PHYSICAL REVIEW A VOLUME 10, NUMBER 2 AUGUST 1974

Radiation-reaction field versus free field

G. S. Agarwal
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay-5, India
(Received 14 November 1973)

Several aspects of the inseparable connection between the radiation-reaction field and the free field are discussed. The importance of the free-field terms in the calculation of two-time-correlation functions is pointed out.

A number of papers[1-5] have recently been devoted to the concept of operator radiation-reaction fields and the role played by such fields in spontaneous emission. The total electromagnetic field at the position of the atom, for the problem of a single two-level atom, is usually written in the form

$$\vec{E}^{\pm}(t) \cong \vec{E}_0^{\pm}(t) + \vec{E}_R^{\pm}(t), \tag{1}$$

where $\vec{E}_0(t)$ is the free field evolving according to the unperturbed Hamiltonian and $\vec{E}_R(t)$ is the radiation-reaction field (the superscripts $\pm$, as usual, denote the positive- and negative-frequency parts of the field operator). The radiation-reaction field is, in general, a functional of the atomic operators; i.e., the operator at time t is related to the atomic operators at earlier times or their derivatives. In the conventional approximations, the reaction field is taken to depend on the atomic operators at time t only. Working with Heisenberg equations of motion, Ackerhalt, Knight, and Eberly[1] interpreted spontaneous emission as due to the reaction-field terms only. It has been further pointed out[2,3] that spontaneous emission can be interpreted as either from $\vec{E}_0(t)$ or from $\vec{E}_R(t)$ or some combination of the two, depending on the ordering procedure adopted between the field operators and the atomic operators. Here we discuss several other aspects of the inseparable connection between the radiation-reaction field and the free field. In particular, we emphasize the following: (i) The damping and Lamb shift can be determined in terms of the correlation functions of the free field [cf. Eq. (7)]; (ii) the free-field terms in the equations of motion, which we refer to as random forces in line with the terminology used in the context of Langevin equations, are important in the computation of two-time-correlation functions even if the so-called normal ordering procedure is adopted [cf. Eq. (23)]; (iii) the correlation functions of the random forces of the Langevin equations are themselves determined in terms of the reaction field [cf. Eq. (26)]. This relation is the so-called second fluctuation-dissipation theorem.

Starting from the usual $-\int \vec{P}(\vec{r})\cdot\vec{E}(\vec{r})d^3r$ interaction [$\vec{P} = \vec{d}\delta(\vec{r})(S^+ + S^-)$, $\vec{d}$ is the dipole moment matrix element, and $S^\pm$ the spin angular momentum operators corresponding to a spin-$\frac{1}{2}$ system], one can show that the fields $\vec{E}_R^{\pm}$ are given, in Born and Markov approximations, by

$$-\vec{d}\cdot\vec{E}_R^+ = \tfrac{1}{2}(\Omega^+ + \Omega^-)S^- - \tfrac{1}{2}(\Omega^+ - \Omega^-)S^+ - i\gamma S^-, \tag{2}$$

$$= -(1/2i)[Q(\omega) - D(\omega)]S^+ - (i/2)[Q(\omega) + D(\omega)]S^-, \tag{3}$$

where γ is $\frac{1}{2}$ the Einstein A coefficient and $\Omega^\pm$ are the Lamb shifts defined by

$$\Omega^{\pm} = \pm\frac{2}{3}\frac{|d|^2}{\pi}\int k^3\,dk\,[(k+k_0)^{-1} \mp (k-k_0)^{-1}],$$
$$k_0 = \omega/c. \tag{4}$$

In (3) we have also introduced the one-sided Fourier transforms of the mean values of the commutator and the anticommutator of the free field, given by

$$Q(\tau) = \langle\{\vec{d}\cdot\vec{E}_0(\tau), \vec{d}\cdot\vec{E}_0(0)\}\rangle,$$
$$Q(\omega) = \int_0^\infty d\tau\, Q(\tau)e^{i\omega\tau}, \tag{5}$$

$$D(\tau) = \langle[\vec{d}\cdot\vec{E}_0(\tau), \vec{d}\cdot\vec{E}_0(0)]\rangle,$$
$$D(\omega) = \int_0^\infty d\tau\, D(\tau)e^{i\omega\tau}, \tag{6}$$

$$Q(\omega) = \gamma + i\Omega^+, \quad D(\omega) = \gamma + i\Omega^- \quad (\omega > 0). \tag{7}$$

Equation (7) shows the role played by the free-field correlations in determining the Lamb shift and the damping.[6] The following symmetry properties of Q and D are of some interest

$$\mathrm{Re}D(-\omega) = -\mathrm{Re}D(\omega), \quad \mathrm{Re}Q(-\omega) = \mathrm{Re}Q(\omega). \tag{8}$$

Note also, that from the dispersion relations and (8), one has

$$\Omega^{\pm}(\omega_0) = \pm\frac{1}{\pi}\int_0^\infty d\omega\,\gamma(\omega)[(\omega+\omega_0)^{-1} \mp (\omega-\omega_0)^{-1}]. \tag{9}$$

It should further be noted that $Q(t)$ and $D(t)$ are connected by the fluctuation-dissipation theorem.

The Heisenberg equation of motion for any atomic operator G is

$$\dot{G} = -i[G, H_0] - i[S^+ + S^-, G](\vec{d}\cdot\vec{E}), \qquad (10)$$

which can be rewritten in several different forms depending on one's taste and the convenience of calculation,

$$\dot{G} = -i[G, H_0] - i(\vec{d}\cdot\vec{E})[S^+ + S^-, G] \qquad (11a)$$

$$= -i[G, H_0] - i[S^+ + S^-, G](\vec{d}\cdot\vec{E}^+) - i(\vec{d}\cdot\vec{E}^-)[S^+ + S^-, G] \qquad (11b)$$

$$= -i[G, H_0] - i(\vec{d}\cdot\vec{E}^+)[S^+ + S^-, G] - i[S^+ + S^-, G](\vec{d}\cdot\vec{E}^-) \qquad (11c)$$

$$= -i[G, H_0] - (i/2)\{\vec{d}\cdot\vec{E}, [S^+ + S^-, G]\}. \qquad (11d)$$

Different forms have been used to interpret spontaneous emission due to reaction fields or due to free fields or some combinations of them. The point worth noticing[2,3] is that the mean-value equation is unique and so different forms should lead to the same equations for the expectation values. Indeed, if the equal-time commutation relations between the matter operators and the field operators were to hold, then one should have from (1)

$$[\vec{E}_R^{\pm}(t), G(t)] = -[\vec{E}_0^{\pm}(t), G(t)], \quad [\vec{E}_0^{+}(t), \vec{E}_R^{+}(t)] = [\vec{E}_0^{-}(t), \vec{E}_R^{-}(t)] = 0, \qquad (12)$$

for any atomic operator G. Note that we are doing an approximate calculation, the equal-time commutation relations should be valid to the same order to which the theory is applicable. If one uses (12) and the fact that the field is in vacuum state at $t=0$, i.e.,

$$\langle G\vec{E}_0^+(t)\rangle = \langle \vec{E}_0^-(t)G\rangle = 0, \qquad (13)$$

then it is easily shown that all the forms of (11) lead to the unique mean-value equation. For example, we have

$$\langle \vec{d}\cdot\vec{E}[S^+ + S^-, G]\rangle = \langle \vec{d}\cdot\vec{E}_R[S^+ + S^-, G] + \vec{d}\cdot\vec{E}_0^+[S^+ + S^-, G]\rangle,$$

which on using (13) becomes

$$\langle \vec{d}\cdot\vec{E}_R[S^+ + S^-, G]\rangle + \langle[\vec{d}\cdot\vec{E}_0^+, [S^+ + S^-, G]]\rangle,$$

which on using (12) reduces to

$$\langle \vec{d}\cdot\vec{E}_R[S^+ + S^-, G]\rangle - \langle[\vec{d}\cdot\vec{E}_R^+, [S^+ + S^-, G]]\rangle = \langle \vec{d}\cdot\vec{E}_R^-[S^+ + S^-, G]\rangle + \langle[S^+ + S^-, G]\vec{d}\cdot\vec{E}_R^+\rangle,$$

$$\langle \dot{G}\rangle = -i\langle[G, H_0]\rangle - i\langle[S^+ + S^-, G]\vec{d}\cdot\vec{E}_R^+\rangle - i\langle \vec{d}\cdot\vec{E}_R^-[S^+ + S^-, G]\rangle. \qquad (14)$$

The same conclusion is arrived at in Ref. 2 in a different manner. Equation (14) is of course trivially obtained[1,7] from (11b) and (13). If we further write $\vec{E}$ in terms of $\vec{E}_0$ and $\vec{E}_R$, then the terms involving $\vec{E}_0$ in (11) act as random forces whose mean values are not necessarily zero; i.e., it is true that

$$\langle \vec{d}\cdot\vec{E}_0\rangle = 0, \qquad (15a)$$

but

$$\langle [S^+ + S^-, G]\vec{d}\cdot\vec{E}_0\rangle \neq 0. \qquad (15b)$$

The situation is similar to the one in the classical theory of Langevin equations and Fokker-Planck equations. It is known that corresponding to a given Langevin equation there is one and only one Fokker-Planck equation, whereas the reverse is not true.[8] The equation of motion for the density operator is unique and the ordering problem does not arise in the density-matrix formulation. Using standard methods it can be shown that the reduced density operator in the interaction picture (which is obtained from the full density operator by taking the trace over the field variables), obeys the equation

$$\frac{\partial\rho}{\partial t} + \frac{1}{2}\int_0^\infty d\tau\,\{Q(\tau)[p(t),[p(t-\tau),\rho(t)]] + D(\tau)[p(t),\{p(t-\tau),\rho(t)\}]\} = 0,$$

$$p(t) = S^+ e^{i\omega t} + \text{H.c.}, \qquad (16)$$

where the conventional Born and Markov approximations have been made. The master equation (16) shows the role played by the correlation functions $Q(\tau)$ and $D(\tau)$ of the free field. On ignoring the counter-rotating terms (16) reduces to[9]

$$\frac{\partial\rho}{\partial t} + i(\omega + \Omega^+)[S^z, \rho] + \gamma(S^+S^-\rho - 2S^-\rho S^+ + \rho S^+S^-) = 0. \qquad (17)$$

The simplest Langevin equations to which (17) is equivalent are given by

$$\dot{S}^{\pm} = \pm i(\omega + \Omega^+)S^{\pm} - \gamma S^{\pm} + F^{\pm}(t), \qquad (18)$$

$$\dot{S}^z = -2\gamma S^+S^- + F^z(t). \qquad (19)$$

These have been obtained from the mean-value equations by just adding the fluctuating forces which have the properties

$$\langle F^{\pm}(t)\rangle = \langle F^z(t)\rangle = 0,$$
$$\langle F^z(t)F^{\pm}(t')\rangle = 2\langle \mathfrak{D}^{z\pm}\rangle\delta(t-t'), \qquad (20)$$
$$\langle F^{\mp}(t)F^{\pm}(t')\rangle = 2\langle \mathfrak{D}^{\mp\pm}\rangle\delta(t-t').$$

The diffusion coefficients are to be obtained from Einstein's relation and are found to be

$$\langle \mathfrak{D}^{z-}\rangle = \langle \mathfrak{D}^{++}\rangle = \langle \mathfrak{D}^{--}\rangle = \langle \mathfrak{D}^{+-}\rangle = 0,$$
$$\langle \mathfrak{D}^{-+}\rangle = \gamma, \quad \langle \mathfrak{D}^{z+}\rangle = \gamma\langle S^{+}\rangle. \tag{21}$$

The random forces appearing in (18) and (19) are associated with free fields. Although these free-field terms in the form (20) do not contribute to one-time mean values, they are nevertheless important in the calculation of two-time-correlation functions. For example, we have from (18)

$$S^{\pm}(t) = e^{(\pm i\omega_0 - \gamma)t} S^{\pm}(0) + \int_0^t d\tau\, F^{\pm}(t-\tau) e^{(\pm i\omega_0 - \gamma)\tau},$$
$$\omega_0 = \omega + \Omega^{+}, \tag{22}$$

and therefore in view of (20)

$$\langle S^{+}(t) S^{-}(t')\rangle = e^{i\omega_0(t-t') - \gamma(t+t')}, \tag{23a}$$

$$\langle S^{-}(t) S^{+}(t')\rangle = \int_0^t dt_1 \int_0^{t'} dt_2 \langle F^{-}(t-t_1) F^{+}(t'-t_2)\rangle \times e^{-\gamma(t_1+t_2) - i\omega_0(t-t')}$$
$$= e^{-i\omega_0(t-t')}(e^{-\gamma(t-t')} - e^{-\gamma(t+t')}), \quad t>t'. \tag{23b}$$

In deriving (23) we assumed that the atom was initially in its *upper state* and hence, for this initial condition, the contribution to $\langle S^{-}(t)S^{+}(t')\rangle$ is solely from the free-field terms. For the initial condition that the atom was in the *ground state*, we find that *both* the free-field terms and the radiation-reaction fields contribute, the result being

$$\langle S^{-}(t) S^{+}(t')\rangle = e^{-(i\omega_0 + \gamma)(t-t')}, \quad t>t'.$$

The above result shows that $\langle S^{-}(t)S^{+}(t)\rangle = 1$. Hence for all times the atom remains in the ground state; i.e., there is no possibility of spontaneous absorption.

To see the connection of the free-field terms to $\vec{E}_R$, we rewrite the Langevin equation in the form [cf. Eq. (14)]

$$\dot{G} = -i[G, H_0] - i[S^{+} + S^{-}, G](\vec{d}\cdot\vec{E}_R^{+}) - i(\vec{d}\cdot\vec{E}_R^{-})[S^{+} + S^{-}, G] + F_G. \tag{24}$$

The random force F_G has zero mean value. The correlation function of F_G is given by Einstein's relation (second fluctuation-dissipation theorem). When G is taken to be S^{-}, then the correlation function can be shown to be

$$\langle F^{+}(t) F^{-}(t')\rangle = 0,$$
$$\langle F^{-}(t) F^{+}(t')\rangle = 2\langle \mathfrak{D}^{-+}\rangle \delta(t-t'), \tag{25}$$

with

$$2\langle \mathfrak{D}^{-+}\rangle = i\langle [S^{+}, S^{-}][\vec{d}\cdot\vec{E}_R^{+}, S^{+}]\rangle + \text{c.c.} \tag{26}$$

Equation (26) shows explicitly the relation of the diffusion coefficient of the random force to the reaction field.

Similar remarks apply to the case of an atom with many levels as well as to the case of emission from a system of many atoms.

---

[1]J. R. Ackerhalt, P. L. Knight, and J. H. Eberly, Phys. Rev. Lett. 30, 456 (1973).

[2]P. W. Milonni, J. R. Ackerhalt, and W. A. Smith, Phys. Rev. Lett. 31, 958 (1973).

[3]I. R. Senitzky, Phys. Rev. Lett. 31, 955 (1973).

[4]R. Bullough, in *Coherence and Quantum Optics*, edited by L. Mandel and E. Wolf (Plenum, New York, 1973), p. 121.

[5]R. Bullough and R. Saunders, J. Phys. A (to be published).

[6]The fact that the Lamb shift of a two-level atom is determined from the anticommutator of the field operator was first pointed by R. Bullough and P. J. Caudrey, J. Phys. A 4, L41 (1971).

[7]The importance of normal ordering was also noted by R. H. Lehmberg, Phys. Rev. A 2, 883 (1970).

[8]See, e.g., M. Lax, Rev. Mod. Phys. 38, 541 (1966).

[9]G. S. Agarwal, Ref. 4, p. 157.