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Eternal life of entropy in non-Hermitian quantum systems

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We find a different effect for the behavior of von Neumann entropy. For this we derive the framework for describing von Neumann entropy in non-Hermitian quantum systems and then apply it to a simple interacting PT-symmetric bosonic system. We show that our model is well defined even in the PT-broken regime with the introduction of a time-dependent metric and that it displays three distinct behaviors relating to the PT symmetry of the original time-independent Hamiltonian. When the symmetry is unbroken, the entropy undergoes rapid decay to zero (so-called "sudden death") with a subsequent revival. At the exceptional point it decays asymptotically to zero and when the symmetry is spontaneously broken it decays asymptotically to a finite constant value ("eternal life").

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I. INTRODUCTION

The information contained within a quantum system is of great importance for various practical implementations of quantum mechanics, most importantly for the development of quantum computers (e.g., Refs. [1–4]). In order to understand the quantum information, one must find a way of measuring the entanglement of a state. Entanglement is a defining feature of quantum mechanics that distinguishes it from classical mechanics, and there has been much work in recent years into the evolution of entanglement with time, particularly the observation of the abrupt decay of entangled states, coined as "sudden death" [5,6]. The decoherence of entanglement [7,8] is a problem for the operation of quantum computers and so understanding the mechanism behind this is an important contribution to the development of future machines. One particular measurement of entanglement and quantum information is the von Neumann entropy (see, for instance, Sec. 11.3 in Ref. [1]). This is well understood in the standard quantum mechanical setting, however, to date there has only been a small amount of work done concerning the proper treatment of entropy in non-Hermitian, parity-time (PT)-symmetric systems [9–12]. These differ from open quantum systems as the energy eigenvalues are real or appear as complex conjugate pairs and do not describe decay.

Non-Hermitian, parity-time (PT)-symmetric quantum mechanics was first popularized when it was shown that non-Hermitian systems with unbroken PT symmetry had real eigenvalues and unitary time evolution [13–17]. This is possible due to the existence of a nontrivial metric operator, and much work has been done on constructing metrics for time-independent systems (e.g., Refs. [18–23]). More recently this has extended to time-dependent systems (e.g., Refs. [24–30]). Of particular interest are non-Hermitian systems with spontaneously broken PT symmetry. These systems possess an exceptional point above which the PT symmetry is broken. In this regime the system exhibits complex energy eigenvalues, becoming ill defined and is therefore ordinarily discarded as nonphysical and useless. However, it has been shown [31–34] that when a time dependence is introduced into the central equations it is possible to make sense of the broken regime via a time-dependent metric. This allows for the definition of a Hilbert space and therefore a well-defined inner product. This will be central to our analysis in non-Hermitian systems as we will be showing how the evolution of entropy changes significantly as we vary the system parameters through the exceptional point.

We will first set up the framework for analyzing the von Neumann entropy in non-Hermitian systems and then we will apply it to a simple model consisting of a bosonic system coupled to a bath.

II. ENTANGLEMENT VON NEUMANN ENTROPY

In order to make calculations of the quantum entropy for non-Hermitian systems, we must first introduce some new quantities when compared to the Hermitian case. In what follows we use natural units, setting $\hbar = 1$. The density matrix for Hermitian systems is defined as an Hermitian operator describing the statistical ensemble of states,

$$\varrho_h = \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right| , \tag{1}$$

where the subscript h indicates it relates to an Hermitian system. $|\phi_i\rangle$ are general pure states, and p_i is the probability that the system is in the pure state $|\phi_i\rangle$, with $0 \leqslant p_i \leqslant 1$ and $\sum_i p_i = 1$. Therefore ϱ_h represents a mix of pure states (a mixed state). If the system is comprised of subsystems A and B, one can define the reduced density operator of these subsystems as the partial trace over the opposing subsystem's

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Hilbert space,

$$\varrho_{h,A} = \mathrm{Tr}_B[\varrho_h] = \sum_i \langle n_{i,B}| \varrho_h |n_{i,B}\rangle , \tag{2}$$

$$\varrho_{h,B} = \mathrm{Tr}_A[\varrho_h] = \sum_i \langle n_{i,A}| \varrho_h |n_{i,A}\rangle , \tag{3}$$

where $|n_{i,A}\rangle$ and $|n_{i,B}\rangle$ are the eigenstates of the subsystems A and B, respectively. In this way one can isolate the density matrix for each subsystem and perform an entropic analysis on them individually. We now want to find the relationship between the ϱ_h and ϱ_H, where the subscript H indicates a non-Hermitian system. The clearest starting point is the von Neumann equation which governs the time evolution of the density matrix. For the Hermitian system it is

$$i\partial_t\varrho_h = [h, \varrho_h], \tag{4}$$

where h is the Hermitian Hamiltonian. We now wish to find the equivalent relation in the non-Hermitian setting. In order to do this we substitute the time-dependent Dyson equation [32,35],

$$h = \eta H\eta^{-1} + i\partial_t\eta\eta^{-1}, \tag{5}$$

into the von Neumann equation. Equation (5) arises when one considers two Hamiltonians, $h = h^\dagger$ and $H \neq H^\dagger$, each satisfying the time-dependent Schrödinger equation with corresponding wave functions $|\phi\rangle$ and $|\psi\rangle$,

$$h\,|\phi\rangle = i\partial_t\,|\phi\rangle , \quad H\,|\psi\rangle = i\partial_t\,|\psi\rangle . \tag{6}$$

The wave functions are related by the Dyson operator $|\phi\rangle = \eta\,|\psi\rangle$, which when substituted into the time-dependent Schrödinger equation, gives the time-dependent Dyson equation. This relates the non-Hermitian Hamiltonian to a Hermitian Hamiltonian. The Dyson operator forms the metric $\rho = \eta^\dagger\eta$ which we will see is essential for the calculation of entropy in non-Hermitian systems. After some manipulation, substituting Eq. (5) into (4) results in the following equation,

$$i\partial_t\varrho_H = [H, \varrho_H], \tag{7}$$

when assuming that the density matrix in the Hermitian system is related to that of the non-Hermitian system via a similarity transformation,

$$\varrho_h = \eta\varrho_H\eta^{-1}. \tag{8}$$

Recalling that $|\phi\rangle = \eta\,|\psi\rangle$, this leads us to the definition of the density matrix ϱ_H for non-Hermitian systems,

$$\varrho_H = \sum_i p_i\,|\psi_i\rangle\,\langle\psi_i|\,\rho, \tag{9}$$

where $|\psi_i\rangle$ are general pure states for the non-Hermitian system. Notice that ϱ_H is a Hermitian operator in the Hilbert space related to the metric $\langle\cdot|\,\rho\,|\cdot\rangle$. It is therefore clear that the existence of a well-defined metric is essential for the calculation of entropy in non-Hermitian systems. These results match those from Ref. [9]. Having defined the density matrix for non-Hermitian systems and found the relation to Hermitian systems we can now consider the entropy. For the total system, the von Neumann entropy is defined as

$$S_h = -\mathrm{tr}[\varrho_h \ln \varrho_h]. \tag{10}$$

This can also be expressed as a sum of the eigenvalues λ_i of the density matrix ρ_h as it is a Hermitian operator,

$$S_h = -\sum_i \lambda_i \ln \lambda_i. \tag{11}$$

As the density matrices for the Hermitian and non-Hermitian systems are related by a similarity transform, they share the same eigenvalues, therefore

$$S_H = S_h. \tag{12}$$

It is important to recall, however, that this relation only holds true for the existence of a well-defined Dyson operator η and metric ρ. Without this, we are unable to form the relation (8). For closed systems, the von Neumann entropy is constant with time. However, we wish to consider the entropy for particular subsystems and for this we must consider the partial trace of the density matrix. In this setting the entropy for subsystem A becomes

$$S_{h,A} = -\mathrm{tr}[\varrho_{h,A} \ln \varrho_{h,A}] = -\sum_i \lambda_{i,A} \ln \lambda_{i,A}, \tag{13}$$

where once again the entropy of the Hermitian subsystem is equal to that of the non-Hermitian subsystem $S_{h,A} = S_{H,A}$ with the existence of η and ρ. The entropy of a particular subsystem is not confined to be constant and we show that it exhibits some very interesting properties when evolved in time.

III. SYSTEM BATH COUPLED MODEL

We now consider a time-independent non-Hermitian Hamiltonian consisting of coupled harmonic oscillators. We have a system composed of $a, a^\dagger$ bosonic operators coupled to a bath of N $q_i, q_i^\dagger$ bosonic operators. The Hamiltonian takes the form

$$H = \nu a^\dagger a + \nu\sum_{n=1}^{N} q_n^\dagger q_n + (g+\kappa)a^\dagger\sum_{n=1}^{N} q_n + (g-\kappa)a\sum_{n=1}^{N} q_n^\dagger, \tag{14}$$

with ν, g, and κ being real time-independent parameters.

A. *PT* symmetry

The Hamiltonian (14) is PT symmetric under the antilinear transformation,

$$PT: \quad i \to -i, \quad a \to -a, \quad a^\dagger \to -a^\dagger, \quad q_n \to -q_n, \quad q_n^\dagger \to -q_n^\dagger, \tag{15}$$

as it commutes with the PT operator for all values of ν, g, and κ,

$$[PT, H] = 0. \tag{16}$$

The energy eigenvalues are

$$E_{m,N}^{\pm} = m(\nu \pm \sqrt{N}\sqrt{g^2-\kappa^2}). \tag{17}$$

In order to ensure boundedness from below the system must have $\nu > \sqrt{N}\sqrt{g^2-\kappa^2}$. Note that there is an exceptional point at $g = \kappa$ and when $\kappa > g$ this system is in the broken

$\mathcal{PT}$ regime. This is clear when studying the first excited state ($m = 1$) expanded in terms of creation operators acting on a tensor product of Fock states. The general state consists of one Fock state for the system of a and $a^\dagger$ bosonic operators and N Fock states for the bath of q_i and $q_i^\dagger$ bosonic operators,

$$|\psi\rangle = |n_a\rangle \otimes |n_{q_1}\rangle \otimes |n_{q_2}\rangle \cdots = |n_a\rangle \bigotimes_{i=1}^{N} |n_{q_i}\rangle \,. \tag{18}$$

When considering the first excited state, we will be dealing with very few nonzero states, and as such we can make some simplifications to the notation. If all the states in the q bath are in the ground state we will represent this with $|\mathbf{0}_q\rangle$. Similarly, if the ith state in the q bath is in the first excited state with the rest in the ground state, we will represent this with a $|1_i\rangle$,

$$|\mathbf{0}_q\rangle = \bigotimes_{i=1}^{N} |0_{q_i}\rangle \,,$$
$$|1_i\rangle = \left[\bigotimes_{j=1}^{i-1} |0_{q_j}\rangle\right] \otimes |1_{q_i}\rangle \otimes \left[\bigotimes_{k=i+1}^{N} |0_{q_k}\rangle\right]. \tag{19}$$

We can now write down the first excited state,

$$\begin{aligned} |\psi_{1,N}^{\pm}\rangle &= \sqrt{\frac{g+\kappa}{2g}}\,|1_a\rangle \otimes |\mathbf{0}_q\rangle \pm \sqrt{\frac{g-k}{2gN}}\,|0_a\rangle \otimes \sum_{i=1}^{N} |1_i\rangle \\ &= \sqrt{\frac{g+\kappa}{2g}}\,|1_a \mathbf{0}_q\rangle \pm \sqrt{\frac{g-\kappa}{2gN}} \sum_{i=1}^{N} |0_a 1_i\rangle \\ &= \sqrt{\frac{g+\kappa}{2g}} a^\dagger\,|0_a \mathbf{0}_q\rangle \pm \sqrt{\frac{g-\kappa}{2gN}} \sum_{i=1}^{N} q_i^\dagger\,|0_a \mathbf{0}_q\rangle \,. \end{aligned} \tag{20}$$

In order for the $\mathcal{PT}$ symmetry to remain unbroken, the wave function must also remain unchanged up to a phase factor when acted on by the $\mathcal{PT}$ operator,

$$\mathcal{PT}\,|\psi_{1,N}^{\pm}\rangle = e^{i\phi}\,|\psi_{1,N}^{\pm}\rangle \,. \tag{21}$$

However, the wave functions are only eigenfunctions of the $\mathcal{PT}$ operator when $\kappa < g$,

$$\mathcal{PT}\,|\psi_{1,N}^{\pm}\rangle = -\,|\psi_{1,N}^{\pm}\rangle \,. \tag{22}$$

When $\kappa > g$, the wave functions are no longer eigenfunctions of the $\mathcal{PT}$ operator,

$$\mathcal{PT}\,|\psi_{1,N}^{\pm}\rangle \neq e^{i\phi}\,|\psi_{1,N}^{\pm}\rangle \,. \tag{23}$$

Therefore we need to employ a time-dependent analysis in order to make sense of the broken regime. To do this we first must solve the time-dependent Dyson equation.

B. Solving the time-dependent Dyson equation

We wish to find the time-dependent metric $\rho(t)$ that allows us to perform an entropic analysis on our model (14). In order to do this we must find the Dyson operator $\eta(t)$ and the equivalent time-dependent Hermitian system $h(t)$. The model (14) is in fact part of a larger family of Hamiltonians belonging to the closed algebra with Hermitian generators,

$$\begin{aligned} N_A &= a^\dagger a, \quad N_Q = \sum_{n=1}^{N} q_n^\dagger q_n, \\ N_{AQ} &= N_A - \frac{1}{N}N_Q - \frac{1}{N}\sum_{n\neq m} q_n^\dagger q_m, \\ A_x &= \frac{1}{\sqrt{N}}\left(a^\dagger \sum_{n=1}^{N} q_n + a\sum_{n=1}^{N} q_n^\dagger\right), \\ A_y &= \frac{i}{\sqrt{N}}\left(a^\dagger \sum_{n=1}^{N} q_n - a\sum_{n=1}^{N} q_n^\dagger\right). \end{aligned} \tag{24}$$

The commutation relations are

$$\begin{aligned} &[N_A, N_Q] = 0, \quad [N_A, N_{AQ}] = 0, \\ &[N_A, A_x] = -iA_y, \quad [N_A, A_y] = iA_y, \\ &[N_Q, A_x] = iA_y, \quad [N_Q, A_y] = -iA_x, \\ &[N_{AQ}, A_x] = -2iA_y, \quad [N_{AQ}, A_y] = 2iA_x. \end{aligned} \tag{25}$$

In terms of this algebra, our original Hamiltonian (14) can be written as

$$H = \nu N_A + \nu N_Q + \sqrt{N} g A_x - i\sqrt{N}\kappa A_y. \tag{26}$$

We are now in a position to begin solving the time-dependent Dyson equation (5). For this we make the ansatz

$$\eta(t) = e^{\beta(t) A_y} e^{\alpha(t) N_{AQ}}, \tag{27}$$

and use the Baker-Campbell-Hausdourff formula to expand the Dyson equation (5) in terms of generators. In order to make the resulting Hamiltonian Hermitian, we must solve two coupled differential equations to eliminate the non-Hermitian terms,

$$\dot{\alpha} = -\tanh(2\beta)[\sqrt{N} g\cosh(2\alpha) + \sqrt{N}\kappa\sinh(2\alpha)], \tag{28}$$

$$\dot{\beta} = \sqrt{N}\kappa\cosh(2\alpha) + \sqrt{N} g\sinh(2\alpha). \tag{29}$$

Equation (29) can be solved for α,

$$\tanh(2\alpha) = \frac{-Ng\kappa + \dot{\beta}\sqrt{\dot{\beta}^2 + N(g^2-\kappa^2)}}{Ng^2 + \dot{\beta}^2}. \tag{30}$$

In principle this could lead to a restriction to the term on the right-hand side of Eq. (30) as $-1 < \tanh(2\alpha) < 1$. However, as we will see, this restriction is obeyed with the final solutions for α and β. Substituting (30) into Eq. (28) gives

$$\ddot{\beta} + 2\tanh(2\beta)\big[Ng^2 - N\kappa^2 + \dot{\beta}^2\big] = 0. \tag{31}$$

Now making the substitution $\sinh(2\beta) = \sigma$, this reverts to a harmonic oscillator equation,

$$\ddot{\sigma} + 4N(g^2 - \kappa^2)\sigma = 0, \tag{32}$$

which is solved with the function

$$\sigma = \frac{c_1}{\sqrt{g^2-\kappa^2}}\sin[2\sqrt{N}\sqrt{g^2-\kappa^2}(t+c_2)], \tag{33}$$

for all values of κ, where c_1 and c_2 are constants of integration. We can now write down expressions for α and β,

$$\tanh(2\alpha) = \frac{\zeta^2 - 1}{\zeta^2 + 1}, \tag{34}$$

$$\sinh(2\beta) = \frac{c_1}{\sqrt{g^2 - \kappa^2}} \sin[2\sqrt{N}\sqrt{g^2 - \kappa^2}(t + c_2)], \tag{35}$$

where ζ is of the form

$$\zeta = \sqrt{2}\sqrt{\frac{g-\kappa}{g+\kappa}}\left[\frac{\sqrt{c_1^2 + g^2 - \kappa^2} + c_1 \cos[2\sqrt{N}\sqrt{g^2-\kappa^2}(t + c_2)]}{\sqrt{c_1^2 + 2(g^2 - \kappa^2) - c_1^2 \cos[4\sqrt{N}\sqrt{g^2-\kappa^2}(t + c_2)]}}\right]. \tag{36}$$

Therefore we have a well-defined solution for $\eta(t)$ from our original ansatz (27) which results in the following time-dependent Hermitian Hamiltonian,

$$h(t) = \nu N_A + \nu N_Q + \mu(t) A_x, \tag{37}$$

where

$$\mu(t) = \frac{(g^2 - \kappa^2)\sqrt{N}\sqrt{c_1^2 + g^2 - \kappa^2}}{c_1^2 + 2(g^2 - \kappa^2) - c_1^2 \cos[4\sqrt{N}\sqrt{g^2 - \kappa^2}(t + c_2)]}. \tag{38}$$

This is real provided $|\frac{c1}{\sqrt{g^2-\kappa^2}}| > 1$. The general time-dependent first excited state is

$$|\phi(t)\rangle = e^{-i\nu t}[A \sin \mu_I(t) + B \cos \mu_I(t)]\,|1_a \mathbf{0}_q\rangle + \frac{e^{-i\nu t}}{\sqrt{N}}[A \cos \mu_I(t) - B \sin \mu_I(t)] \sum_{i=1}^{N} |0_a 1_i\rangle\,, \tag{39}$$

with $A^2 + B^2 = 1$ and

$$\mu_I(t) = \int^t \mu(s) ds = \frac{1}{2} \arctan\left(\frac{\sqrt{c_1^2 + g^2 - \kappa^2}\tan[2\sqrt{N}\sqrt{g^2 - \kappa^2}(t + c_2)]}{\sqrt{g^2 - \kappa^2}}\right). \tag{40}$$

Now we have a full solution for $\eta(t)$ and therefore $\rho(t) = \eta(t)^\dagger \eta(t)$. This allows us to calculate the entropy for our non-Hermitian system (14). The easiest route to take is to work with the resulting Hermitian system (37) as it was shown in Sec. II that the entropy in both systems is equivalent when $\eta(t)$ is well defined. It is important to note that if the $\eta(t)$ ever becomes ill defined, then our analysis of the Hermitian system does not correspond to the original non-Hermitian Hamiltonian as we cannot form a metric $\rho(t)$.

IV. THREE TYPES OF ENTROPY EVOLUTION

We now calculate the entropy of the system and show how varying the parameters N, g, and κ affect its evolution with time. We prepare our system in an entangled first excited state (39) at time $t = 0$—this is equivalent to a single qubit entangled with itself,

$$|\phi(0)\rangle = \sin\gamma\,|1_a \mathbf{0}_q\rangle + \frac{\cos\gamma}{\sqrt{N}} \sum_{i=1}^{N} |0_a 1_i\rangle\,, \tag{41}$$

for which we choose $A = \sin\gamma$, $B = \cos\gamma$, and $c_2 = 0$. Therefore the general state at time t is

$$|\phi(t)\rangle = e^{-i\nu t}[\sin\gamma \sin\mu_I(t) + \cos\gamma\cos\mu_I(t)]\,|1_a \mathbf{0}_q\rangle + \frac{e^{-i\nu t}}{\sqrt{N}}[\sin\gamma\cos\mu_I(t) - \cos\gamma \sin\mu_I(t)] \sum_{i=1}^{N} |0_a 1_i\rangle\,. \tag{42}$$

Now we form the density matrix for the system (a) with a partial trace over the external bosonic bath (q),

$$\varrho_a(t) = \mathrm{Tr}_q[\varrho_h(t)] = \begin{pmatrix} [\sin\gamma\sin\mu_I(t) + \cos\gamma\cos\mu_I(t)]^2 & 0 \\ 0 & [\sin\gamma\cos\mu_I(t) - \cos\gamma\sin\mu_I(t)]^2 \end{pmatrix}. \tag{43}$$

We can now calculate the von Neumann entropy of the system using this reduced density matrix. First, we read off the eigenvalues of $\varrho_a(t)$ as it is diagonal,

$$\lambda_1(t) = [\sin\gamma\sin\mu_I(t) + \cos\gamma\cos\mu_I(t)]^2,$$
$$\lambda_2(t) = [\sin\gamma\cos\mu_I(t) - \cos\gamma\sin\mu_I(t)]^2, \tag{44}$$

and substitute these into the expression for the entropy,

$$S_{h,a}(t) = S_{H,a}(t) = -\lambda_1(t)\ln[\lambda_1(t)] - \lambda_2(t)\ln[\lambda_2(t)]. \tag{45}$$

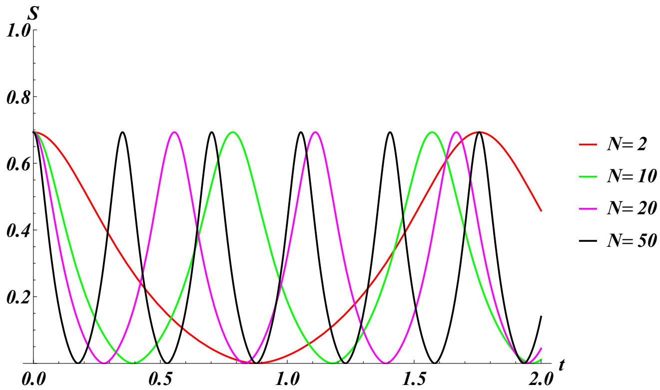

FIG. 1. von Neumann entropy as a function of time and varied bath size, with $c_1 = 1$, $g = 0.7$, $\kappa = 0.3$.

With this expression we are free to choose the initial state of our system with a given value of γ. If the initial state of our system is maximally entangled state with $\gamma = \pi/4$, then we observe how the entanglement entropy evolves with time. This is most applicable to quantum computing as in that context one would like to preserve the entangled state. We will now vary the parameters N, g, and κ to see how they affect the evolution of entropy with time. Of particular interest is the exceptional point $g = \kappa$ where the non-Hermitian system enters the broken PT regime in the time-independent setting. It is in this area that the evolution we see differs from the standard evolution of entropy in Hermitan quantum mechanics.

Figure 1 shows how the entropy evolves when $\kappa < g$. This is equivalent to the unbroken PT regime of the non-Hermitian model. In this setting the entropy experiences so-called "sudden death" similar to Ref. [6]. The entropy rapidly decays from a maximum value to zero with a subsequent revival after the initial death. When the number of oscillators in the bath increases, the moment of vanishing entropy occurs at an earlier time.

Figure 2 depicts the entropy evolution when $\kappa = g$. This is equivalent to the exceptional point of the non-Hermitian model. As $\kappa = g$, any dependence on either κ or g disappears as they only appear in the combination $g^2 - \kappa^2$ in the entropy. In this specific setting, the system decays asymptotically from maximal entropy to zero. The half-life of this decay decreases with the number of oscillators in the bath.

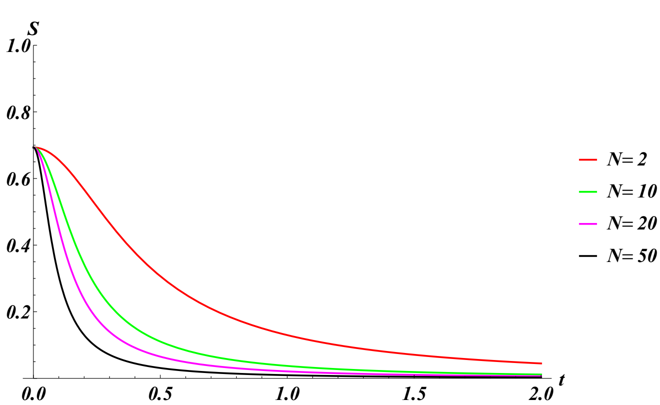

FIG. 2. von Neumann entropy as a function of time and varied bath size, with $c_1 = 1$, $g = \kappa$.

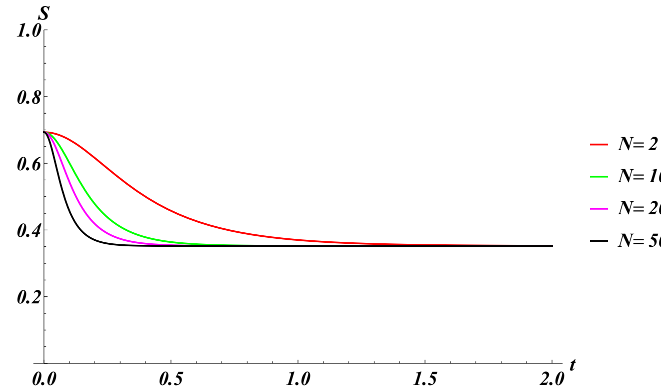

FIG. 3. von Neumann entropy as a function of time and varied bath size, with $c_1 = 1$, $g = 0.3$, $\kappa = 0.7$. The asymptote is at $S_{t\to\infty} \approx 0.3521$.

Figure 3 now shows the results of entropy evolution when $g < \kappa$. This is the spontaneously broken PT regime of the original time-independent non-Hermitian model. In this case the system once again decays asymptotically but in this instance the decay is to a nonzero value of entropy. In this way, the entropy is preserved eternally. Once again the half-life decreases with increasing N. The finite value that is asymptotically approached independently of N is

$$S_{t\to\infty} = -\tfrac{1}{2}(1+\xi)\ln\left[\tfrac{1}{2}(1+\xi)\right] - \tfrac{1}{2}(1-\xi)\ln\left[\tfrac{1}{2}(1-\xi)\right], \tag{46}$$

where

$$\xi = \frac{\sqrt{c_1^2 + g^2 - \kappa^2}}{c_1}. \tag{47}$$

We see the condition for the asymptote to exist is $|\frac{c1}{\sqrt{g^2-\kappa^2}}| > 1$, which matches the reality condition of μ in Eq. (38).

We have found three significantly different phenomena at $\kappa > g$, $\kappa = g$, and $\kappa < g$. Specifically, we see a change from rapid decay of entropy to zero, to asymptotic decay to zero, through to asymptotic decay to a nonzero entropy. This can be interpreted as crossing the PT exceptional point into the spontaneously broken regime of the original time-independent non-Hermitian system. However, with the existence of a time-dependent metric, the broken regime is no longer truly broken as we are able to provide a well-defined interpretation.

V. CONCLUSION

We derived a framework for the von Neumann entropy in non-Hermitian quantum systems and applied it to a simple system bath coupled bosonic model. In order to analyze the model we were required to find a time-dependent metric and we chose to solve the time-dependent Dyson equation for this. This method also gave us the equivalent Hermitian system which we worked with to perform the analysis as the framework showed the entropy was equivalent in both systems. The PT symmetry of the non-Hermitian system played an important role in the characterization of the regimes of different qualitative behavior in the evolution of the von Neumann entropy. We found three different types of behavior depending on whether we are in the PT-unbroken regime, at the exceptional point, or in the spontaneously broken PT regime. In the unbroken regime, the entropy underwent a rapid decay to zero. At subsequent times it was revived and continued this oscillatory behavior indefinitely. At the exceptional point, the entropy decayed asymptotically to zero, and in the spontaneously broken regime, the entropy decayed asymptotically from a maximum to a finite minimum (46) that remained constant with time.

Our findings may have implications for maintaining entanglement in quantum computers when the computer is operated in the spontaneously broken PT regime. The challenge here is to construct a system in a laboratory that mimics that of the non-Hermitian system presented here. However, non-Hermitian systems have been realized in quantum optical experiments (e.g., Refs. [36,37]) and so it is certainly possible that the same could be carried in quantum computing.

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