4 = 1-2\alpha_2, \quad \alpha_3 = 2(\beta-\alpha_1). \tag{11}$$

After the application of no-interaction condition, the general transformation is a three-parameter family which can be written as

$$\mathcal{T}_g^{\prime(2)} = \begin{pmatrix} \alpha_1 & \alpha_2 & 1-\alpha_1-\alpha_2 \\ 2(\beta-\alpha_1) & 1-2\alpha_2 & 2(-\beta+\alpha_1+\alpha_2) \\ 1+\alpha_1-2\beta & \alpha_2 & 2\beta-\alpha_1-\alpha_2 \end{pmatrix}. \tag{12}$$

Further application of evolution principle (8) to Eq. (12) relates the two parameters α_1 and α_2 by the equations

$$\alpha_1 = \beta^2, \quad \alpha_2 = 2(\beta-\beta^2). \tag{13}$$

The general transformation $\mathcal{T}_g^{(2)}$ which satisfies all the three physicality conditions is a single-parameter family represented by parameter β and given as

$$\mathcal{T}_g^{\prime(2)}(\beta) = \begin{pmatrix} \beta^2 & 2\beta(1-\beta) & (1-\beta)^2 \\ 2\beta(1-\beta) & 1-4\beta(1-\beta) & 2\beta(1-\beta) \\ (1-\beta)^2 & 2\beta(1-\beta) & \beta^2 \end{pmatrix}. \tag{14}$$

B. Quantum mechanical characterization

The input-output relation for boson annihilation operators for general two-mode BS can be written as

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = U_{\mathrm{BS}} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \tag{15}$$

where $(\hat{a}_1, \hat{a}_2)$ and $(\hat{b}_1, \hat{b}_2)$ are the input-output boson annihilation operators. The output bosonic commutation relation $[\hat{b}_i, \hat{b}_j] = \delta_{ij}$ restricts the elements of the transformation U with $|u_{11}|^2 + |u_{12}|^2 = 1$, $|u_{21}|^2 + |u_{22}|^2 = 1$ and $u_{11}u_{11}^* + u_{22}u_{22}^* = 0$. These restrictions imply that the BS transformation has to be an unimodular representation of subgroup SU(2) [46], and the simplest representation of SU(2) that realizes the BS transformation is

$$U_{\mathrm{BS}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{16}$$

The corresponding two-particle $\mathcal{T}_{\mathrm{QM}}^{(2)}$ and single-particle $\mathcal{T}_{\mathrm{QM}}^{(1)}$ transformations in the GPT can be written as

$$\mathcal{T}_{\mathrm{QM}}^{(1)} = \begin{pmatrix} \sin^2\theta & \cos^2\theta \\ \cos^2\theta & \sin^2\theta \end{pmatrix} \tag{17}$$

and

$$\mathcal{T}_{\mathrm{QM}}^{(2)} = \begin{pmatrix} \sin^4\theta & 2\sin^2\theta\cos^2\theta & \cos^4\theta \\ 2\sin^2\theta\cos^2\theta & \cos^2 2\theta & 2\sin^2\theta\cos^2\theta \\ \cos^4\theta & 2\sin^2\theta\cos^2\theta & \sin^4\theta \end{pmatrix}. \tag{18}$$

The constraint on the general transformation of two noninteracting identical particles in two modes that satisfies all three conditions of physicality is characterized in Sec. IV A and the allowed single-parameter general transformation is given in Eq. (14). Any transformation that satisfies all three physicality conditions can be realized in a quantum mechanical way by a BS transformation (18) by substituting $\beta = \sin^2\theta$ in Eq. (14). It is important to note that the given QM realization is not unique. The general BS transformation can be represented by including the phase factors. There can be many BS unitaries that can realize the given GPT transformation.

V. IMPOSSIBLE PROCESS

Karczewski *et al.* [30] provided an example of a process involving $N = 3$ in an $M = 3$ system with a transformation $\mathcal{T}^{(3)}$ that exceeds the bunching probability bound given in QM, even though the transformation $\mathcal{T}^{(3)}$ satisfies the no-interaction condition (5) and the matrix representing $\mathcal{T}^{(3)}$ is doubly stochastic. The given transformation (process) is not realizable in QM, similar to the maximally nonlocal PR box, which cannot be realized in QM. Here we report a class of transformations $\mathcal{T}^{(2)}$ which satisfies double stoachasticity and no-interaction conditions that cannot be realized in QM. The class of transformations $\mathcal{T}^{\prime(2)}$ in Eq. (12) that violate the evolution condition given in Eq. (13) cannot be realized within QM.

An elementary example involving $N = 2$ in $M = 2$, which satisfies the no-interaction condition (7) and is doubly stochastic but has no quantum mechanical unitary transformation that can realize such a process, can be given by the transformation $\mathcal{T}_{(\mathrm{imp})}^{(2)}$:

$$\mathcal{T}_{(\mathrm{imp})}^{(2)} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}. \tag{19}$$

It can be easily verified that $\mathcal{T}_{(\mathrm{imp})}^{(2)}$ is doubly stochastic and satisfies the no-interaction condition (7) with respect to $\mathcal{T}_{\mathrm{BS}}^{(1)}$ in Eq. (1). We can use the evolution principle to rule out $\mathcal{T}_{(\mathrm{imp})}^{(2)}$, as the transformation $\mathcal{T}^{(2)}_{(\mathrm{imp})}$ violates it. This can be seen by noting that $P^{(20)}_{(20)} \neq P^{(10)}_{(10)} P^{(10)}_{(10)}$.

VI. EVOLUTION CONDITION AS PHYSICAL CONDITION

The notion of a physicality condition is obtained by noting the general principle satisfied by the required physical theory. For example, the most general physicality conditions like no signaling and the laws of thermodynamics arise from the fact that all known physical theories satisfy these conditions. The quest for deriving QM from well-formulated physical principles began with a more preliminary task of bounding the nonlocal correlations to Tsirelson's value (bound) [21]. In the process, many physicality conditions were formulated which bound nonlocal correlations to quantum value [22–26]. The basic idea is that these principles lead to the predictions satisfied by classical and quantum mechanical theories. On the contrary, violating any of these principles lead to predictions that are not in consonance with that of either classical or quantum theories.

Similarly, as noted in Ref. [30] and shown in Ref. [45], the evolution principle is satisfied by classical and quantum theories, making it a viable candidate for physical principle. If the proposed physicality principle is satisfied by a broad range of physical theories, then the principle will have a broad range of theoretical validity. As a first step in this direction, we show that in a two-mode beam-splitter scenerio, the evolution principle is satisfied by more generalized statistics of identical particles interpolated between fermions and bosons generated by deformed Fermi and Bose algebras [31–33]. The extended algebra are called q-deformed algebra and the excitations are called *quons*. The operator algebraic relation between creation ($\hat{a}^\dagger$) and annhilation ($\hat{a}$) operators is given as

$$\hat{a}_i \hat{a}_j^\dagger - q \hat{a}_j^\dagger \hat{a}_i = \delta_{ij} I, \tag{20}$$

where $-1 \leqslant q \leqslant 1$ is the deformation factor. This algebraic relations leads to the following relations:

$$\hat{a}_i |0\rangle = 0, \quad \hat{a}_i^\dagger \hat{a}_i |n_i\rangle = v_n |n_j\rangle ,$$
$$\hat{a}_i^\dagger |n_i\rangle = \sqrt{v_{n+1}} |(n+1)_i\rangle , \quad \hat{a}_i |n_i\rangle = \sqrt{v_n} |(n-1)_i\rangle ,$$
$$(\hat{a}_i^\dagger)^n |0\rangle = \sqrt{v_n v_{n-1} \dots v_1} |n_i\rangle , \tag{21}$$

where

$$v_n = \begin{cases} \sum_{m=0}^{n-1} v^m = 1 + q v_{n-1}, & n \geqslant 2, \\ 1, & n = 1. \end{cases} \tag{22}$$

Consider the linear transformation as BS operation given in Eq. (15), where $(\hat{a}_1, \hat{a}_2)$ and $(\hat{b}_1, \hat{b}_2)$ are the input-output quon annhilation operators. Denote $\sqrt{T} = u_{11} = u_{22}$ and $\sqrt{R} = u_{12} = -u_{21}$ as reflection and transmission coefficients of BS with $R + T = 1$.

The single quon in the two-mode scenario transforms according to the Eq. (15) as

$$|\phi_{\mathrm{in}}\rangle = \hat{a}_1^\dagger |00\rangle \to u_{11} |1, 0\rangle + u_{21} |0, 1\rangle . \tag{23}$$

Accordingly, the single-quon statistics are $P^{(1,0)}_{(1,0)} = T$, $P^{(1,0)}_{(0,1)} = R$, $P^{(0,1)}_{(1,0)} = R$, $P^{(0,1)}_{(0,1)} = T$. Beginning with a single quon in both modes, we get

$$|\phi_{\mathrm{in}}\rangle = \hat{a}_1^\dagger \hat{a}_2^\dagger |00\rangle \to [u_{11}u_{12}(\hat{b}_1^\dagger)^2 + u_{21}u_{22}(\hat{b}_1^\dagger)^2 + u_{11}u_{12}\hat{b}_1^\dagger \hat{b}_2^\dagger + u_{21}u_{12}\hat{b}_2^\dagger \hat{b}_1^\dagger] |00\rangle . \tag{24}$$

From this, we get $P^{(1,1)}_{(2,0)} = P^{(1,1)}_{(0,2)} = RT(1+q)$ and $P^{(1,1)}_{(2,0)} = R^2 + T^2 - 2qRT$. Similarly, we get $P^{(2,0)}_{(2,0)} = P^{(0,2)}_{(0,2)} = T^2$, $P^{(2,0)}_{(0,2)} = P^{(0,2)}_{(2,0)} = R^2$, and $P^{(2,0)}_{(1,1)} = P^{(0,2)}_{(1,1)} = 2RT$. From this, it is easy to see that the quon statistics satisfies evolution principle (8) for two quons in two modes. For example, note $P^{(2,0)}_{(2,0)} = P^{(1,0)}_{(1,0)} \times P^{(1,0)}_{(1,0)} = T^2$.

The evolution principle is satisfied by quon statistics, which raises the question of what other principle(s) it satisfies. In order to investigate that, let us write the general transformation matrix $\mathcal{T}_q^{(2)}$ for two quons in two modes:

$$\mathcal{T}_q^{(2)} = \begin{pmatrix} T^2 & 2RT & R^2 \\ RT(1+q) & R^2 + T^2 - 2qRT & RT(1+q) \\ R^2 & 2RT & T^2 \end{pmatrix}. \tag{25}$$

From this, it is immediately clear that the transformation $\mathcal{T}_q^{(2)}$ is not doubly stochastic in general but it is for $q = 1$, which is bosonic statistics. The transformation $\mathcal{T}_q^{(1)}$ for general quon statistics is given as

$$\mathcal{T}^{(1)}_{\mathrm{QM}} = \begin{pmatrix} T & R \\ R & T \end{pmatrix}. \tag{26}$$

The satisfaction of the no-interaction condition (7) requires that $\mathcal{R}^{(2)}\mathcal{T}_q^{(2)} = \mathcal{T}_q^{(1)}\mathcal{R}^{(2)}$. $\mathcal{R}^{(2)}\mathcal{T}_q^{(2)}$ and $\mathcal{T}_q^{(1)}\mathcal{R}^{(2)}$ are given respectively as

$$\mathcal{R}^{(2)}\mathcal{T}_q^{(2)} = \frac{1}{2}\begin{bmatrix} 2T^2 + RT(1+q) & R^2 + T^2 + 2RT(2-q) & 2T^2 + RT(1+q) \\ 2T^2 + RT(1+q) & R^2 + T^2 + 2RT(2-q) & 2T^2 + RT(1+q) \end{bmatrix} \tag{27}$$

and

$$\mathcal{T}_q^{(1)}\mathcal{R}^{(2)} = \frac{1}{2}\begin{pmatrix} 2T & 1 & 2R \\ 2R & 1 & 2T \end{pmatrix}. \tag{28}$$

From this, it is clear that no-interaction condition is satisfied if and only if $q = 1$. The GPT version of general quon statistics fails to satisfy both the double stochasticity condition and the no-interaction condition.

VII. CONCLUSION

The nonlocal nature of quantum correlations is one of the salient features that dramatically deviates from our understanding of classical correlations which purely originate from human ignorance. Maximal nonlocality beyond QM exhibited by the PR box has laid the foundation for exploring the possibility of finding physical principles behind many properties of QM and provided a methodological application in terms of device-independent characterization of information-theoretic tasks.

The indistinguishable nature of quantum identical particles deviates our worldview of nature from a classical description. Recent computational advantages in the case of boson sampling [47] have solely emerged from the indistinguishable nature of identical quantum particles. Formulating and studying identical particles in GPT and identifying more general principles that single out quantum mechanical statistics are important.

In this work, we consider an elementary setting of two noninteracting identical particles in two modes, in which we provide a class of transformations that satisfies a well-defined notion of physicality conditions yet cannot be realized in QM. The principle proposed by Karczewski *et al.* [30], where they considered three particles in three modes, has been applied in this work for the case of a more elementary system of two identical particles in two modes to rule out such an impossible process. Any statistical process involving two identical particles in two modes that satisfies all the three physicality conditions is shown to be realizable in QM by using a suitable BS configuration. The physicality condition is identified by its general applicability in a wide range of theories. We investigate the physicality conditions on a broad range of physical theories involving generalized statistics of identical particles interpolated between fermions and bosons called quons. Although bosons satisfy all three conditions, the general statistics of two quons in two modes satisfies the evolution principle but fails to satisfy the double-stoachasticity condition and the no-interaction condition. This provokes more general questions to identify the physicality conditions which will be necessary and sufficient for realizing the statistical process of many identical particles in multimode settings.

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