Thermalization of test particles in plasmas*

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A general form of the scattering distribution function is obtained through the use of the Schrödinger equation. The formulation is quite adequate in the study of particle-wave interactions, plasma radiation, fusion reactions, and nonelastic scattering as well as elastic scattering. The distribution function is obtained from a first Born approximation and is separated into two independent functions: a space-time-correlation function and a potential function. The former is of great interest in simulation of the dynamics of many-body systems and in the interpretation of scattering experiments. The potential function allows the use of different interaction potentials without the labor of repeating the calculations and with considerable reduction in the computation time. Analytic expressions for the rate of energy transfer from test particles to background plasmas are obtained together with the proper thermalization time. The dynamic behavior of the plasma is taken into consideration. The present formulation is compared with other different approaches. Results which are available in the literature lack the mathematical rigor of the treatment given here and are bound to stringent physical limitations. Consequently, the behavior of a test particle thermalizing in a background plasma is different from that anticipated in several studies, especially in the area of high-temperature plasmas.

I. INTRODUCTION

Analytic expressions have been derived for energy-transfer rates from fast test particles to slow background plasma particles. Classical,[1-3] semiclassical,[4] and quantum-mechanical[5-8] derivations result in differences in the argument of a logarithmic term common to all of the results obtained. Although the differences do not affect the order of magnitude of the rate of energy transfer, nevertheless they lead to different conclusions concerning other parameters, such as the slowing-down time and the dispersion of the initial distribution. In addition, there is a disagreement between experimental[9,10] and theoretical results[3] which tend to overestimate the energy-transfer rates. This discrepancy persists even when the theoretical results are corrected for magnetic field effects.[3,11,12] Other extreme cases have previously been considered such as infinite-mass test particles[7] and slow test particles in plasmas of very low kinetic temperature.[5]

Growing interest in neutral-beam heating in fusion devices[13] and plasma processing of ores and scrap material[14] requires information about the behavior of test particles injected into a background plasma whose electron velocity is greater than that of the test particle. This is aside from the fact that in studying internal heating of thermonuclear plasmas by charged fusion reaction products, plasma electrons are expected to be faster than test ions while plasma ions are slower. Understanding the behavior of test particles in this range is also necessary for studying the time history of fusion plasmas,[15] injection of neutral atoms in toroidal systems[16] and in mirrorlike devices,[17,18] and other plasma processes.[14,19]

To study energy exchange rates in a D-T fusion reactor, Rose[1] adapted a classical expression from the stellar dynamics work of Chandrasekhar[20] for the energy transfer from fast α particles to ions and electrons. His results are accurate within the limitations of the model that has been used. Other investigators have considered the effects of plasma oscillations and quantum mechanical corrections in an effort to obtain more accurate results[21,22]; however, close collisions were not properly treated in the analytical formulation. The dispersion of the initial impulse of the test particles is not taken into account in the numerical calculations.[23] In a recent work[24,25] a phase shift method is used with a cutoff length equal to the Debye radius; however, this work has also been subject to criticism.[5] Comprehensive reviews of different approaches and limitations have been reported by de Witt[5] and more recently by Kihara and Aono.[8]

In an earlier paper the slowing down of a fast test particle in a plasma is reported by Husseiny and Forsen[4] wherein investigation is limited to scattering by Maxwellian particles whose most probable speed is much less than the velocity of the test particle. In the present work these studies are extended to include the energy transfer to background plasmas from relatively slower test particles. However, a more general formalism is used here to allow for the investigation of the behavior of test particles of arbitrary velocity.

Starting from the two-body Schrödinger equation the general form of the scattering distribution is obtained in Sec. II, in a manner similar to that used by Van Hove.[26] The angular and energy distribution of scattering is evaluated for a first Born approximation. The result is then separated into two factors: a space-time-correlation G function which for given energy and momentum transfer depends only on the plasma properties, and a $\mathfrak{V}$ function which depends on the properties of the scattered particles and on the interaction potentials.

In Sec. III, elastic scattering of test particles in fully ionized plasmas is considered. The plasma ions are assumed to be totally stripped of their electrons. To evaluate the energy distribution function for elastic scattering, we adopted a method which has been suggested earlier by Husseiny and Forsen[27] for the calculation of scattering probabilities. This involves the separation of scattering events into two groups: events that lead to small energy-transfer increments and events corresponding to relatively larger energy-transfer increments. An interaction potential screened by Debye sphere is used to describe the former group, while an inverse potential of the Coulomb-type is used for the latter group. Energy distribution functions for elastic scattering are then obtained in an integral form for both potentials.

The integrals are evaluated in Sec. IV assuming fixed scattering centers; that is, in the slowing-down energy range. Thus, a comparison is made between the results of the present approach with the outcome of other methods of calculation. The thermalization range is then considered in Sec. V, wherein the thermal motion of the plasma is taken into account. The scattering distribution functions are used in Sec. VI to evaluate the total energy-transfer cross sections for elastic scattering of fast test particles with a faster electron background and with a slower ion background. Rates of energy transfer are then obtained in Sec. VII as first moments of the distribution functions. The mean energy-transfer time is derived in Sec. VIII for both the slowing down and the thermalization of test particles in plasmas. Effects of the Born approximation used in the derivation of the present results are discussed in Sec. IX. Corrections for plasma oscillations in both cases of interactions of test particles with ions and electrons are discussed in Sec. X. Finally, conclusions are given in Sec. XI and comparison between energy-transfer rates to ions and electrons in a Maxwellian plasma is made. A brief discussion is given to show the mathematical rigor of the present formulation in comparison to other formulations.

II. GENERAL FORMULATION

Let us consider the interaction of a fast test particle x with mass m_x and a plasma particle s with mass m_s through a potential $\mathfrak{V}(|\vec{r}_s - \vec{r}_x|)$, where $\vec{r}_s$ and $\vec{r}_x$ are the position vectors of the plasma particle and the test particle respectively. The Schrödinger equation for this two-particle system is

$$[-(\hbar^2/2m_s)\nabla_s{}^2 - (\hbar^2/2m_x)\nabla_x{}^2 + \mathfrak{V}(|\vec{r}_s - \vec{r}_x|)]\Psi(\vec{r}_s, \vec{r}_x) = (E+U)\Psi(\vec{r}_s, \vec{r}_x), \quad (1)$$

where $\Psi(\vec{r}_s, \vec{r}_x)$ is the wave function of the two-particle system, E is the initial energy of the incident test particle, and U is the initial energy of the plasma particle. Expanding $\Psi(\vec{r}_s, \vec{r}_x)$ in terms of the complete set of eigenfunctions of the unperturbed plasma particles gives

$$\Psi(\vec{r}_s, \vec{r}_x) = S_\nu \mathfrak{F}_\nu(\vec{r}_x)\phi_\nu(\vec{r}_s), \quad (2)$$

where $\phi_\nu(\vec{r}_s)$ is the wave function of the plasma particle that satisfies the equation

$$-(\hbar^2/2m_s)\nabla_s{}^2\phi_\nu(\vec{r}_s) = U_\nu\phi_\nu(\vec{r}_s) \quad (3)$$

and $\mathfrak{F}_\nu(\vec{r}_x)$ is the wave function of the x particle corresponding to the νth state.

Equation (1) can be transformed by means of Eqs. (2) and (3) to a set of coupled integrodifferential equations after multiplying by $\phi_\nu^*(\vec{r}_s)$ and integrating over all $\vec{r}_s$:

$$(\nabla_x{}^2 + k_\nu{}^2)\mathfrak{F}_\nu(\vec{r}_x) = (2m_x/\hbar^2) \times \int d\vec{r}_s \mathfrak{V}(|\vec{r}_s - \vec{r}_x|)\Psi(\vec{r}_s, \vec{r}_x)\phi_\nu^*(\vec{r}_s),$$

$$\nu = 0, 1, 2, \ldots, \quad (4)$$

where $\vec{k}_\nu$ is the wave vector corresponding to the νth state of energy of the test particle. Thus,

$$k_0^2 = 2m_x E/\hbar^2 \quad (5)$$

for the initial state and

$$k_\nu^2 = (2m_x/\hbar^2)(E + U - U_\nu) \quad (6)$$

for the final state of energy $E + U - U_\nu$. The Green's function

$$G(\vec{r}_x, \vec{r}_x') = -\frac{\exp(ik_\nu|\vec{r}_x - \vec{r}_x'|)}{4\pi|\vec{r}_x - \vec{r}_x'|} \quad (7)$$

satisfies Eq. (4) with the right-hand side replaced by $\delta(\vec{r}_x - \vec{r}_x')$, and thus can be used to convert this set of equations into the set of integral equations

$$\mathfrak{F}_\nu(\vec{r}_x) = -\frac{m_x}{2\pi\hbar^2}\int d\vec{r}_x' \int d\vec{r}_s \left(\frac{\exp(ik_\nu|\vec{r}_x - \vec{r}_x'|)}{|\vec{r}_x - \vec{r}_x'|}\right) \times \mathcal{V}(|\vec{r}_s - \vec{r}_x'|)\Psi(\vec{r}_s, \vec{r}_x')\phi_\nu^*(\vec{r}_s). \quad (8)$$

This set of equations can be solved by the formal method of iteration. If $\Psi(\vec{r}_s, \vec{r}_x)$ is replaced by the unperturbed function $\phi_0(\vec{r}_s)\exp(ik_0\hat{n}_0 \cdot \vec{r}_x)$ as a first approximation, the scattered amplitude is then given by

$$f_\nu^{(1)}(\theta_x, \Phi_x) = -\frac{m_x}{2\pi\hbar^2}\int d\vec{r}_x' \int d\vec{r}_s \exp[i(k_0\hat{n}_0 - k_\nu\hat{n}) \cdot \vec{r}_x'] \times \mathcal{V}(|\vec{r}_s - \vec{r}_x'|)\phi_0(\vec{r}_s)\phi_\nu^*(\vec{r}_s), \quad (9)$$

where the superscript (1) refers to the order of approximation.

The second-order approximation is obtained by inserting the wave function which is found by the first-order approximation, Eq. (8); and the scattered amplitude becomes

$$f_\nu^{(2)}(\theta_x, \Phi_x) = f_\nu^{(1)}(\theta_x, \Phi_x) + \left(\frac{m_x}{2\pi\hbar^2}\right)^2 \int d\vec{r}_x' \int d\vec{r}_s \int d\vec{r}_x'' \int d\vec{r}_s' \times \left(\frac{\exp[ik_\nu(|\vec{r}_x'| - \hat{n}\cdot\vec{r}_x')]}{|\vec{r}_x'|}\right)\mathcal{V}(|\vec{r}_s - \vec{r}_x'|) \times \exp[i(k_0\hat{n}_0 - k_\nu\hat{n})\cdot\vec{r}_x'']\,\mathcal{V}(|\vec{r}_s' - \vec{r}_x''|) \times \phi_0(\vec{r}_s')\phi_\nu^*(\vec{r}_s')\phi_\nu^*(\vec{r}_s). \quad (10)$$

Additional terms need to be included as the desired degree of accuracy increases considering that the series formed by successive approximations does not diverge. The validity criterion for using only the first term in the series is discussed in Sec. IX. In fact, this is just the first-order Born approximation which is considered here.

The differential cross section per unit solid angle for the x particle to change its energy from E to E' owing to scattering by a system of n_s particles is given in the first Born approximation as

$$\sigma(\xi \to \xi', \vec{\Omega} \to \vec{\Omega}') = m_x^2/(2\pi\hbar^2)^2(\xi'/\xi)^{1/2}\mathcal{V}^2(|\vec{\kappa}|) \times \sum_{a,b}\rho_{sa}|\langle b|\sum_{\gamma=1}^{n_s} e^{i\vec{\kappa}\cdot\vec{r}_{s\gamma}}|a\rangle|^2 \times \delta(\xi - \xi' - \mu_{sb} + \mu_{sa}), \quad (11)$$

where ξ and ξ' are the dimensionless energies of the x particle before and after scattering respectively. These are taken below as the ratios of E and E' to the kinetic temperature T_s of the s particles in energy units. The ratios μ_{sb} and μ_{sa} are those of the s-particle energies corresponding to the final state b and the initial state a to the mean plasma kinetic temperature, $\mathcal{V}(|\vec{\kappa}|)$ is the Fourier transform of the interaction potential $\mathcal{V}(|\vec{r}_s - \vec{r}_x|)$, and the momentum $\vec{\kappa}$ transferred from the test particle is given by

$$\vec{\kappa} = \vec{k} - \vec{k}', \quad (12)$$

where $\vec{k}'$ is the final wave vector of the test particle. The initial and final states include the initial and final spin states of the x-particle and s-particle systems. The Boltzmann factor ρ_{sa} accounts for the thermal motion of the plasma and thus it statistically weighs the initial states. It also includes the distribution of initial spin states of the s particles. The summation over γ is taken over the position vectors of the number of s particles per unit volume n_s and the δ function accounts for energy conservation. To account for different species of plasma particles Eq. (11) may be summed over s.

A result that is similar in form has been found using the Fermi pseudopotential for thermal neutron scattering.[26,28] In fact, this result is quite general and we notice that the matrix elements are independent of the potential which can therefore take any form. The advantage of this property in the present formulation has been discussed in detail elsewhere in connection with slow neutron scattering[28,29] and elastic scattering of x rays by the electrons of an atom.[30,31]

This formulation can be applied to elastic scattering, inelastic scattering, excitation, ionization, and can be extended to include the plasma oscillations. However, in this work we will limit ourselves to the problem of elastic scattering.

Expressing the δ function in its Fourier integral form, summing over the final states and averaging over the distribution of initial states, Eq. (11) becomes

$$\sigma(\xi \to \xi', \vec{\Omega} \to \vec{\Omega}') = (m_x/2\pi\hbar^2)^2(\xi'/\xi)^{1/2}\mathcal{V}^2(|\vec{\kappa}|) \times \int_{-\infty}^{\infty} dt\, \mathcal{G}(\kappa, t)\exp[-i(\xi - \xi')T_s t/\hbar], \quad (13)$$

where the function $\mathcal{G}(\kappa, t)$ is given by

$$\mathcal{G}(\kappa, t) = \exp[-\tfrac{1}{2}(\kappa/m_s)(T_s t^2 - i\hbar t)]. \quad (14)$$

The function $\mathcal{G}(\kappa, t)$ is the Fourier transform of the space-time pair-correlation function $G(r, t)$, which is given by

$$G(r,t)=\{m_s/[2\pi(T_s t^2-i\hbar t)]\}^{3/2}\times\exp[-\tfrac{1}{2}m_s r^2/(T_s t^2-i\hbar t)]\,. \quad (15)$$

This is a complex quantity and has a Gaussian form as a function of r, which is independent of the properties of the test particles. In the classical limit, that is as $\hbar\to 0$, Eq. (15) reduces to

$$G_{\rm cl}(r,t)=\left(\frac{m_s}{2\pi T_s t^2}\right)^{3/2}\exp\left(\frac{-m_s r^2}{2T_s t^2}\right), \quad (16)$$

where $G_{\rm cl}(r,t)$ is the classical space-time pair-correlation function. The G function is simply interpreted as the average density distribution at a time $t'+t$ as seen from a point where a particle passes at time t'. Equation (15) is valid for all times, since we are dealing with Brownian particles.[32]

The general properties of the space-time G function have been extensively discussed in the literature.[26,28,33,34] This function is useful in the interpretation of plasma scattering experiments[35] and in simulation of the dynamics of many-body systems,[32,33] and can be adapted to various applications in the field of plasma simulation.

III. SCATTERING DISTRIBUTIONS

Integrating Eq. (13) over the time domain we obtain

$$\sigma(\xi\to\xi',\vec{\Omega}\to\vec{\Omega}')=\left(\frac{2m_x^7}{\pi^3\kappa T_s}\right)^{1/2}\frac{\epsilon_{sx}}{\hbar^4}\left(\frac{\xi'}{\xi}\right)^{1/2}\mathcal{V}^2(\kappa)\times\exp\left[-\left(\frac{\hbar\kappa}{2\epsilon_{sx}(2m_xT_s)^{1/2}}-\frac{\epsilon_{sx}(2m_xT_s)^{1/2}(\xi-\xi')}{4\hbar\kappa}\right)^2\right], \quad (17)$$

where $\epsilon_{sx}=(m_s/m_x)^{1/2}$. For plasmas of thermonuclear interest $\epsilon_{ii'}\sim 1$ for ions and $\epsilon_{ei}\sim 10^{-2}$ for electrons. This equation gives the angular and energy distribution for interactions via an arbitrary potential. Thus, elastic scattering distribution functions of test particles in a plasma background can be readily obtained from Eq. (17) by specifying the proper interaction potentials.

Customarily, an inverse Coulomb potential is used to describe elastic scattering between charged particles. The divergent nature of the inverse potential appears for zero momentum transfer and for zero energy transfer. Thus, the conditions that $\kappa>0$ and $(E-E')>0$ have to be imposed. An *ad hoc* cutoff must be assumed in order to calculate the elastic scattering distribution function. This cutoff is tantamount to excluding scattering events that result in very small increments of energy transfer. Other cutoff methods tend to overlook contributions from other scattering events. Consequently, in this work we use a technique that has been suggested by Husseiny and Forsen[27] to include all possible scattering angles. The approach involves the use of a screened potential for interactions leading to small scattering angles and an inverse potential for large-angle scattering events. Describing the interactions by these potentials involves nontrivial assumptions.[8] In addition, these potentials are restricted to interactions involving electrons and ions fully stripped of their electrons.[36,37] The merits of the technique employed here *vis-à-vis* the use of a single potential with some *ad hoc* cutoff are discussed in Ref. 27. The accuracy of the energy-transfer rates obtained by each technique is considered in Sec. XI.

The Fourier transform of a potential screened by a Debye radius λ_D can be shown to yield the form

$$\mathcal{V}(|\vec{\kappa}|)=4\pi\zeta\lambda_D^2/(|\vec{\kappa}|^2\lambda_D^2+1), \quad (18)$$

where ζ/λ_D is the strength of the potential. For plasmas of thermonuclear interest, the potential strength has a value of the order of 0.1 meV. In the limit of $\lambda_D\to\infty$, that is for no screening, we have the inverse potential, namely,

$$\mathcal{V}(|\vec{\kappa}|)=4\pi\zeta/\kappa^2. \quad (19)$$

The screened potential also reduces to the inverse potential for encounters taking place at distances $|\vec{r}_x-\vec{r}_s|\ll\lambda_D$, whatever the value of λ_D is. At distances of the order λ_D, the potential is a Coulomb potential reduced by $1/e$. For close encounters, the momentum transfer corresponds to $|\vec{\kappa}|\gg(1/\lambda_D)$, and consequently Eq. (18) reduces to Eq. (19). Thus the inverse potential may be used for energy transfers corresponding to

$$\sqrt{\xi}-\sqrt{\xi'}>\Delta, \quad (20)$$

where

$$\Delta=\hbar/(2m_xT_s\lambda_D^2)^{1/2}. \quad (21)$$

For plasmas of thermonuclear interest $\Delta\sim 10^{-8}$. Even if the inequality in Eq. (20) is replaced by an equality, the results of utilizing the inverse potential in Eq. (17) will not be in great error providing that the first Born approximation is still valid. The use of Eq. (19) in the first Born approximation together with a cutoff of the interaction range to a distance equal to the de Broglie wavelength $\lambda\!\!\!^-$, is equivalent to Fokker-Planck-type calculations. The use of Eq. (18) in the same approximation with a short-range cutoff is equivalent to the Lenard-Balescu-type calculations provided that the dynamical behavior of the plasma is taken into account. Close encounters are treated correctly for all plasma temperatures and incident energies if Eq. (18) is used to calculate the scattering cross section quantum mechanically without approximations. If, in addition, the excitation of plasma oscillations are neglected we are lead to the Boltzmann-type calculations.

With the screened potential of Eq. (18) used in Eq. (17), one may integrate over the angular dependence to obtain the cross section for changing the energy of the x particle from ξ to ξ'. Thus, we have

$$\sigma(\xi\to\xi')=\sqrt{\pi}\,\epsilon_{sx}(\zeta/T_s)^2\Delta^{-4}(\xi'/\xi)^{1/2}\int_{-1}^{1}d\cos\theta \times[\xi+\xi'-2(\xi\xi')^{1/2}\cos\theta]^{-1} \times[\xi+\xi'-2(\xi\xi')^{1/2}\cos\theta+\Delta^2]^{-2} \times\exp(-[(1-\epsilon_{sx}^2)\xi+(1+\epsilon_{sx}^2)\xi'-2(\xi\xi')^{1/2}\times\cos\theta]^2/\{4\epsilon_{sx}^2[\xi+\xi'-2(\xi\xi')^{1/2}\cos\theta]\}), \tag{22}$$

where θ is the deflection angle in the laboratory frame of reference. The integral in Eq. (22) can be put in a more convenient form by changing the variable of integration from $\cos\theta$ to $\mathcal{U}=\kappa\lambda_D$; thus

$$\sigma(\xi\to\xi')=\sqrt{\pi}\,\frac{1}{2\partial_{sx}\xi}\left(\frac{\zeta}{\Delta T_s}\right)^2\times\int_{\mathcal{U}_-}^{\mathcal{U}_+}d\mathcal{U}\left(\frac{\exp[-(\partial_{sx}\mathcal{U}-\mathcal{A}/\mathcal{U})^2]}{(1+\mathcal{U}^2)^2}\right), \tag{23}$$

where $\partial_{sx}=\Delta/(2\epsilon_{sx})$, $\mathcal{U}_+$ and $\mathcal{U}_-$ correspond to $\cos\theta=-1$ and $\cos\theta=+1$ respectively; that is

$$\mathcal{U}_\pm=|\sqrt{\xi}\pm\sqrt{\xi'}|/\Delta, \tag{24}$$

$$\mathcal{A}=(\xi-\xi')/(4\partial_{sx}). \tag{25}$$

For an interaction via an inverse potential the result is

$$\sigma(\xi\to\xi')=\sqrt{\pi}\,(2\partial_{sx}\xi)^{-1}(\zeta/\Delta T)^2\times\int_{\mathcal{U}_-}^{\mathcal{U}_+}d\mathcal{U}\,\mathcal{U}^{-4}\exp[-(\partial_{sx}\mathcal{U}-\mathcal{A}/\mathcal{U})^2]. \tag{26}$$

Equation (26) is also an approximation for Eq. (23) in the case of close encounters, that is, for $\mathcal{U}\gtrsim\mathcal{U}_-\gg1$, which is equivalent to the conditions given by Eq. (20).

The integral in Eq. (23) is evaluated in the Appendix, and the scattering distribution in a screened potential is given by

$$\begin{aligned}\sigma(\xi\to\xi')=\frac{\pi}{\partial_{sx}\xi}\left(\frac{\zeta}{2\Delta T_s}\right)^2\Bigg\{&e^{\xi-\xi'}[\mathrm{erf}(\partial_{sx}\mathcal{U}_++\mathcal{A}/\mathcal{U}_+)-\mathrm{erf}(\partial_{sx}|\mathcal{U}_-|+\mathcal{A}/|\mathcal{U}_-|)]\\&\times\Bigg[(\mathcal{A}+\tfrac{1}{2}\partial_{sx}\mathcal{B}_-^2-\mathcal{B}_-^3)-(1-4\mathcal{A}\mathcal{B}_-+2\mathcal{B}_-^2)\sum_{\beta=0}^{\infty}\sum_{\alpha=0}^{2\beta+3}\sum_{\iota=0}^{2\beta+3-\alpha}\frac{(-1)^{2\beta+3-\alpha+\iota}}{(2\beta+3)!!2^{3\beta-2\alpha+6}}\binom{2\beta+3}{\alpha}\\&\times\binom{2\beta+3-\alpha}{\iota}\xi^{2\beta+3-\alpha-\iota}\xi'^{\iota}\partial_{sx}^{2\alpha-2\beta-3}\Bigg]-[\mathrm{erf}(\partial_{sx}\mathcal{U}_+-\mathcal{A}/\mathcal{U}_+)-\mathrm{erf}(\partial_{sx}|\mathcal{U}_-|-\mathcal{A}/|\mathcal{U}_-|)]\\&\times\Bigg[(\mathcal{A}-2\partial_{sx}\mathcal{B}_+^2+\mathcal{B}_+^3)+(1-4\partial_{sx}\mathcal{B}_++2\mathcal{B}_+^2)\sum_{\beta=0}^{\infty}\sum_{\alpha=0}^{2\beta+3}\sum_{\iota=0}^{2\beta+3-\alpha}\frac{(-1)^{\iota}}{(2\beta+3)!!2^{3\beta-2\alpha+6}}\\&\times\binom{2\beta+3}{\alpha}\binom{2\beta+3-\alpha}{\iota}\xi^{2\beta+3-\alpha-\iota}\xi'^{\iota}\partial_{sx}^{2\alpha-2\beta-3}\Bigg]+(2/\sqrt{\pi})(2\partial_{sx}\mathcal{B}_--\mathcal{B}_-^2)\\&\times\left(\frac{\mathcal{U}_+\exp[-(\partial_{sx}\mathcal{U}_+-\mathcal{A}/\mathcal{U}_+)^2]}{1+\mathcal{U}_+^2}-\frac{|\mathcal{U}_-|\exp[-(\partial_{sx}|\mathcal{U}_-|-\mathcal{A}/|\mathcal{U}_-|)^2]}{1+|\mathcal{U}_-|^2}\right)\\&+(2/\sqrt{\pi})\sum_{\beta=0}^{\infty}\sum_{\alpha=0}^{2\beta+3}\sum_{l=0}^{2\beta+3-\alpha}\sum_{\iota=0}^{2\beta+3-\alpha}(-1)^{(\alpha+1)/2+\iota}\delta_{2\beta+3-\alpha,2l+1}\binom{2\beta+3}{\alpha}\binom{2\beta+3-\alpha}{\iota}\\&\times\xi^{2\beta+3-\alpha-\iota}\xi'^{\iota}\exp[\partial_{sx}^2+\tfrac{1}{2}(\xi-\xi')]\\&\times\Bigg[|\mathcal{U}_-|\exp[-(\mathcal{A}/|\mathcal{U}_-|)^2]\left(\exp[-(1+|\mathcal{U}_-|^2)\mathcal{R}^2]+\frac{1}{1+|\mathcal{U}_-|^2}\right)^{(\alpha-1)/2}\Bigg|_{\mathcal{R}=\partial_{sx}}^{\mathcal{R}=\infty}\\&-\mathcal{U}_+\exp[-(\mathcal{A}/\mathcal{U}_+)^2]\left(\exp[-(1+\mathcal{U}_+^2)\mathcal{R}^2]+\frac{1}{1+\mathcal{U}_+^2}\right)^{(\alpha-1)/2}\Bigg|_{\mathcal{R}=\partial_{sx}}^{\mathcal{R}=\infty}\Bigg]\Bigg\},\end{aligned} \tag{27}$$

where $\mathcal{B}_\pm = \partial_{sx} \pm \mathcal{A}$ and the other parameters are used in the same sense as in the Appendix.

Equation (26), on the other hand, can be solved in closed form and the result is

$$\begin{aligned}\sigma_I(\xi \to \xi') = \frac{\pi\zeta^2}{\epsilon_{sx}^2 T_s^2 \xi(\xi-\xi')^3} \{ & [1+\tfrac{1}{2}(\xi-\xi')][\mathrm{erf}(\partial_{sx}\mathfrak{U}_+ - \mathcal{A}\mathfrak{U}_+) - \mathrm{erf}(\partial_{sx}|\mathfrak{U}_-| - \mathcal{A}/|\mathfrak{U}_-|)] \\ & - e^{\xi-\xi'}[1-\tfrac{1}{2}(\xi-\xi')][\mathrm{erf}(\partial_{sx}\mathfrak{U}_+ + \mathcal{A}/\mathfrak{U}_+) - \mathrm{erf}(\partial_{sx}|\mathfrak{U}_-| + \mathcal{A}/|\mathfrak{U}_-|)] \\ & + (4/\sqrt{\pi})\mathcal{A}\mathfrak{U}_+^{-1}\exp[-(\partial_{sx}\mathfrak{U}_+ - \mathcal{A}/\mathfrak{U}_+)^2] - (4/\sqrt{\pi})\mathcal{A}|\mathfrak{U}_-|^{-1}\exp[-(\partial_{sx}|\mathfrak{U}_-| - \mathcal{A}/|\mathfrak{U}_-|)^2]\}, \end{aligned} \tag{28}$$

where the subscript I is used in the left-hand side to assign the cross section to energy transfer via an inverse potential. The parameter $\hbar$ cancels out in the derivation of Eq. (28) and consequently the results are valid in the classical limit as expected.

In the present model Eq. (28) is used in the range $|\sqrt{\xi} - \sqrt{\xi'}| \gg \Delta$ and Eq. (27) is used for the range $0 \lesssim |\sqrt{\xi} - \sqrt{\xi'}| \gtrless \Delta$. The solution of Eq. (27) for a screened potential can be expressed as the summation of the solution given by Eq. (28) for the inverse potential and additional terms which originate from the screening effect; that is,

$$\begin{aligned}\sigma_S(\xi \to \xi') = \sigma_I(\xi \to \xi') + \frac{2\pi\zeta^2 e^{(\partial_{sx}+\mathcal{A})^2}}{\epsilon_{sx}^2 T_s^2 \xi(\xi-\xi')^3}\int_{\partial_{sx}}^{\infty} d\mathcal{R}\{ & e^{-(\mathcal{R}-\mathcal{A})^2}[\mathrm{erf}(\mathcal{R}\mathfrak{U}_+ + \mathcal{A}/\mathfrak{U}_+) - \mathrm{erf}(\mathcal{R}|\mathfrak{U}_-| + \mathcal{A}/|\mathfrak{U}_-|)] \\ & \times([1-\tfrac{1}{8}(\xi-\xi')^2]\mathcal{R} - 2\mathcal{A}\mathcal{R}^2 + 2\mathcal{A}^2\mathcal{R}^3) - e^{-(\mathcal{R}+\mathcal{A})^2} \\ & \times[\mathrm{erf}(\mathcal{R}\mathfrak{U}_+ - \mathcal{A}/\mathfrak{U}_+) - \mathrm{erf}(\mathcal{R}|\mathfrak{U}_-| - \mathcal{A}/|\mathfrak{U}_-|)] \\ & \times([1+\tfrac{1}{8}(\xi-\xi')^2]\mathcal{R} + 2\mathcal{A}\mathcal{R}^2 + 2\mathcal{A}^2\mathcal{R}^3) - (4/\sqrt{\pi})\mathcal{A}\mathfrak{U}_+^{-1}\mathcal{R} \\ & \times\exp(-[(1+\mathfrak{U}_+^2)\mathcal{R}^2 - (1+\mathfrak{U}_+^{-2})\mathcal{A}^2]) \\ & + (4/\sqrt{\pi})\mathcal{A}|\mathfrak{U}_-|^{-1}\mathcal{R}\exp(-[(1+|\mathfrak{U}_-|^2)\mathcal{R}^2 - (1+|\mathfrak{U}_-|^{-2})\mathcal{A}^2])\}, \end{aligned} \tag{29}$$

where the subscript S is used to refer to scattering probabilities calculated by means of the potential given in Eq. (18).

IV. SLOWING-DOWN APPROXIMATION

In the slowing-down range, the plasma ions and the very cold electrons can be regarded as fixed scattering centers, that is $\epsilon_{sx}^2 \xi \gg 1$. For plasmas of thermonuclear interest, only a small fraction of the electron population have velocities less than those of the fusion reaction charged products and yet most of the plasma ions can be treated as fixed scattering centers until the reaction products degrade in energy to the limit at which such assumptions become invalid. For energetic injection into relatively cold plasmas, this approximation will be valid in some situations for most of the plasma particles until the injected particles start to thermalize with the background plasma. In this limit Eqs. (23) and (26) can be solved for the screened potential to give

$$\sigma(E \to E') \simeq \begin{cases} \pi\zeta^2/[\epsilon_{sx}^2 E T_s^2(\xi - \xi' + 4\partial_{sx}^2)^2], \\ \quad [(\epsilon_{sx}^2 - 1)/(\epsilon_{sx}^2 + 1)]^2 E \leqslant E' \leqslant E \\ 0, \text{ otherwise} \end{cases} \tag{30}$$

and for the inverse potential

$$\sigma(E \to E') \simeq \begin{cases} \pi\zeta^2/[\epsilon_{sx}^2 E(E-E')^2], \\ \quad [(\epsilon_{sx}^2 - 1)/(\epsilon_{sx}^2 + 1)]^2 E \leqslant E' < E \\ 0, \text{ otherwise.} \end{cases} \tag{31}$$

The result of Eq. (30) agrees with a result obtained semiclassically using a binary cross section of the shielded type which is calculated in the limit $\epsilon_{sx}^2 \xi \gg 1$ by the first Born approximation.[4] As expected, Eq. (30) reduces to Eq. (31) when $(\xi - \xi') \gg 4\partial_{sx}^2$. For electrons the limits may be better written as $0 < E - E' < 4\epsilon_{sx}^2 E$, where the upper limit corresponds to a head-on collision. In Eq. (31) the upper limit is not specified since the orbital model upon which Eq. (19) is based ceases to be valid as κ approaches zero and the scattering starts to be a diffraction process. One may also expect that Eq. (31) can be derived classically since the first Born approximation yields the classical result as the screening length becomes infinite. In fact, Eq. (31) can be recovered in the limit of $\lambda_D \to \infty$ from screened potential scattering distributions corresponding to any order of Born approximation. It is believed that if we include all finite terms of all orders in the complete Born series in the calculation of the screened potential scattering amplitude, the inverse potential scattering amplitude can still be obtained exactly in the limit of infinite screening radius.[38] This statement has been a major point of dispute between several investigators.[38-43]

V. THERMAL-RANGE APPROXIMATION

When the speeds of the interacting particles are comparable, test particles tend to gain and lose energy from the plasma particles until they come into thermal equilibrium. To study scattering in this range of energy we need to consider two separate groups of events: first, small deflection angle events and second, relatively large energy transfer increments per encounter which correspond to events that result in rather large deflections.

The quantity ∂_{sx} is small and in most practical cases we are not interested in very low energy test particles. Consequently, we limit our discussion to energies in excess of the quantity $T_s \partial_{sx}$.

For energy-transfer increments smaller than $4\partial_{sx}$, the series in Eq. (27) can be truncated to few terms, that is,

$$\begin{aligned}\sigma(\xi \to \xi') \simeq \frac{\pi\epsilon_{sx}\zeta^2}{2\Delta^3 \xi T_s^2} \Big(& e^{\xi-\xi'} \{\mathcal{A}[1 - \tfrac{1}{2}(\xi - \xi')] + \tfrac{2}{3}\partial_{sx}^3\} [\operatorname{erf}(\partial_{sx}\mathcal{U}_+ + \mathcal{A}/\mathcal{U}_+) - \operatorname{erf}(\partial_{sx}|\mathcal{U}_-| + \mathcal{A}/|\mathcal{U}_-|)] \\ & - \{\mathcal{A}[1 + \tfrac{1}{2}(\xi - \xi')] - \tfrac{2}{3}\partial_{sx}^3\} [\operatorname{erf}(\partial_{sx}\mathcal{U}_+ - \mathcal{A}/\mathcal{U}_+) - \operatorname{erf}(\partial_{sx}|\mathcal{U}_-| - \mathcal{A}/|\mathcal{U}_-|)] \\ & - [1 + 6\mathcal{U}_+^2(\mathcal{A}^2 - \partial_{sx}^2)]\, 2\exp[-(\partial_{sx}\mathcal{U}_+ - \mathcal{A}/\mathcal{U}_+)^2]/3\sqrt{\pi}\,\mathcal{U}_+^3 + 2|\mathcal{U}_-|[1 + 6(1 + |\mathcal{U}_-|^2)(\mathcal{A}^2 - \partial_{sx}^2)] \\ & \times \exp[-(\partial_{sx}|\mathcal{U}_-| - \mathcal{A}/|\mathcal{U}_-|)^2]/3\sqrt{\pi}\,(1 + |\mathcal{U}_-|^2)^2 \Big), \quad |\xi - \xi'| < 4\partial_{sx}. \end{aligned} \tag{32}$$

Equation (27) converges very slowly unless the energy transfer per encounter is negligibly small. Thus, for encounters resulting in large increments of energy transfer, the solution of Eq. (23) is better expressed in terms of the series

$$\begin{aligned}\sigma(\xi \to \xi') = \frac{\pi\zeta^2}{\epsilon_{sx}^2 T_s^2 \xi(\xi - \xi')^3 (1 + 1/\mathcal{U}_0^2)^2} \Bigg[& \bigg([1 + \tfrac{1}{2}(\xi - \xi')][1 + 2/(1 + \mathcal{U}_0^2) + 3/(1 + \mathcal{U}_0^2)^2 + \cdots] - \frac{4[3 + 6(\xi - \xi') + (\xi - \xi')^2]}{(\xi - \xi')(1 + \mathcal{U}_0^2)} \\ & \times [1 + 3/(1 + \mathcal{U}_0^2) + \cdots] \bigg) [\operatorname{erf}(\partial_{sx}\mathcal{U}_+ - \mathcal{A}/\mathcal{U}_+) - \operatorname{erf}(\partial_{sx}|\mathcal{U}_-| - \mathcal{A}/|\mathcal{U}_-|)] \\ & + e^{\xi - \xi'} \bigg([\tfrac{1}{2}(\xi - \xi') - 1][1 + 2/(1 + \mathcal{U}_0^2) + 3/(1 + \mathcal{U}_0^2)^2 + \cdots] \\ & \qquad + \frac{4[3 - 6(\xi - \xi') + (\xi - \xi')^2]}{(\xi - \xi')(1 + \mathcal{U}_0^2)} [1 + 3/(1 + \mathcal{U}_0^2) + \cdots] \bigg) \\ & \times [\operatorname{erf}(\partial_{sx}\mathcal{U}_+ + \mathcal{A}/\mathcal{U}_+) - \operatorname{erf}(\partial_{sx}|\mathcal{U}_-| + \mathcal{A}/|\mathcal{U}_-|)] + (4/\sqrt{\pi}) \\ & \times \mathcal{A}\{\mathcal{U}_+^{-1}\exp[-(\partial_{sx}\mathcal{U}_+ - \mathcal{A}/\mathcal{U}_+)^2] - |\mathcal{U}_-|^{-1}\exp[-(\partial_{sx}|\mathcal{U}_-| - \mathcal{A}/|\mathcal{U}_-|)^2]\} \\ & \times \left(1 + \frac{2[1 - 6/(\xi - \xi')]}{1 + \mathcal{U}_0^2} + \frac{3[1 - 12/(\xi - \xi')]}{(1 + \mathcal{U}_0^2)^2} + \cdots \right) \\ & - (32/\sqrt{\pi})\mathcal{A}^3 (1 + \mathcal{U}_0^2)^{-1} (\xi - \xi')^{-1} [1 + 3/(1 + \mathcal{U}_0^2) - \cdots] \\ & \times \{\mathcal{U}_+^{-3}\exp[-(\partial_{sx}\mathcal{U}_+ - \mathcal{A}/\mathcal{U}_+)^2] - |\mathcal{U}_-|^{-3}\exp[-(\partial_{sx}|\mathcal{U}_-| - \mathcal{A}/|\mathcal{U}_-|)^2]\} \Bigg], \end{aligned} \tag{33}$$

where $\mathcal{U}_0^2 = (\xi - \xi')/(2\partial_{sx})^2$ is the value of $\mathcal{U}^2$ at the peak of the exponential in the integral of Eq. (23). The series in Eq. (33) is obtained by expanding the denominator of the integrand by a Taylor series about $\mathcal{U}_0^{-2}$ and integrating term by term. The solution given by this equation actually reduces to Eq. (28) for $\mathcal{U}_0^2 > 1$, and therefore it can be used for energy transfer increments larger than $4\partial_{sx}$ to account for large deflections.

VI. ENERGY-TRANSFER CROSS SECTION

The total cross section for elastic scattering of an incident charged particle by plasma species is the normalization factor of the scattering distribution function $\sigma(\xi \to \xi')$ that describes such interaction. This was obtained semiclassically in Ref. 27 for all possible values of the ratio between the velocities of the interacting particles and plasma particles. In this reference it was suggested to use the first Born cross section for distant collisions up to some small angle χ_0 below which classical calculations are invalid. For larger angles, the Rutherford cross section is used. The same technique will be used here to obtain the total cross section for elastic scattering in the energy range $\epsilon_{sx}^2 < \xi < \epsilon_{sx}^{-2}$, which applies primarily to fast electrons. In this range one may use the approximate scattering distribution given in Eq. (32) for small energy transfer with the $\mathfrak{U}$ values given by

$$\mathfrak{U}_- = (\sqrt{\xi} - \sqrt{\xi'})/\Delta \tag{34}$$

which corresponds to no deflection, and

$$\mathfrak{U}_+ = [(\sqrt{\xi} - \sqrt{\xi'})^2 + \theta_0^2 \sqrt{\xi\xi'}]^{1/2}/\Delta = \mathfrak{U}_m \tag{35}$$

for an angle of deflection $\theta_0 \simeq \epsilon_{sx} \ll 1$ in the laboratory coordinate system.

For larger scattering angles, and consequently larger energy transfer, Eq. (33) is used with the $\mathfrak{U}$ values

$$\mathfrak{U}_- = \mathfrak{U}_m \tag{36}$$

and

$$\mathfrak{U}_+ = (\sqrt{\xi} + \sqrt{\xi'})/\Delta, \tag{37}$$

which corresponds to head-on collision. The resulting cross section is found to be

$$\sigma(E) \simeq 2\sqrt{\pi}\, \zeta^2 \epsilon_{sx}/(3\Delta^2 T_s^2 \sqrt{\xi}). \tag{38}$$

This result agrees, within a small numerical factor, with results obtained semiclassically.[27] Equation (38) is also valid for interaction with plasma ions providing the energies are such that the condition $\epsilon_{sx}^2 \xi < 1$ is still applicable.

For $\epsilon_{sx}^2 \xi > 1$, the total cross section is found to be

$$\sigma(E) = \pi\zeta^2/\Delta^2 T_s^2 \xi, \tag{39}$$

a result that can be derived directly from Eqs. (30) and (31) for the slowing-down approximation. In both limits the values of the cross section are dominated by the contribution from low energy-transfer terms, that is, $(\xi - \xi') < 4\partial_{sx}$. This is due to the nature of elastic scattering of charged particles via screened potentials. In such cases, the small deflections contribute the most to the scattering probabilities.

VII. RATES OF ENERGY TRANSFER

The rate of encounters between the x particles and the plasma electrons times the first moment of the scattering distribution yields the energy-transfer rate, that is,

$$\frac{dE}{dt} = -n_e \sigma \left(\frac{2\xi T_e^3}{m_x}\right)^{1/2} \frac{\int_0^\infty d\xi'(\xi - \xi')\sigma(\xi \to \xi')}{\int_0^\infty d\xi'\, \sigma(\xi \to \xi')}. \tag{40}$$

Using the same technique as in the scattering cross section calculations (Sec. VI), one finds for the rate of energy transfer

$$\left.\frac{dE}{dt}\right|_{x\to e} \simeq -\frac{4n_e}{3m_x}\left(\frac{2\pi m_e}{T_e}\right)^{1/2} \zeta_{ex}^2 \left(\xi - \frac{3}{2}\right)(\ln\Lambda_{1e} - 4) + O(\epsilon_{ex}^2), \quad \epsilon_{ex}\xi_{ex}^{1/2} < 1 \tag{41}$$

where

$$\Lambda_{1s} = 1 + 1/(\epsilon_{sx}^2 \xi). \tag{42}$$

In the presence of a large number of x particles in the plasma at different energies, gain and loss of energy due to encounters between the x particles and the plasma species will take place randomly until the x particles come into kinetic equilibrium around an average temperature T_x. The rate of energy transfer at $\xi < 1.5$ will be less sensitive to changes in E and instead will depend rather strongly on the electron kinetic temperature.

To complete the picture, we derive the rate of energy transfer to plasma ions in the range $\epsilon_{ix}\xi_{ix}^{1/2} > 1$ using Eq. (28); that is,

$$\left.\frac{dE}{dt}\right|_{x\to i} = -\frac{2\pi m_i}{m_i}\left(\frac{2m_x}{E}\right)^{1/2} \zeta_{ix}^2 \ln\Lambda_{2i}, \quad \epsilon_{ix}\xi_{ix}^{1/2} > 1 \tag{43}$$

where

$$\Lambda_{2s} = \frac{\epsilon_{sx}^2 \xi^{1/2}/\Delta}{1 + \epsilon_{sx}^2}. \tag{44}$$

The same expression has been derived semiclassically by Husseiny and Forsen.[4]

The two energy-transfer rates given in Eqs. (41) and (43) can be added to give the total energy transfer rate to the plasma for $(m_x T_i/m_i) < E < (m_x T_e/m_e)$. For energies in the range $(m_x T_e/m_e) < E < \infty$, Eq. (41) is replaced by an equation similar to Eq. (43) with the exchange of plasma ion parameters with plasma electron parameters. For $E < (m_x T_i/m_i)$, the use of the scattering distributions obtained in Sec. III will yield proper results; however, in practice such energy transfer rates are meaningless. A more valuable quantity will be the relaxation rate which has been studied by many investigators.[44]

VIII. MEAN-ENERGY TRANSFER TIME

The mean energy-transfer time is defined here as the time required for a charged test particle x to change its energy from E_{x0} to E_{xf} via encounters with the background plasma. This characteristic time is customarily referred to as the slowing-down time when the thermal speed of the background particles is on the average slower than the velocity of the test particle. In other situations, the time required to affect the energy change is called the thermalization time. Generally the mean energy-transfer time is obtained by integrating the proper energy-transfer-rate equation.

In the slowing down range, that is, when $\epsilon_{sx}\xi^{1/2} > 1$, Eq. (43) can be integrated to give the slowing down time τ_L:

$$n_s\tau_L \simeq \frac{1.25\times10^{11}A_s}{Z_x^2Z_s^2A_x^{1/2}} \times\left(\frac{E_{x0}^{3/2}\mathrm{H}(a_s^3E_{x0}^{3/2})}{\ln(a_sE_{x0}^{1/2})} - \frac{E_{xf}^{3/2}\mathrm{H}(a_s^3E_{xf}^{3/2})}{\ln(a_sE_{xf}^{1/2})}\right)\ \mathrm{sec/cm^3}, \tag{45}$$

where

$$a_s = 1.72\times10^{18}[A_s/(A_s+A_x)] \times[A_xT_e/(Z_s^2n_e)]^{1/2}\ \mathrm{keV^{-1/2}},\quad T_e>0 \tag{46a}$$

and

$$a_s = 1.72\times10^{4}[A_s/(A_s+A_x)](A_x/Z_s^2)^{1/2}\ \mathrm{keV^{-1/2}},\quad T_e\simeq0. \tag{46b}$$

A_s and A_x are the atomic numbers for species s and x, respectively, and E and T are given in keV. The function $\mathrm{H}(y)$ is given by the series

$$\mathrm{H}(y) = 1+\frac{1!}{y}+\frac{2!}{y^2}+\cdots. \tag{47}$$

Equation (45) is an approximate version of an expression that was derived earlier using a different approach.[4]

The thermalization time τ_T, on the other hand, is obtained from Eq. (41) for encounters with fast background electrons:

$$(n_s\tau_T)^{-1} \simeq 5\times10^{-13}(Z_x^2/A_x)T_e^{-3/2}\left\{\left[\ln\left(\frac{3.5+\ln(A_xT_e)-\ln E_{xf}}{3.5+\ln(A_xT_e)-\ln E_{x0}}\right)\right]^{-1} + \left[2.375\ln\left(\frac{27+\ln(T_e/A_xn_e)+\ln E_{x0}}{27+\ln(T_e/A_xn_e)+\ln E_{xf}}\right)\right]^{-1}\right\}\ \mathrm{cm^3/sec},\quad E_{x0}\leqslant 33A_xT_e. \tag{48}$$

It is observed that the thermalization time strongly depends on the kinetic energy of the background plasma electrons while the slowing-down time is more sensitive to changes in the test particle energy. Thus, in a multispecies plasma, the energy of the test particle can be transferred preferentially to one species rather than the others by the independent adjustment of T_e and E_{x0}. For example as T_e approaches zero all the test particle energy is transferred to electrons and the mean energy-transfer time is essentially the slowing-down time with the species s taken as electrons.

IX. VALIDITY OF FIRST BORN APPROXIMATION

The validity of the first Born approximation has been discussed extensively in the literature.[38,42] This approximation has been used to derive Eq. (11) and consequently any resulting error from the use of such approximation may propagate through the development of the scattering distributions given by Eqs. (23) and (26).

The use of the first Born approximation is restricted by two conditions. First, the screening radius must exceed the de Broglie wavelength of the incident wave. This is well satisfied for ions and electrons in all cases of interest in thermonuclear plasmas. Thus the relatively large screening radius will allow the incident wave to be scattered without significant distortion. In addition, the impact parameter of the closest approach has to be larger than the de Broglie wavelength, that is,

$$v_e \gg 2.25\times10^6 Z_x\ \mathrm{m/sec}, \tag{49}$$

where Z_x is the charge number of the x particle. For electron kinetic temperatures $T_e \gtrsim 10$ keV, condition (49) is well satisfied.

For neutral beam heating of low-temperature plasmas we find that classical results are invalid as will be shown in Sec. XI. Higher-order corrections to the first Born approximation are negligible even though condition (49) is only weakly satisfied for this case—for example, $v_e \gtrsim 10^7$ cm/sec in present day TOKAMAK devices.[45]

In plasma torches where the incident particles are heated by the background plasma the first Born approximation is found to be adequate for nonideal plasmas where the coefficient of fugacity is close to unity.[14] This situation is of interest in the utilization of plasma processes in materials separation.

The scattering amplitude using the second-order approximation of Eq. (10) has been evaluated for a screened potential in the cm system.[46] The contribution of the second term in the iteration process is calculated for forward and backward scattering and we find that it is negligibly small. Thus the first approximation is legitimate in the range chosen for these studies.

X. EFFECTS OF PLASMA OSCILLATIONS

The formulation given in Sec. II correctly treats the scattering problem within the limitations of the first Born approximations. However, the retardation effects in the dynamic screened potential are neglected in Sec. III and the excitation of plasma oscillations has not been considered in developing the different expressions for the scattering parameters. To consider these effects on the scattering distribution a different expression for the $\mathcal{G}$ function or its Fourier transform has to be obtained using the proper Heisenberg position operator. The expression given in Eq. (14) for $\mathcal{G}$ is actually the same if we represent the plasma by a harmonic oscillator and take the limit as the frequency of oscillation vanishes. To include contributions from the collective energy loss, the logarithmic term given by Eq. (44) for $\epsilon_{sx}\xi^{1/2} \gg 1$ needs to be modified to

$$\Lambda_{2s} = \epsilon_{sx}^3 \xi / [\Delta(1+\epsilon_{sx}^2)] . \tag{50}$$

For $\epsilon_{sx}\xi^{1/2} < 1$ the contribution of the plasma oscillations to the energy transfer rate vanishes.[5] Thus the results obtained here represent the collective rate of energy transfer in the range specified in their derivation.

XI. CONCLUDING REMARKS

The angular and energy distribution function of interaction via an arbitrary potential is obtained in Eq. (17) using a first Born approximation. Nevertheless, the formulation allows for higher-order approximations. For elastic scattering and at the energy ranges of interest in this work the use of the first Born approximation is found to be within the desired accuracy, as shown in Sec. IX. The result of Eq. (17) is the basic relation obtained in this work and is given in a form which can be tailored to evaluate the confinement time for fast fusion devices.[47] The result can also be used to calculate the transport coefficients for multicomponent plasmas. These are of special interest in separation processes wherein interactions take place between ions of arbitrary state of ionization and between neutrals and charged particles.[14] The quantum mechanical formulation improves upon the classical results[48,49] and is expected to provide close agreement with observations.[50–53] In addition, using the result of the present formalism with a proper model of the interaction potential in calculating the collision terms in the Boltzmann equations would give a rather accurate description of interactions between weakly ionized particles[37] and would adequately treat the Ramsauer effect[54] which significantly influences the plasma transport properties.[55] Thus, proper corrections for the neoclassical theory[56,57] can be sought.

Limiting the analysis to elastic scattering of test particles in fully ionized plasmas, the angular dependence of the distribution function is removed and the energy-dependent scattering probabilities are obtained. These are given by Eq. (28) for scattering via a Coulomb potential and in Eq. (29) for a Debye screened potential. The two probability functions are used to evaluate the relaxation parameters and the relations necessary for the analysis of test particle behavior in plasmas under various constraints.

In the slowing-down range the elastic scattering probability functions given by Eqs. (30) and (31) agree with the semiclassical results obtained in Ref. 4.

Taking the thermal motion of plasma particles into consideration, the elastic scattering probability function is given in a closed form by Eq. (32) for encounters leading to small energy-transfer increments. For encounters resulting in large increments of energy transfer, the scattering probability function is given by the series of Eq. (33). These results are the necessary relations which are required to evaluate any of the relaxation and energy-transfer parameters for the thermalization of test particles in Maxwellian plasmas.

The elastic scattering cross section is obtained from Eqs. (32) and (33) and is given in Eq. (38) in the thermalization range and by Eq. (39) for fixed scattering centers. These are in fair agreement with previous semiclassical results.[4,27]

For plasmas of thermonuclear interest, Eqs. (41) and (43) give handy expressions for the calculations of the energy-transfer rates from test particles to faster electrons and to slower ions, respectively. These expressions can be directly used to give the amount of energy transfer as well as the energy distribution of test particles released in plasmas. The result of Eq. (41) differs from the analytical expressions which have been used in the literature to calculate the energy-transfer rates in the thermalization range.[1,15] The disagreement is essentially due to the fact that the available analytical results are obtained in the classical limit using *ad hoc* cutoffs. However, comparison of Eq. (43) with earlier semiclassical results[4] obtained in the infinite mass approximation shows, as we expected, a perfect agreement.

The characteristic time of energy transfer between test particles and background plasmas is given by Eq. (45) for the slowing-down range. This can be regarded as the mean energy-transfer time between fast test ions and slower plasma particles. As we expected, the results are similar to the semiclassical expression which has been derived under the same conditions.[4] The agreement is due to the fact that the mean energy-transfer time is evaluated using the relation of the energy-transfer rate. In addition, inspection of Eq. (45) shows a dependence on the argument of the logarithm which appears in the corresponding expression of the energy-transfer rate, Eq. (43). Consequently, the mean energy-transfer time for test particles released in faster electrons background is not expected to agree with classical relations based on *ad hoc* cutoff.

To provide some basis for comparison between our results in the thermalization range and the classical analytical results which are often used in thermonuclear plasma calculations, we may reconsider the scattering distribution in Eq. (28) for scattering by an inverse potential. This is independent of $\hbar$ and the classical results are the same as those obtained in the first Born approximation. Choosing the values of $\mathfrak{U}_+$ and $\mathfrak{U}_-$ as given by Eq. (24) and calculating the energy-transfer rates by means of Eq. (28) in the energy range $\epsilon_{ex}^2 < \xi < \epsilon_{ex}^2$, gives the result

$$\frac{dE}{dt} = -\frac{8n_e}{3m_i}\left(\frac{2\pi m_e}{T_e}\right)^{1/2}\zeta^2(\xi - \tfrac{3}{2})\ln\Lambda_{3e}, \quad (51)$$

where

$$\Lambda_{3s} = 2\epsilon_{sx}^3\xi^{1/2}/\Delta. \quad (52)$$

The divergence of Eq. (28) at $\xi' = \xi$ is handled by cutting off the integral at $\xi' = \xi \pm 2\partial_{sx}$. Aside from the argument of the logarithm, Eq. (51) agrees with the classical result[58] for which the logarithmic argument is given by

$$\Lambda_{4s} = 12\pi n_e \lambda_{eD}^3 \quad (53)$$

or nine times the number of particles in a Debye sphere.

To investigate the α heating in a D-T reactor, Rose[1] used the results of Ref. 58 for energy transfer to electrons. For energy transfer to ions he used an equation that differs from Eq. (43) in the argument of the logarithmic term, which is taken as Λ_{4i}. Neglecting quantum mechanical effects as well as collective effects, his results showed that the energy transfer to ions and electrons will be equal at $\xi = 44.2$ or at $T_e = 80$ keV. Ions are expected to be dominantly heated by α at $T_e \geq 33$ keV if the equations for the rate of energy transfer are used in the energy balance while taking into account the finite plasma confinement time.[1] These values exceed those obtained in this work by more than a factor of 2.3. Therefore one expects that for electron kinetic temperatures significantly below 33 keV, the ions get most of the α heating, in agreement with previous investigations.[59,60]

The disagreement between the results of the present work and those currently used in thermonuclear plasma calculations are credited to the use of the quantum mechanical formulation and the fact that all scattering events are included in evaluating the scattering probability. The merits of these features can be examined by considering the condition for the validity of the classical results, that is

$$v_r \ll 2.25\times 10^6 Z_x, \quad (54)$$

where v_r is the relative velocity of the interacting particles. For α heating in thermonuclear plasmas of $T_e > 10$ keV such a condition is not applicable as shown in Sec. IX. For neutral heating of present generation TOKAMAK devices one expects that $v_r \gtrsim 10^7$ for the interaction of energetic beam protons with electrons. For interactions with plasma protons $v_r \gtrsim 10^6$ and therefore the classical results do not apply in any of the above situations.

In addition, cutoff techniques *vis-à-vis* the technique used in this work are susceptible to errors that have been discussed elsewhere.[27,61] To assess the uncertainties associated with these errors in the values of cross sections, energy-transfer rates and mean energy-transfer times, we may consider the mathematical development of some of the cutoff methods.

The cutoff parameters are often chosen to be dependent on variables such as the reduced mass or the energy of the test particles. The integrations involved in the derivation of the energy-transfer rates cover intervals that vary with the mass of the plasma species and with the test-particle energy. Consequently, final expressions are so inconsistent that it is meaningless to attempt to draw comparisons between the amount of energy which can be transferred to plasma electrons and that transferred to plasma ions or between thermalization rates at the initial energy and at some later time. These difficulties are encountered in the use of the Lenard-Balescu equation and also in the use of the Landau or RMJ form of the Fokker-Planck equations. For example, if an *ad hoc* short range cutoff is chosen as $\hbar/\mu v_r$, where v_r is the relative velocity of the test ion with respect to the velocity of the plasma particle, the integral over the inverse impact parameter covers a volume of integration given by a sphere of radius $\mu v_r/\hbar$. For a given integrand and a given relative velocity, the radius of the sphere in the case of test particle encounters with electrons is about four orders of magnitude smaller than the radius of integration in the case of encounters with plasma ions. The results of the integral in both cases is very sensitive to the difference between the intervals of the two integration processes and hence the results are largely affected.

Nevertheless, the choice of the cutoff in the work of Rose and Clark[58] which has been preceded by that of Spitzer[44] and Chandrasekhar[20] in a different context has merit over other *ad hoc* cutoffs chosen by other investigators.[7,23,24] Namely, the cutoff chosen in Refs. 44 and 58 depends only on the plasma electrons parameters.[62] Thus, for given electron parameters their results may be used to determine the energy-transfer rates from test particles to different plasma species, to follow the energy degradation of fast test particles, or to determine energy balance and heating rates. However, this is a special case and the results obtained here are applicable to a wide range of practical situations.

Another error which frequently appears in the literature is the inconsistency in the choice of the limits of the double integrals occurring in energy-transfer rate calculations. That is, the integral over the final energy or momentum and the integration over scattering angles or over the momentum transfer. In some of the investigations reviewed in the introduction,[7,22] the integration over the final momentum is broken into two intervals while that over the scattering angle includes angles from zero to π in each interval. The fact that the double integral is bounded and the integration area includes discontinuities makes it necessary to adjust the limits of both integrals consistently whenever one of them is broken to intervals.

In contrast, the mathematical rigor is not compromised in the approximations used in this work and the limits of the integrals are consistent as shown in Eqs. (34) through (37).

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APPENDIX

To evaluate the integral in the derivation of the scattering distributions via screened potential, Eq. (23), we may use the auxiliary integral

$$(1+\mathcal{U}^2)^{-2} = 2e^{(1+\mathcal{U}^2)\partial_{sx}^2}\int_{\partial_{sx}}^{\infty} d\mathcal{R}\,(\mathcal{R}^3 - \partial_{sx}\mathcal{R})e^{-(1+\mathcal{U}^2)\mathcal{R}^2}.$$

Thus the integral becomes

$$\int_{\mathcal{U}_-}^{\mathcal{U}_+} d\mathcal{U}\exp\{-[(\xi-\xi')/(4\partial_{sx}\mathcal{U})]^2\}\int_{\partial_{sx}}^{\infty} d\mathcal{R}\,(\mathcal{R}^3 - \partial_{sx}\mathcal{R}) \times e^{-(1+\mathcal{U}^2)\mathcal{R}^2}.$$

Reversing the order of integration and integrating over $\mathcal{U}$, we get

$$\tfrac{1}{4}\sqrt{\pi}\,e^{(\partial_{sx}+\mathcal{A})^2}\int_{\partial_{sx}}^{\infty} d\mathcal{R}\,(\mathcal{R}^2-\partial_{sx}^2)\{e^{-(\mathcal{R}-\mathcal{A})^2}[\mathrm{erf}(\mathcal{R}\mathcal{U}_+ + \mathcal{A}/\mathcal{U}_+) - \mathrm{erf}(\mathcal{R}|\mathcal{U}_-| + \mathcal{A}/|\mathcal{U}_-|)] + e^{-(\mathcal{R}+\mathcal{A})^2}[\mathrm{erf}(\mathcal{R}\mathcal{U}_+ - \mathcal{A}/\mathcal{U}_+) - \mathrm{erf}(\mathcal{R}|\mathcal{U}_-| - \mathcal{A}/|\mathcal{U}_-|)]\}.$$

The integration over $\mathcal{R}$ is then carried out term by term using the recurrence relations

$$I_n^{\mp} = \int_{\mathcal{B}_-}^{\infty} d\mathcal{R}\,(\mathcal{R}\mp\mathcal{A})^n e^{-\mathcal{R}^2}\,\mathrm{erf}(\mathcal{R}\mathcal{U}_\pm \pm \mathcal{A}/\mathcal{U}_\pm)$$

$$= -\frac{\mathcal{B}_\mp^{n+1}}{n+1}e^{-\mathcal{B}_\mp^2}\,\mathrm{erf}(\partial_{sx}\mathcal{U}_\pm \pm \mathcal{A}/\mathcal{U}_\pm) - \frac{2}{\sqrt{\pi}}\mathcal{U}_\pm e^{-\mathcal{A}(1+1/\mathcal{U}_\pm^2)}\int_{\partial_{sx}}^{\infty} d\mathcal{R}\,\frac{(\mathcal{R}\pm\mathcal{A})^{n+1}}{n+1}e^{-(1+\mathcal{U}_\pm^2)\mathcal{R}^2} + \frac{2I_{n+2}}{(n+1)}$$

and

$$\int dX\,X^\alpha e^{-VX^2} = \tfrac{1}{2}(-1)^{(\alpha-1)/2}[e^{-VX^2}+1/V]^{(\alpha-1)/2}, \qquad \alpha = 1, 3, 5, 7, \ldots$$

where the superscript $(\alpha-1)/2$ refers to the $[(\alpha-1)/2]$th derivative with respect to V.

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