Radiative corrections to the decay rate of orthopositronium*

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The e^8 corrections to the decay rate of orthopositronium are calculated. For this calculation, as for bound-state problems in general, special attention must be given to infrared and Coulomb terms. The final result of this calculation is that the e^8 corrections yield a decrease in the lifetime of approximately 0.4% which represents a discrepancy of $2\frac{1}{2}$ standard deviations with the current experimental value.

The decay rate of orthopositronium as calculated by considering only diagram (a) of Fig. 1 is[1]

$$\Gamma_a = (2/9\pi)(\alpha^6 mc^2/\hbar)(\pi^2 - 9)\,, \tag{1}$$

where Γ_a is the reciprocal of the orthopositronium lifetime, m is the electron mass, α is the fine-structure constant, c is the velocity of light, and $\hbar$ is Planck's constant divided by 2π. The decay rate of orthopositronium has been measured[2] with an accuracy that is sensitive to the first radiative corrections to Eq. (1).

The method used to calculate these radiative corrections to the decay rate of orthopositronium[3,4] is essentially the same as the method used to calculate the radiative corrections to the decay rate of parapositronium.[5] In particular, the formulation of the decay matrix element and the resolution of the infrared and Coulomb problems are as in the case of parapositronium.

For an electron and a positron in a bound state the decay matrix element for electron-positron pair annihilation must be weighted over the relative momentum distribution of the electron and the positron to give the decay matrix element from the bound state:

$$\mathfrak{M} = \int \chi_{ab}(p) M(p)\, dp\ . \tag{2}$$

In Eq. (2), $\chi_{ab}(p)$ is the relative momentum distribution between particles "a" and "b" and $M(p)$ is (with an important exception which is discussed below) the plane-wave annihilation matrix element for the sum of all diagrams in Fig. 1. The relative momentum distribution is given by the Bethe-Salpeter equation. Alekseev[6] has given a convenient nonrelativistic solution of the Bethe-Salpeter equation. Retaining only components of the wave function (relative momentum distribution) through relative order α inclusive, the 16-component wave function is written as the direct product of four-component column matrices:

$$\chi_{ab}(p) = \delta^4(p_0 \mp \tfrac{1}{2}E \mp \vec{p}^{\,2}/2m)\begin{pmatrix} 1 \\ -\vec{\sigma}^a\cdot\vec{p}/2m\end{pmatrix} \times \begin{pmatrix} 1 \\ \vec{\sigma}^b\cdot\vec{p}/2m\end{pmatrix}\chi_{ab}^{(\mathrm{spin})}\,\varphi_0(\vec{p}),\quad t \gtrless 0 \tag{3}$$

where the components of $\vec{\sigma}$ are the Pauli matrices, $\chi_{ab}^{(\mathrm{spin})}$ is the direct product of the spin wave functions, $\varphi_0(\vec{p})$ is the Schrödinger wave function in momentum space and the indices a and b indicate that the various operators act only on particle a or b. In addition, E is the binding energy and p_0 is the zero component of the relative momentum four-vector. Writing $M(p)$ in terms of four two-

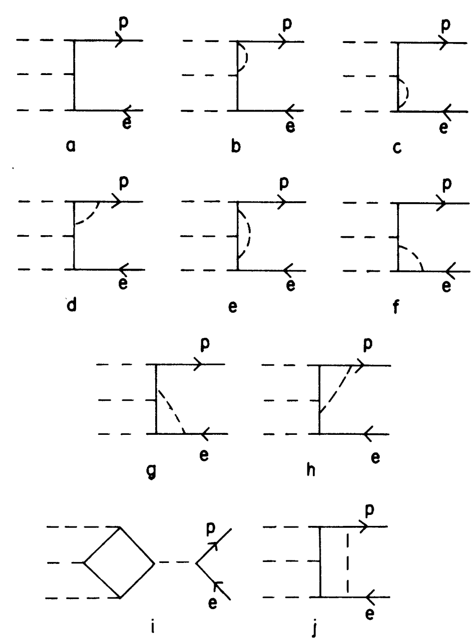

FIG. 1. Feynman diagrams of order e^3 and e^5 for the annihilation of an electron-positron pair. The solid and dotted lines represent electron-positron and photon lines, respectively. Time increases from right to left.

dimensional square matrices and noting that to the required order the δ function in Eq. (3) may be evaluated at $p_0=0$ since $E\sim 0(\alpha^2)$ and $\vec{p}^2/2m \sim O(\alpha^2)$, we obtain for Eq. (2)

$$\mathfrak{M}=\int d^3p\left(\chi_p^T M_{11}(\vec{p})\chi_e+\chi_p^T\frac{\vec{\sigma}_p^T\cdot\vec{p}}{2m}M_{21}(\vec{p})\chi_e-\chi_p^T\frac{\vec{\sigma}_e\cdot\vec{p}}{2m}M_{12}(\vec{p})\chi_e+O(\vec{p}^2\sim\alpha^2)M_{22}(\vec{p})\right)\varphi_0(\vec{p}), \tag{4}$$

where χ_e and χ_p are the two-component electron and positron spin wave functions, respectively. To obtain the bound-state decay matrix element to relative order α we may expand the plane-wave matrix element in powers of $\vec{p}$, and retain only the constant and linear terms in $\vec{p}$ since $v/c\sim O(\alpha)$ for hydrogenic systems. Then expanding $M_{11}(\vec{p})$, $M_{21}(\vec{p})$, and $M_{12}(\vec{p})$ about $\vec{p}=0$ we obtain

$$\begin{aligned}\mathfrak{M}&=\int d^3p\left[\chi_p^T\left(M_{11}(0)+p_i\frac{\partial M_{11}(0)}{\partial p_i}\right)\chi_e+\chi_p^T\frac{\vec{\sigma}_p^T\cdot\vec{p}}{2m}M_{21}(0)\chi_e-\chi_p^T M_{12}(0)\frac{\vec{\sigma}_e\cdot\vec{p}}{2m}\chi_e\right]\varphi_0(\vec{p})\\&=\left[\chi_p^T M_{11}(\vec{p}=0)\chi_e\psi(\vec{x}=0)+i\chi_p^T\frac{\partial M_{11}(\vec{p}=0)}{\partial p_i}\chi_e\frac{\partial\psi(\vec{x}=0)}{\partial x_i}\right.\\&\quad\left.+\left(\chi_p^T\frac{(\sigma_p^T)_i}{2m}M_{21}(\vec{p}=0)\chi_e-\chi_p^T M_{12}(\vec{p}=0)\frac{(\sigma_e)_i}{2m}\chi_e\right)\frac{\partial\psi(\vec{x}=0)}{\partial x_i}\right],\end{aligned} \tag{5}$$

where $\psi(\vec{x})$ is the coordinate space wave function corresponding to $\varphi_0(\vec{p})$. The orthopositronium ground-state wave function is even in $\vec{p}$: $\varphi_0(\vec{p})=\varphi_0(-\vec{p})$. Thus,

$$\mathfrak{M}=\chi_p^T M_{11}(0)\chi_e\psi(\vec{x}=0)\ . \tag{6}$$

We mention that the last three terms of Eq. (5) contribute to the decay amplitude for positronium in a P state. In Eq. (6), $\chi_p^T M_{11}(0)\chi_e$ is just the plane-wave matrix element with the initial particles at rest. Thus, the bound-state decay matrix squared is

$$|\mathfrak{M}|^2=|\psi(0)|^2|M(0)|^2\ , \tag{7}$$

where $\psi(0)$ is the Schrödinger wave function evaluated at contact and $M(0)$ is the plane-wave annihilation matrix element for the sum of all diagrams in Fig. 1 evaluated for the initial electron and positron at rest.

In constructing the plane-wave matrix element squared, it is understood that we average over initial spins and multiply by $\frac{4}{3}$ since the singlet contribution is zero in the nonrelativistic limit and the statistical weight of the triplet states is $\frac{3}{4}$. In addition, we must integrate over final photon energies, sum over final photon polarizations, and multiply by the Bose factor 1/3!.

The detailed setting up of the phase-space integrals is given in the text by Jauch and Rohrlich.[7] Additional details are given by Holt[3] and Stroscio.[4] The reciprocal of the orthopositronium lifetime is

$$\Gamma=\frac{1}{72\pi}\frac{\alpha^6mc^2}{\hbar}\int_0^1 dK_1\int_{1-K_1}^1 dK_2\,X(K_1,K_2)\ , \tag{8}$$

where in terms of the final photon energies (ω_1, ω_2, and ω_3), $K_1=\omega_1/m$, $K_2=\omega_2/m$. Furthermore, ω_3 has been eliminated by use of the energy conservation relation $\omega_1+\omega_2+\omega_3=2m$. $X(K_1,K_2)$ is now defined in terms of the diagrams in Fig. 1.

The amplitude of the diagram of order e^3 in Fig. 1 is denoted by N and amplitudes of the diagrams of order e^5 are denoted by Q. Letting

$$\langle f|M^{(3)}|i\rangle=\bar{v}(p_2)Nu(p_1)\ , \tag{9a}$$

$$\langle f|M^{(5)}|i\rangle=\bar{v}(p_2)Qu(p_1)\ , \tag{9b}$$

where p_1 and p_2 are the electron and positron momentum four-vectors, respectively, and u and v are the Dirac spinors we have,

$$\begin{aligned}\langle f|M^{(3)}+M^{(5)}|i\rangle^2&=|\bar{v}(p_2)(N+Q)u(p_1)|^2\\&=[\bar{v}(p_2)(N+Q)u(p_1)]\\&\quad\times[\bar{u}(p_1)(\bar{N}+\bar{Q})v(p_2)]\ .\end{aligned} \tag{10}$$

Including the averages over initial spins gives,

$$\langle f|M^{(3)}+M^{(5)}|i\rangle^2\to\frac{1}{4}\sum_{s_1\pm}\sum_{s_2\pm}[\bar{v}_\alpha(p_2,s_2)(N+Q)_{\alpha\beta}\,u_\beta(p_1,s_1)][\bar{u}_\gamma(p_1,s_1)(\bar{N}+\bar{Q})_{\gamma\delta}v_\delta(p_2,s_2)]\ . \tag{11}$$

Then using,

$$\sum_{s\pm}u_\alpha(p_1,s)\bar{u}_\beta(p_1,s)=\left(\frac{\not{p}_1+m}{2m}\right)_{\alpha\beta}\ , \tag{12a}$$

$$\sum_{s\pm}v_\alpha(p_2,s)\bar{v}_\beta(p_2,s)=-\left(\frac{-\not{p}_2+m}{2m}\right)_{\alpha\beta}\ , \tag{12b}$$

we have

$$\langle f|M^{(3)}+M^{(5)}|i\rangle^2=-\tfrac{1}{4}\,\mathrm{tr}\left[(N+Q)\left(\frac{\not p_1+m}{2m}\right)(\bar N+\bar Q)\left(\frac{-\not p_2+m}{2m}\right)\right]$$

$$=-\tfrac{1}{4}\,\mathrm{tr}\left[N\left(\frac{\not p_1+m}{2m}\right)\bar N\left(\frac{-\not p_2+m}{2m}\right)\right]-\tfrac{1}{4}\,\mathrm{tr}\left[N\left(\frac{\not p_1+m}{2m}\right)\bar Q\left(\frac{-\not p_2+m}{2m}\right)\right.$$

$$\left.+Q\left(\frac{\not p_1+m}{2m}\right)\bar N\left(\frac{-\not p_2+m}{2m}\right)\right]+O(e^{10})\,. \qquad (13)$$

Defining N' and Q' by

$$N=-ie^3\epsilon_\mu\epsilon_\nu\epsilon_\lambda N'^{\mu\nu\lambda}\,, \qquad (14a)$$

$$Q=-ie^3\epsilon_\mu\epsilon_\nu\epsilon_\lambda Q'^{\mu\nu\lambda}\,, \qquad (14b)$$

we have

$$\langle f|M^{(3)}+M^{(5)}|i\rangle^2=-e^6\epsilon_\mu^{\lambda_1}\epsilon_\nu^{\lambda_2}\epsilon_\lambda^{\lambda_3}\epsilon_\rho^{\lambda_1}\epsilon_\sigma^{\lambda_2}\epsilon_\tau^{\lambda_3}\left\{\mathrm{tr}\left[N'^{\mu\nu\lambda}\left(\frac{\not p_1+m}{2m}\right)\bar N'^{\rho\sigma\tau}\left(\frac{-\not p_2+m}{2m}\right)\right]\right.$$

$$+\,\mathrm{tr}\left[N'^{\mu\nu\lambda}\left(\frac{\not p_1+m}{2m}\right)\bar Q'^{\rho\sigma\tau}\left(\frac{-\not p_2+m}{2m}\right)+Q'^{\rho\sigma\tau}\left(\frac{\not p_1+m}{2m}\right)\right.$$

$$\left.\left.\times\bar N'^{\mu\nu\lambda}\left(-\frac{\not p_2+m}{2m}\right)\right]+O(e^{10})\right\}. \qquad (15)$$

Then using

$$\sum_\lambda \epsilon_\mu^\lambda\epsilon_\nu^\lambda\Rightarrow -g_{\mu\nu} \qquad (16)$$

to sum over photon polarizations, and noting that the two terms in the trace of order e^8 are complex conjugates of each other we have to the required order,

$$X=\frac{4m^4}{e^6}\sum_{s_1,s_2}\ \sum_{\lambda_1,\lambda_2,\lambda_3}\langle f|M^{(3)}+M^{(5)}|i\rangle^2$$

$$=m^4\,\mathrm{tr}\left[N'\left(\frac{\not p_1+m}{2m}\right)\bar N'\left(\frac{-\not p_2+m}{2m}\right)\right]$$

$$+2m^4\,\mathrm{tr}\left[\mathrm{Re}Q'\left(\frac{\not p_1+m}{2m}\right)\bar N'\left(\frac{-\not p_2+m}{2m}\right)\right], \qquad (17)$$

where the factor of $4m^4/e^6$ is inserted so that Eq. (17) corresponds to the integrand of Eq. (8).

Letting N correspond to the Feynman amplitude of diagram (a) in Fig. 1 we have,

$$N=-ie^3\epsilon_\mu\epsilon_\nu\epsilon_\eta\sum\left[\frac{\gamma^\mu(\not r_3+m)\gamma^\nu(\not q_1+m)\gamma^\eta}{4m^2K_3'K_1}\right], \qquad (18)$$

where $\sum$ means six permutations over photons, and

$$r_i=-p_2+k_i\,, \qquad (19a)$$

$$q_i=p_1-k_i\,, \qquad (19b)$$

$$K_i=p_1\cdot k_i/m^2\,, \qquad (19c)$$

$$K_i'=p_2\cdot k_i/m^2\,, \qquad (19d)$$

with k_1, k_2, and k_3 representing the four-momenta of the final-state photons. In addition, we write

$$Q=Q_{bc}+Q_{def}+Q_{gh}+Q_i+Q_j\,, \qquad (20)$$

where Q_{bc}, for example, represents the sum of the Feynman amplitudes of diagrams (b) and (c). Then in the Feynman gauge,

$$Q_{bc}=-ie^3\epsilon_\mu\epsilon_\nu\epsilon_\eta\sum_{\substack{\text{six}\\ \text{perms}}}\left(\frac{\gamma^\mu\Sigma(r_3)\gamma^\nu(\not q_1+m)\gamma^\eta}{K_3'K_1}+\frac{\gamma^\mu(\not r_3+m)\gamma^\nu\Sigma(q_1)\gamma^\eta}{K_3'K_1}\right), \qquad (21)$$

where,

$$\Sigma(q_i)=-2m^2K_i\,\mathrm{Re}\,i\Sigma_f(q_i)\,, \qquad (22a)$$

$$\Sigma(r_i)=-2m^2K_i'\,\mathrm{Re}\,i\Sigma_f(r_i)\,, \qquad (22b)$$

and

$$\Sigma_f(p)=\frac{\alpha}{2\pi mi}\left\{\frac{1}{2(1-\rho)}\left(1-\frac{2-3\rho}{(1-\rho)}\ln\rho\right)+\frac{(\not p+m)}{m}\left[\frac{1}{2\rho(1-\rho)}\left(-4+3\rho+\frac{4-4\rho-\rho^2}{1-\rho}\ln\rho\right)+\frac{2}{\rho}\int_{\lambda/m}^1\frac{dx}{x}\right]\right\}, \qquad (23)$$

with $\rho = -(p^2 - m^2)/m^2$ and λ the photon "mass." Equation (23) is given incorrectly in the literature in several places.[3,7] Upon minor rearrangement of the terms in the square bracket this result agrees with that of Chou and Dresden[8] who also note that Eq. (23) is given incorrectly in various works.

In addition,

$$Q_{def} = -ie^3\epsilon_\mu\epsilon_\nu\epsilon_\lambda \sum_{\substack{\text{six}\\ \text{perms}}} \left(\frac{\Lambda^\mu(-p_2, r_3)(\not r_3 + m)\gamma^\nu(\not q_1 + m)\gamma^\eta}{K_3' K_1} + \frac{\gamma^\mu(\not r_3 + m)\Lambda^\nu(r_3, q_1)(\not q_1 + m)\gamma^\eta}{K_3' K_1} + \frac{\gamma^\mu(\not r_3 + m)\gamma^\nu(\not q_1 + m)\Lambda^\eta(q_1, p_1)}{K_3' K_1}\right), \tag{24}$$

where after ultraviolet renormalization,

$$\Lambda_\mu(p', p) \to \Lambda_{\mu f} = -\frac{\alpha}{2\pi}\left(\Lambda_{1\mu} + \gamma_\mu\Lambda_2 - 2\gamma_\mu\Lambda_3\right), \tag{25}$$

with

$$\Lambda_{1\mu} = \int_0^1 dx\, x\int_0^1 dy\, K_\mu/a^2, \tag{26}$$

$$a^2/m^2 = x^2 + \rho x(1-x)(1-y) + \rho' x(1-x)y, \tag{27}$$

$$K_\mu = (\not p' - m)\gamma_\mu(\not p - m)(1-x) + \gamma_\mu C + (\not p' - m)M_\mu' + M_\mu(\not p - m) + k_\mu m x(1+x)(2y-1) + i\sigma_{\mu\nu}k^\nu m x(1-x). \tag{28}$$

The various factors in Eq. (28) are

$$C = (1-x)(1-x+xy)(p^2 - m^2) + (1-x)(1-xy)(p'^2 - m^2), \tag{29a}$$

$$M_\mu' = m(1-x^2)\gamma_\mu - (1-x)(1-xy)P_\mu - (1-x+2xy)(1-xy)k_\mu, \tag{29b}$$

$$M_\mu = m(1-x^2)\gamma_\mu - (1-x+xy)(1-x)P_\mu + (1-x+xy)(1+x-2xy)k_\mu, \tag{29c}$$

$$\sigma_{\mu\nu} = \tfrac{1}{2}i(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \tag{29d}$$

where

$$p = -(p^2 - m^2)/m^2, \tag{30a}$$

$$p' = -(p'^2 - m^2)/m^2, \tag{30b}$$

$$P_\mu = p_\mu' + p_\mu, \tag{30c}$$

$$k_\mu = p_\mu - p_\mu'. \tag{30d}$$

In Eq. (25) we also have,

$$\Lambda_2 - 2\Lambda_3 = -2\int_{\lambda/m}^1 \frac{dx}{x} + \frac{3}{2} + \frac{1}{2(1-\rho)(1-\rho')} + \frac{1}{\rho' - \rho}\left(\frac{\rho'(\rho'^2 + 4\rho' - 4)}{2(1-\rho')^2}\ln\rho' - \frac{\rho(\rho^2 + 4\rho - 4)}{2(1-\rho)^2}\ln\rho\right). \tag{31}$$

For diagrams (g) and (h) we have,

$$Q_{gh} = -ie^3\epsilon_\mu\epsilon_\nu\epsilon_\eta \sum_{\substack{\text{six}\\ \text{perms}}} \left(\frac{\Gamma'^{\mu\nu}(-p_2, k_3, k_2, q_1)(\not q_1 + m)\gamma^\eta}{4m^4 K_3' K_1} + \frac{\gamma^\mu(\not r_3 + m)\Gamma'^{\nu\eta}(r_3, k_2, k_1, p_1)}{4m^4 K_3' K_1}\right), \tag{32}$$

where

$$\Gamma'^{\mu\nu}(p', k', k, p) = \mathrm{Re}\, i2p_1\cdot k'\Gamma^{\mu\nu}(p', k', k, p). \tag{33}$$

Furthermore,

$$\Gamma_{\sigma\tau}(p', k', k, p) = \frac{\alpha}{2\pi}\frac{i}{2}\int_0^1 dv\, v(1-v)\int_0^1 dx\int_0^1 dy\left(\frac{N_{\sigma\tau}^{(0)}}{(a^2)^2} + \frac{N_{\sigma\tau}^{(2)}}{a^2}\right), \tag{34}$$

where

$$N_{\sigma\tau}^{(0)} = \gamma_\nu[-\not k' + m + \not p' vx + \not k' v - \not k(1-v)y]\gamma_\sigma(m + \not p' vx + \not k' v - \not k(1-v)y)\gamma_\tau[\not p - \not p' - \not k' + m + \not p' vx + \not k' v - \not k(1-v)y]\gamma^\nu, \tag{35a}$$

$$N_{\sigma\tau}^{(2)} = \gamma_\nu[\gamma_\sigma\gamma_\tau(\not p + m) + (\not p' + m)\gamma_\sigma\gamma_\tau + \gamma_\tau(\not p' + \not k')\gamma_\sigma - 2mg_{\sigma\tau}]\gamma^\nu - \gamma_\nu[\gamma_\sigma\gamma_\tau\gamma_p + \gamma_p\gamma_\sigma\gamma_\tau + \gamma_\tau\gamma_p\gamma_\sigma]\gamma^\nu[p'^p(1-vx) + k'^p(1-v) + k^p(1-v)y], \tag{35b}$$

$$a^2 = m^2(1-vx) - p'^2 vx(1-vx) - 2p'\cdot k' v(1-v)x - 2p'\cdot k v(1-v)xy - 2k\cdot k' v(1-v)y. \tag{35c}$$

For the photon-photon scattering diagram we have followed the treatment of Karplus and Neuman.[9] In addition, Shima's calculation[10] of photon splitting in a static field differs from the present work only in that a static photon is replaced by an internal photon. For diagram (i) we have

$$Q_i = -e\epsilon^{\mu}\epsilon^{\nu}\epsilon^{\lambda}\gamma^{\sigma}\frac{1}{k_4^2}\Pi_{\mu\nu\sigma\lambda}(k_1,k_2,k_3,k_4), \tag{36}$$

where k_4 is the four-momentum of the internal photon in diagram (i) of Fig. 1 and,

$$\begin{aligned}\Pi_{\mu\nu\sigma\lambda}(k_1,k_2,k_3,k_4) = {} & F_{\mu\nu\sigma\lambda}(k_1,k_2,k_3,k_4) \\ & + F_{\nu\mu\sigma\lambda}(k_2,k_1,k_4,k_3) \\ & + F_{\lambda\nu\sigma\mu}(k_3,k_2,k_4,k_1),\end{aligned} \tag{37}$$

where

$$F_{\mu\nu\sigma\lambda}(k_1,k_2,k_3,k_4) = -\frac{12e^4}{(2\pi)^4}\int d^4p\int_0^1 dx\int_0^x dy\int_0^y dz\,\frac{\mathrm{tr}[\gamma_\mu(\not p-\not l_1+m)\gamma_\nu(\not p-\not l_2+m)\gamma_\sigma(\not p-\not l_3+m)\gamma_\lambda(\not p-\not l_4+m)]}{(p^2-a^2)^4}, \tag{38}$$

where

$$l_1 = p_0, \tag{39a}$$

$$l_2 = p_0 + k_2, \tag{39b}$$

$$l_3 = p_0 - k_1 - k_3, \tag{39c}$$

$$l_4 = p_0 - k_1, \tag{39d}$$

$$p_0 = -k_1(1-y) - k_2(z-y) - k_3(x-y), \tag{39e}$$

$$\begin{aligned}a^2 = {} & 2k_1\cdot k_2(1-y)(y-z) + 2k_1\cdot k_3 y(x-y) \\ & + 2k_2\cdot k_3(x-y)(y-z) - m^2 .\end{aligned} \tag{39f}$$

The sum over the six permutations of photons is accomplished here by the three terms in Eq. (37). The other three terms differ only by the direction of the fermion loop momentum in diagram (i), Fig. 1, and may therefore be accounted for by a factor of 2 which has been inserted in Eq. (38). Equation (38) also contains a minus sign associated with the time ordering of the closed fermion loop.

Diagram (j) corresponds to the amplitude,

$$Q_j = -ie^3\epsilon_\mu\epsilon_\nu\epsilon_\eta\sum_{\substack{\text{six}\\\text{perms}}}K^{\mu\nu\eta}(-p_2,k_3,k_2,k_1,p_1), \tag{40}$$

where

$$K^{\mu\nu\eta}(-p_2,k_3,k_2,k_1,p_1) = -\frac{\alpha}{2\pi}\int_0^1 dv\,v(1-v)\int_0^1 du\,u^3\int_0^1 dy\int_0^1 dx\left(\frac{M^{\mu\nu\eta(0)}}{(a^2)^3}-\frac{M^{\mu\nu\eta(2)}}{4(a^2)^2}+\frac{M^{\mu\nu\eta(4)}}{2a^2}\right), \tag{41}$$

with

$$\begin{aligned}M^{\mu\nu\eta(0)} = {} & -4m^2\gamma^\mu(\not p_3-\not l_0+m)\gamma^\nu(\not q_1-\not l_0+m)\gamma^\eta + 2m[\gamma^\mu(\not p_3-\not l_0+m)\gamma^\nu(\not q_1-\not l_0+m)\gamma^\eta\not l_0 + \not l_0\gamma^\mu(\not p_3-\not l_0+m)\gamma^\nu(\not q_1-\not l_0+m)\gamma^\eta] \\ & + \gamma^\sigma\not l_0\gamma^\mu(\not p_3-\not l_0+m)\gamma^\nu(\not q_1-\not l_0+m)\gamma^\eta\not l_0\gamma_\sigma,\end{aligned} \tag{42a}$$

$$\begin{aligned}M^{\mu\nu\eta(2)} = {} & -2\gamma^\sigma(-\not p_2-\not l_0+m)\gamma^\mu(\not p_3-\not l_0+m)\gamma^\nu\gamma^\eta\gamma_\sigma \\ & + \gamma^\sigma(-\not p_2-\not l_0+m)\gamma^\mu\gamma^\tau\gamma^\nu(\not q_1-\not l_0+m)\gamma^\eta\gamma_\tau\gamma_\sigma - 2\gamma^\sigma(-\not p_2-\not l_0+m)\gamma^\mu\gamma^\nu\gamma^\eta(\not p_1-\not l_0+m)\gamma_\sigma \\ & + \gamma^\sigma\gamma^\tau\gamma^\mu(\not p_3-\not l_0+m)\gamma^\nu(\not q_1-\not l_0+m)\gamma^\eta\gamma_\tau\gamma_\sigma + \gamma^\sigma\gamma^\tau\gamma^\mu(\not p_3-\not l_0+m)\gamma^\nu\gamma_\tau\gamma^\eta(\not q_1-\not l_0+m)\gamma_\sigma \\ & - 2\gamma^\sigma\gamma^\mu\gamma^\nu(\not q_1-\not l_0+m)\gamma^\eta(\not p_1-\not l_0+m)\gamma_\sigma,\end{aligned} \tag{42b}$$

$$M^{\mu\nu\eta(4)} = -2(\gamma^\eta\gamma^\nu\gamma^\mu - 2\gamma^\nu g^{\mu\eta} + \gamma^\mu\gamma^\nu\gamma^\eta), \tag{42c}$$

$$\begin{aligned}a^2/u^2 = {} & m^2u - 2k_1\cdot k_2 v(1-uv)x - 2k_1\cdot k_3 vx[u(1-v)y+(1-u)] \\ & - 2k_2\cdot k_3 v[u(1-v)y+(1-u)] + 2p_2\cdot k_1(1-u)vx + 2p_2\cdot k_2(1-u)v \\ & + 2p_2\cdot k_3(1-u)[1-(1-v)y] + \lambda^2(1-u)/u,\end{aligned} \tag{42d}$$

$$l_0 = p_1 + p_2(1-u) - k_1(1-uvx) - k_2(1-uv) - k_3[1-u+u(1-v)y]. \tag{42e}$$

The terms in Eq. (42a) without powers of l_0 in the numerator: (i) contain an infrared factor that cancels the infrared factors in Eqs. (23) and (31), (ii) contain a Coulomb term that is linearly divergent in the nonrelativistic limit [see Eqs. (47) and (48)], and (iii) are finite only after we subtract the infrared and Coulomb terms and sum over all photon permutations as indicated in Eq. (40). Likewise, the terms in Eq. (42a) with one power of l_0 in the numerator must be written in "subtracted form" after the sum over photon permutations is performed in order to demonstrate cancellation of infinities.

The actual calculation of the diagrams in Fig. 1 was accomplished by using the symbol manipulation program REDUCE[11] for most of the algebraic calculations, and Gauss-Legendre integration algorithms to perform some of the necessary integrations (other integrals were done by hand). The results of these calculations in the Feynman gauge are,

$$\Gamma_{bc} = [4.785 \pm 0.010 + 4\ln(\lambda/m)](\alpha/\pi)\Gamma_a , \quad (43)$$

$$\Gamma_{def} = [-2.90 \pm 0.15 - 6\ln(\lambda/m)](\alpha/\pi)\Gamma_a , \quad (44)$$

$$\Gamma_{gh} = (-3.4 \pm 0.4)(\alpha/\pi)\Gamma_a , \quad (45)$$

$$\Gamma_i = (-0.5 \pm 0.2)(\alpha/\pi)\Gamma_a , \quad (46)$$

$$\Gamma_j = [3.8 \pm 0.4 + 2\ln(\lambda/m) + (\pi^2/v) + O(v^2 \ln(\lambda/m))](\alpha/\pi)\Gamma_a , \quad (47)$$

where Γ_a is given by Eq. (1) and the subscripts on Γ_{bc}, for example, indicate that both diagrams (b) and (c) of Fig. 1 are included. In addition, λ is the "photon mass" and v is the relative velocity of the electron and the positron. The numerical error estimates are determined by comparing numerical calculations with analytic expressions and by comparing Gauss-Legendre integration results for different numbers of integration points. We mention that diagrams (b)–(f), and most of (i) have been calculated independently by both of us. All differences between these calculations have been explicitly isolated and reconciled.

It is observed that all infrared factors in Eqs. (43)–(47) cancel with the exception of the term of order $v^2\ln(\lambda/m)$. This term is, however, of order $\alpha^2\ln(\lambda/m)$ since the Bohr momentum of positronium is of order α. This treatment of the infrared problem[12–14] could be improved by taking into account binding in intermediate states.[15,16]

The term π^2/v in Eq. (47) is due to the Coulomb part of the virtual photon interaction in diagram (j) of Fig. 1. The divergence of the Coulomb term represents a failure of the Born approximation and is corrected in the present calculation by the use of the Sommerfeld factor. It is known that a method of modifying plane-wave cross sections to *include* the effect of the Coulomb interaction is to multiply by the Sommerfeld factor, which to lowest order in α is given by,[5]

$$S = 1 + \pi\alpha/v . \quad (48)$$

Thus the Coulomb corrected cross section σ_c is given in terms of the plane-wave cross section σ by

$$\sigma_c = \sigma(1 + \pi\alpha/v) . \quad (49)$$

The last factor in Eq. (49) is exactly the Coulomb factor multiplying Γ_a in Eq. (47). Thus to *remove* the effect of the Coulomb interaction, which is included in the bound-state wave function of Eq. (7), we subtract the factor $(\pi\alpha/v)\,\Gamma_a$ from Eq. (47). That we should include only the transverse photon contribution from diagram (j) is also made evident by an examination of the Bethe-Salpeter equation.[3,4,5,17]

The photon-photon contribution, diagram (i) of Fig. 1, has been previously calculated by two different methods.[18] Our result for Γ_i, which was calculated using the method developed by Karplus and Neuman[9] for the photon-photon scattering part, is in agreement with the previous calculation[18] for the case where the Karplus-Neuman method is used. We mention that our error estimate is larger than that given in Ref. 18.

From Eq. (1) and Eqs. (43)–(47), the total decay rate to relative order α is

$$\Gamma = \frac{2}{9\pi}\,\frac{\alpha^6 mc^2}{\hbar}\,(\pi^2 - 9)[1 + (\alpha/\pi)(1.8 \pm 0.6)] = (0.7241 \pm 0.0010) \times 10^7\ \mathrm{sec}^{-1}. \quad (50)$$

This is to be compared with the experimental value,[2]

$$\Gamma_{\mathrm{expt}} = (0.7275 \pm 0.0015) \times 10^7\ \mathrm{sec}^{-1} . \quad (51)$$

There is thus a discrepancy in the amount of approximately $2\frac{1}{2}$ standard deviations in the experimental error. Future theoretical work, with the purpose of reducing the numerical error estimates of the present calculation, could possibly consist of replacing the present Gaussian integration algorithms by Monte Carlo algorithms and calculating the binding contributions via the method of binding in intermediate states. Comparison of these results with the present results would, at least, provide an independent check of the present work.

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