New effects in the interaction of second sound with superfluid vortex lines*

James B. Mehl

Department of Physics, University of Delaware, Newark, Delaware 19711

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The force terms in the wave equation for second sound which arise from the interaction of rotons and vortex lines are recalculated. When the phase of the perturbation in the roton drift velocity owing to dragging of the rotons by the vortex lines is taken into account, new force terms are found. One of these terms predicts a decrease of the second-sound velocity proportional to the vortex density, in agreement with recent experiments on rotating helium and heat currents. The other term predicts a new contribution to the attenuation of second sound in rotating helium which may be observable.

I. INTRODUCTION

Explanation of measurements of second-sound attenuation in rotating helium II,[1] and of second-sound attenuation and temperature gradients in heat currents[2] was made possible by the addition of frictional force terms to the two-fluid hydrodynamic equations of Landau.[3] The frictional forces arise because of interactions between elementary excitations (primarily rotons) and superfluid vortex lines. The form of the force terms applicable to rotating helium was deduced by Hall and Vinen[4] in a calculation that related the force on vortex lines owing to roton scattering to the average superfluid and normal-fluid velocities. Vinen, constructing an approximate theory of superfluid turbulence along similar lines, developed a theoretical basis for the force terms applicable to heat currents.[5,6]

In this work, we reconsider the calculation of the force terms, and find that inclusion of a previously neglected effect leads to a satisfactory explanation of the decrease of the second-sound velocity which has been observed in rotation[7,8] and in heat currents.[9,10]

When superfluid vortices are subject to oscillating forces in a second-sound wave, the vortices oscillate, dragging the rotons in the vicinity of the line so that the average roton velocity near the vortex core differs from the normal-fluid velocity averaged over a region containing many vortex lines. This velocity perturbation was treated by Hall and Vinen, who found that the perturbation had a large effect on the calculated magnitude of the force term. In this work, we consider the effect of the relative phase of the velocity perturbation and the force on the normal fluid near the vortex. This consideration leads to two additional terms in the wave equation for second sound in rotating helium. One of the terms, and a similar term which applies to heat currents, predicts a decrease in the velocity of second sound which depends on the vortex density. The magnitude of the predicted effect is in close agreement with one of the experiments in rotating helium[8] and in satisfactory agreement with the experiments in heat currents.[10] The other new term predicts an additional contribution to the attenuation of second sound in rotating helium which may be observable.

In Sec. II expressions for the force on rectilinear vortex lines and for the macroscopic force terms in rotating helium are derived. The effects on second sound in rotating helium are considered in Sec. III. In Sec. IV, a discussion of the effects of superfluid turbulence on second sound is given.

II. MACROSCOPIC FORCE TERMS IN UNIFORMLY ROTATING HELIUM II

In this section, expressions for the macroscopic force terms arising from the interaction of second sound with vortex lines are derived. The expressions are given in terms of phenomenological scattering parameters which describe the force on a vortex line owing to roton scattering from the line. Although attempts have been made to calculate these parameters from first principles,[4,11,12] the parameters are usually treated phenomenologically.

The calculation is based on the treatment of the dynamics of a rectilinear vortex by Hall and Vinen,[4] Hall,[13,14] and Hillel, Hall, and Lucas.[15] Originally,[13] the force per unit length of line was assumed to be entirely due to the scattering of rotons from the line. In a reconsideration of the theory in 1970,[14] Hall proposed that a nonscattering contribution to the force, first predicted by Iordanskii,[16] acted only on the normal fluid. This led to a prediction that second sound would couple to first sound, but not vice versa. Later, Lucas[17] failed to observe a coupling of the required magnitude. This led to a reexamination of the theory by Hillel *et al.*,[15] who concluded that both the scattering force and the Iordanskii force are balanced by a Magnus force on the superfluid, so the vector sum of the forces acting on the normal fluid and superfluid is zero. Forces of this form will be assumed in this paper.[18]

We first calculate the frictional forces acting near a line, being careful to take into account the relative phases of the various oscillatory quantities. A time dependence $e^{i\sigma t}$ is assumed. We neglect vortex wave effects, and assume the lines are straight and parallel to the axis of rotation. For simplicity, we assume that the normal-fluid and superfluid velocities $\vec{v}_n$ and $\vec{v}_s$, the roton drift velocity at the vortex core, $\vec{v}_R$, and the vortex velocity $\vec{v}_L$ are normal to the axis of rotation, although the results of the calculation can be extended to apply to the case of arbitrary $\vec{v}_n$ and $\vec{v}_s$.[19] We also assume, following Hall,[14] that $\vec{f}_n$, the force on the normal fluid near the vortex lines is given by

$$\vec{f}_n = -D\vec{v}_{RL} - (D' - \rho_n\kappa)\hat{\omega}\times\vec{v}_{RL}\,, \tag{1}$$

where D and D' are parameters proportional to the scattering cross sections, $\hat{\omega}$ is a unit vector in the direction of the angular velocity $\vec{\omega}$, $\vec{v}_{RL} = \vec{v}_R - \vec{v}_L$, κ is the quantum of circulation, and ρ_n is the normal fluid density. The transverse force term proportional to $\rho_n\kappa$ is the Iordanskii force. Following Hillel *et al.*,[15] we assume a balance of forces near the line, with the force on the superfluid provided by the Magnus effect:

$$\vec{f}_s = -\vec{f}_n = \rho_s\kappa\hat{\omega}\times(\vec{v}_s - \vec{v}_L)\,, \tag{2}$$

where ρ_s is the superfluid density. This equation can be used to relate $\vec{v}_L$ to $\vec{v}_s$. Hall and Vinen[4] use a result from classical hydrodynamics, with an approximate correction for the finite roton-roton mean-free-path L, to relate the roton-drift velocity perturbation $(\vec{v}_R - \vec{v}_n)$ to $\vec{f}_n$:

$$\vec{v}_R - \vec{v}_n = \vec{f}_n/E\,, \tag{3}$$

where

$$\frac{1}{E} = \frac{-\ln(\frac{1}{2}\lambda L) - 1 - \frac{1}{4}i\pi}{4\pi\eta}\,, \tag{4}$$

η is the normal-fluid viscosity, and $\lambda = (\rho_n\sigma/\eta)^{1/2}$ is proportional to the reciprocal of the hydrodynamic penetration depth. This expression gives an approximate description of the dragging of rotons near the oscillating lines. The imaginary part of $1/E$, neglected by Hall and Vinen, corresponds to a velocity perturbation which is out of phase with $\vec{f}_n$. (Such phase shifts generally arise in the treatment of oscillatory motions of viscous fluids.) For typical experimental temperatures and second-sound frequencies, $-\ln(\frac{1}{2}\lambda L) - 1 \approx 7$, so the imaginary part of $1/E$ has a magnitude about a factor of 10 smaller than the real part. Although it is difficult to estimate the errors which may be introduced by the use of this expression for E (Hall[14] estimates the uncertainty at about 20%), the use of (4) leads to a calculated value of the decrease of the second-sound velocity in rotation which is in surprisingly good agreement with experiment.

Taking the vector product of $\hat{\omega}$ and (2), and using (1), we find

$$\vec{v}_s - \vec{v}_L = (D''/\rho_s\kappa)\vec{v}_{RL} - (D/\rho_s\kappa)\hat{\omega}\times\vec{v}_{RL}\,, \tag{5}$$

where $D'' = D' - \rho_n\kappa$. From (1) and (3) we obtain

$$\vec{v}_n - \vec{v}_L = \left(1 + \frac{D}{E}\right)\vec{v}_{RL} + \left(\frac{D''}{E}\right)\hat{\omega}\times\vec{v}_{RL}\,. \tag{6}$$

By subtracting (6) from (5) we obtain

$$\vec{q} = \vec{v}_s - \vec{v}_n = -y\vec{v}_{RL} + x\hat{\omega}\times\vec{v}_{RL}\,, \tag{7}$$

where

$$x = -\frac{D}{\rho_s\kappa} - \frac{D''}{E}\,, \quad y = 1 + \frac{D}{E} - \frac{D''}{\rho_s\kappa}\,. \tag{8}$$

To obtain $\vec{v}_{RL}$ in terms of $\vec{q}$, we invert (7) by taking the vector product of $\hat{\omega}$ and (7) and eliminating $\hat{\omega}\times\vec{v}_{RL}$ from the two resulting equations. The result is

$$\vec{v}_{RL} = -\frac{y\vec{q} + x\hat{\omega}\times\vec{q}}{x^2 + y^2}\,. \tag{9}$$

Substituting this into the expressions for the forces, we obtain

$$\vec{f}_s = -\vec{f}_n = -\frac{(yD - xD'')\vec{q} + (yD'' + xD)\hat{\omega}\times\vec{q}}{x^2 + y^2}\,. \tag{10}$$

Unfortunately, no useful simplification is achieved by separation of the real and imaginary parts of the factors involving x and y. Note, however, that such a separation will lead to two force terms proportional to q and two terms proportional to $\dot{q}$. One of the latter terms predicts a dependence of the second-sound velocity on vortex density; the other may have observable effects on the attenuation of second sound in rotating helium.

Note also that an expression for $\vec{v}_L$ can be obtained from (5) and (9), and that $\vec{v}_R$ can be obtained from (3) and (10). Generally, the phases of $\vec{f}_s$, $\vec{v}_R$, $\vec{v}_L$, and $\vec{q}$ are all different.

The macroscopic forces per unit volume are obtained by multiplying (10) by $2\omega/\kappa$, the density of the vortex lines per unit area. The macroscopic force constants (the various B's) are defined by the following equation:

$$\vec{F}_s = -\vec{F}_n = \frac{2\omega}{\kappa}\vec{f}_s$$
$$= \frac{\rho_s\rho_n}{\rho}[B_1\hat{\omega}\times(\vec{\omega}\times\vec{q}) + B_2\hat{\omega}\times(\vec{\omega}\times\dot{\vec{q}}/\sigma)$$
$$+B_1'\vec{\omega}\times\vec{q} + B_2'\vec{\omega}\times\dot{\vec{q}}/\sigma)] \,, \tag{11}$$

where $\vec{F}_s$ and $\vec{F}_n$ are the superfluid and normal-fluid forces per unit volume. The form of expression presented in (11) will be valid even if $\vec{q}$ is not normal to $\hat{\omega}$ providing that the cross sections for roton-vortex scattering in the $\hat{\omega}$ direction can be neglected, as experimental evidence indicates.[6,19] The macroscopic force constants are given by

$$B_1 + iB_2 = \left(\frac{2\rho}{\rho_s\rho_n\kappa}\right)\frac{yD - x(D' - \rho_n\kappa)}{x^2+y^2}$$
$$= \left(\frac{2\rho}{\rho_s\rho_n\kappa}\right)\frac{X}{X^2+Y^2}\,,$$
$$B_1' + iB_2' = -\left(\frac{2\rho}{\rho_s\rho_n\kappa}\right)\frac{xD + y(D' - \rho_n\kappa)}{x^2+y^2}$$
$$= \left(\frac{2\rho}{\rho_s\rho_n\kappa}\right)\frac{Y}{X^2+Y^2}\,, \tag{12}$$

where

$$X = \frac{D}{D^2 + (D' - \rho_n\kappa)^2} + \frac{1}{E}\,,$$
$$Y = \frac{-(D' - \rho_n\kappa)}{D^2 + (D' - \rho_n\kappa)^2} + \frac{1}{\rho_s\kappa}\,. \tag{13}$$

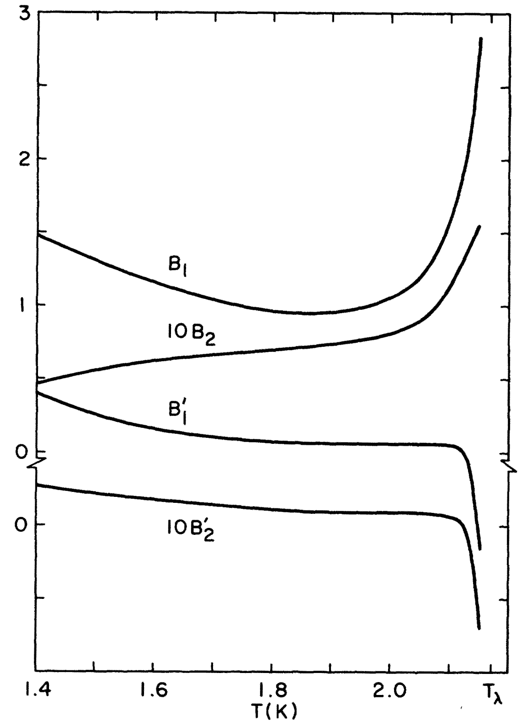

FIG. 1. Temperature dependence of the mutual friction parameters. The values of B_1 and B_1' were determined from an average of all available experimental data, and the values of B_2 and B_2' were calculated using the experimental values of B_1 and B_1', as described in the text. (The curve of $10B_2'$ is displaced 0.5 unit downward for clarity.)

If the imaginary terms in (12) are neglected, (11) reduces to the form for $\vec{F}_s$ first proposed by Hall and Vinen,[4] who determined values of B_1 from measurements of the increase of second-sound attenuation owing to rotation. The parameter B_1' can be determined from measurements of the frequency shift of certain modes of a second-sound resonator.[20,21] Lucas[21] has summarized all published measurements of B_1 and B_1'. The data presented in his paper were averaged to obtain the B_1 and B_1' curves in Fig. 1. These data were used to calculate the scattering parameters D and D' by inverting (12). (Omission of the imaginary parts of x and y has a negligible effect on B_1 and B_1'.) The values of D and D' were then used in a calculation of B_2 and B_2'. The results are shown in Fig. 1. The uncertainty in B_2 arising from scatter in the B_1 and B_1' data is about 10% at high and low temperatures, rising to approximately 20% near 1.9 K. The uncertainty in B_2' arising from the same sources is somewhat smaller in magnitude. Although the uncertainty in E arising from approximations in the derivation of (4) is hard to estimate, it is worth considering the effect of changes of E on the force parameters. If D and D' are held constant in (12), a change in the real part of $1/E$ will affect all the force parameters. However, if (12) is used to calculate B_2 and B_2' from experimental values of B_1 and B_1' as described above, a change in the real part of $1/E$ will shift the values of D and D' but will not affect B_2 and B_2'. A small fractional increase in the imaginary part of $1/E$ will, on the other hand, increase the calculated values of B_2 and B_2' by the same fraction.

Since E depends on the second-sound frequency through the λ-term in (4), (12) predicts that all of the force parameters will have a weak frequency dependence. For example, at 1.65 K, increasing σ from 5 to 10 kHz will increase the calculated values of B_1 and B_2 by 2.5% and 5%, respectively. This frequency dependence of B_1 has escaped detection; the range of scatter of all reported B_1 measurements is on the order of 10%.

Note that if the Iordanskii contribution in (1) is omitted, the $(D' - \rho_n\kappa)$ terms in (12) are replaced by D'. Thus the value of D' calculated from experimental values of B_1 and B_1' will be shifted, but the calculated values of B_2 and B_2' will not be affected by use of the new values in the modified form of (12).

Different expressions for the force parameters are found if the calculation is based on Hall's 1970 theory.[14] According to this theory, the force on the normal fluid near a vortex line is still given by (1), but only the scattering portion of the force is balanced by a Magnus force on the superfluid. Thus, different expressions for $\vec{F}_n$ and $\vec{F}_s$ are found. These are given in terms of additional macroscopic force parameters. The combinations of these parameters which have the same effect on second sound as those given by (12) are

$$B_1 + iB_2 = \left(\frac{2\rho}{\rho_s \rho_n \kappa}\right) \frac{yD - x(D' - \rho_s \rho_n \kappa/\rho)}{x^2 + y^2},$$
$$B_1' + iB_2' = -\left(\frac{2\rho}{\rho_s \rho_n \kappa}\right) \frac{xD + y(D' - \rho_s \rho_n \kappa/\rho)}{x^2 + y^2}, \tag{14}$$

where x is given correctly by (8), but a term ρ_n/ρ_s must be subtracted from the expression for y in (8). When these expressions are used to calculate D and D' from B_1 and B_1', the results agree with those obtained using (12) only at low temperatures. However, if the new values of D and D' are used in (14) to calculate B_2 and B_2', the differences are found to be insignificant except near T_λ, where (14) predict (i) a larger value of B_2, and (ii) a positive value of B_2'.

III. SECOND SOUND IN UNIFORMLY ROTATING HELIUM II

A. Wave equation

The wave equation for second sound is obtained from the linearized hydrodynamic equations

$$\rho_s \frac{\partial \vec{v}_s}{\partial t} = -\frac{\rho_s}{\rho}\vec{\nabla}p + \rho_s S \vec{\nabla} T + \vec{F}_s,$$
$$\rho_n \frac{\partial \vec{v}_n}{\partial t} = -\frac{\rho_n}{\rho}\vec{\nabla}p - \rho_s S \vec{\nabla} T - \vec{F}_s, \tag{15}$$

where S and p are the entropy and pressure. Together with the conservation equations for mass and entropy, these lead to the wave equation

$$\ddot{\vec{q}} = u_2^2 \vec{\nabla}\vec{\nabla}\cdot\vec{q} + B_1 \hat{\omega}\times(\vec{\omega}\times\dot{\vec{q}}) + B_2\hat{\omega}\times(\vec{\omega}\times\ddot{\vec{q}}/\sigma)$$
$$+ (B_1' - 2)\vec{\omega}\times\dot{\vec{q}} + B_2'\vec{\omega}\times\ddot{\vec{q}}/\sigma, \tag{16}$$

where we have included the one term linear in $\vec{\omega}$ which arises from transformation to a rotating coordinate system. Ordinary dissipative effects have been neglected.

B. Second-sound velocity in rotating liquid helium

Consider the effect of the B_2 term on the velocity of second sound. [The combined effect of all the friction terms in (16) will be considered in Sec. III C.] If we neglect the other friction terms, the wave equation is, for $\vec{q}\perp\hat{\omega}$,

$$\ddot{\vec{q}} = u_2^2 \vec{\nabla}\vec{\nabla}\cdot\vec{q} - B_2\omega\ddot{\vec{q}}/\sigma. \tag{17}$$

Thus the wave velocity $u_2(\omega, \sigma)$ depends on the angular velocity and second-sound frequency:

$$u_2(\omega, \sigma) = u_2(1 + B_2\omega/\sigma)^{-1/2} \approx u_2(1 - \tfrac{1}{2}B_2\omega/\sigma). \tag{18}$$

The approximation in (18) is valid since B_2 is of order 0.1 at all temperatures at typical second-sound frequencies, and $\omega/\sigma \ll 1$ under usual experimental conditions. Although the B_2 term appears to dominate (18) as σ approaches zero, this limit cannot be realized experimentally, for B_2 decreases (weakly) with decreasing σ owing to the λ term in the expression for E[(4)], and second sound cannot propagate for $\sigma \leq 2\omega$.[22]

The dependence of u_2 on both ω and σ in (18) has been verified by Lynall and Mehl[8] in measurements at 1.65 K. A comparison of their results with (18) is presented in Table I, which also lists the measurement of Vidal, leRay, and Francois at 1.63 K.[7] The u_2-vs-ω data of Lynall and Mehl are linear for $\omega < 4$ rad sec^{-1}, the maximum angular velocity used in their experiment. The slopes of their u_2-vs-ω plots are listed in Table I. The un-

TABLE I. Comparison of the predictions of Eq. (18) with experiments.

Investigators	T (K)	$\sigma/2\pi$ (Hz)	$-\Delta u_2(u_2\omega)^{-1}$ (ppm/rad sec^{-1}) Experiment	Calculated
Lynall and Mehl (Ref. 8)	1.65	654	6.8 ± 0.2	7.5 ± 0.4
		1198	4.1 ± 0.1	4.1 ± 0.2
		1935	2.0 ± 0.4	2.8 ± 0.1
Vidal *et al.* (Ref. 7)	1.63	740	13 ± 3	7.1 ± 0.8

certainties are estimated from the scatter of the measurements, with no allowance for possible systematic errors. The uncertainties in the calculated slopes stated in Table I are based on scatter in the B_1 and B_1' data used in the calculation of B_2. (For the 1.65-K data, this uncertainty is somewhat less than reported in Sec. II, since a value of B_1 measured by Lynall and Mehl in the same experiment was used.) Any uncertainty in the imaginary part of $1/E$ in (4) will contribute further to the uncertainty in B_2. This source of uncertainty may account for the slight discrepancy between the theoretical and experimental slopes in Table I.

The data of Vidal *et al.*[7] show a linear decrease of u_2 with ω at small ω, with a more rapid decrease above $\omega = 3\ \mathrm{rad\,sec^{-1}}$. The slope stated in Table I is based on a fit to the three points at $\omega < 3\ \mathrm{rad\,sec^{-1}}$. Vidal, leRay, Francois, and Lhuillier later report a quadratic dependence of Δu_2 on ω.[10] However, more recent measurements by the same group show a linear dependence for $\omega < 6\ \mathrm{rad\,sec^{-1}}$.[23] A decrease in the size of the effect with increasing second-sound frequency was also reported, although the frequency dependence was not stated.

The result of Vidal *et al.* shown in Table I is in disagreement with both the experiment of Lynall and Mehl and with the theory presented here. While their estimate of the uncertainty in individual measurements, 5 ppm, is about five times larger than the scatter in the measurements of Lynall and Mehl, the difference does not appear to be sufficient to account for the disagreement.

Vidal *et al.*[10] interpret their measurements in terms of a phenomenological model in which the B_2 term in the expression for $\vec{F}_s$ is replaced by a term proportional to $\vec{\nabla}T$. Since the temperature gradient $\vec{\nabla}T$ is proportional to $\vec{q}$ in a second-sound wave, both theories will predict a decrease of u_2. The theory presented here has the additional advantage of predicting the magnitude of the effect in terms of independently determinable parameters.

C. Other effects in rotating helium II

The effect of the B_2' term can be seen if we consider the combined effect of all the terms on the modes of a cylindrical second-sound resonator. This convenient geometry was used in the experiments of Lucas[21] and Lynall and Mehl.[8] Consider a cylinder of radius a with the cylinder axis coincident with the axis of rotation. The normal modes can be determined from the eigenvalue equation that is obtained from (16) if we take the time derivatives. [We assume a time dependence $e^{i\sigma t}$.] We obtain

$$-u_2^2\vec{\nabla}\vec{\nabla}\cdot\vec{q} + \sigma\omega(iB_1 - B_2)\vec{q} + \sigma[i(2-B_1') + B_2']\,\vec{\omega}\times\vec{q} = \sigma^2\vec{q}. \tag{19}$$

The effect of the friction terms can be calculated using perturbation theory.[24] The unperturbed equation

$$-u_2^2\vec{\nabla}\vec{\nabla}\cdot\vec{q} = \sigma^2\vec{q}\,, \tag{20}$$

has eigenvectors

$$\vec{q}_{ns\pm}(r,\theta) = \vec{\nabla}J_s(k_{ns}r)\exp(\pm is\theta)\,, \tag{21}$$

where $J_s(u)$ is the s-order Bessel function. The boundary condition at the cylinder wall, $dJ_s(u)/du = 0$ at $u = k_{ns}a$, determines the values of k_{ns} and the zero-order eigenfrequencies $\sigma_{ns} = k_{ns}u_2$. (We are neglecting the possibility of modes with a component of $\vec{q}$ in the $\hat{\omega}$ direction.) If we let P stand for the perturbing operators in (19), the first-order shift in the eigenfrequencies is given by

$$\delta(\sigma^2) = \int \vec{q}^{\,*}_{ns\pm}\cdot P\vec{q}_{ns\pm}\,dV\,. \tag{22}$$

Using (22), we find

$$\delta\sigma_{ns\pm} = i\omega\left(\tfrac{1}{2}B_1 \mp \frac{sB_2'}{k_{ns}^2a^2 - s^2}\right) - \omega\left(\tfrac{1}{2}B_2 \mp \frac{s(2-B_1')}{k_{ns}^2a^2 - s^2}\right)\,. \tag{23}$$

The unperturbed modes include degenerate pairs for $s \neq 0$. The real term in (23) predicts, in addition to the frequency decrease corresponding to (18), a splitting of the normal frequencies of the two modes in a degenerate pair. This splitting is due to the combined effect of the Coriolis acceleration and the B_1' term. Experimental values of B_1' have been determined from observations of this splitting by Lucas,[21] and also from observation of the splitting in a square resonator by Snyder and Linekin.[20]

The imaginary part of $\sigma_{ns\pm}$ corresponds to exponential decay, and is thus related to the quality factor Q of the resonator. Measurements of Q are used to determine B_1. Our calculation shows that such measurements only determine B_1 directly if an $s=0$ mode is used. Otherwise, an effective value proportional to the linear combination of B_1 and B_2' in (23) is determined. The Q's of the (+) and (−) modes of a degenerate pair will differ due to the B_2' term. This effect is largest for the $n=1$ modes. For the (1 1±) mode the factor $s(k_{ns}^2a^2 - s^2)^{-1}$ equals 0.418; for the (1 2±) modes it is 0.375. The fractional separation of the effective B_1 values for these degenerate mode pairs is thus approximately equal to $1.6B_2'/B_1$. Calculated values of the ratio B_2'/B_1 are less than 0.02

at all temperatures where data is available. The size of the effect is thus on the same order as the effect on B_1 of doubling the second-sound frequency. Refinements in experiments may make it possible to observe both effects. Since Lucas used the (1 2+) mode for his B_1 measurements at temperatures below 2.12 K,[21,25] the effect would shift his data by a small amount which is much less than the scatter in his measurements.

IV. SECOND SOUND IN THE PRESCENCE OF SUPERFLUID TURBULENCE

A. Force terms

Although small heat currents due to normal fluid-superfluid counterflow can be successfully described by the unmodified equations of two-fluid hydrodynamics, additional terms must be added to describe the increase in friction which occurs with large heat currents.[6] Vinen[5] attributed the increased friction to superfluid turbulence, and developed a theory to explain the results of his studies of the temperature gradients and second-sound attenuation associated with heat currents. In Vinen's model an equation for the frictional force per unit volume is obtained by multiplying an expression similar to (10) by an effective vortex density $\frac{2}{3}L$. An irregular, tangled mass of vortex lines is assumed. The length of vortex line per unit volume is L, and the factor of $\frac{2}{3}$ arises from an average over orientations. The vortex lines are assumed to move, on the average, with the superfluid velocity $\vec{v}_s$. This is reasonable since the effect of the Magnus force (2) might be expected to average to zero when averaged over the random orientations of vortex lines in turbulence.

In this section we will attempt to account for the observed decrease of the second-sound velocity in the presence of heat currents[9,10] by an extension of Vinen's theory. Because of the approximate nature of Vinen's theory, we should not expect to find close agreement between theoretical expressions and experimental results. The agreement turns out to be satisfactory, however.

In order to apply (10) to turbulence, we must first eliminate the effects of the Magnus force by expressing $\vec{f}_s$ in terms of $\vec{v}_L - \vec{v}_n$, and then approximating $\vec{v}_L$ by $\vec{v}_s$. We first consider the modification of (10), and then use Vinen's expression for L to find the macroscopic force terms in the second-sound wave equation.

The Magnus effect was included in our calculations by combining (5) and (6) to get (7). If instead, we invert (6) directly, we obtain an expression similar to (9):

$$\vec{v}_{RL} = -\frac{y_L(\vec{v}_L - \vec{v}_n) + x_L\hat{\omega}\times(\vec{v}_L - \vec{v}_n)}{x_L^2 + y_L^2}, \tag{24}$$

where

$$x_L = -(D' - \rho_n\kappa)/E, \quad y_L = 1 + D/E. \tag{25}$$

The force $\vec{f}_s$ can now be expressed in terms of $\vec{v}_L - \vec{v}_n$. The expression differs from (10) only in replacement of x by x_L, y by y_L, and $\vec{q}$ by $\vec{v}_L - \vec{v}_s$. In applying the equation to turbulence, we assume that the transverse terms average to zero, and that the macroscopic forces are obtained by multiplying the analog of (10) by $\frac{2}{3}L$. The effect on the wave equation can be obtained by replacing the vortex density $2\omega/\kappa$ by $\frac{2}{3}L$ in (16), dropping the transverse terms, and assuming that $\vec{v}_L$ equals $\vec{v}_s$ on the average:

$$\ddot{\vec{q}} = u_2^2\vec{\nabla}\vec{\nabla}\cdot\vec{q} - \tfrac{1}{3}B_{L1}\kappa L\dot{\vec{q}} - \tfrac{1}{2}B_{L2}\kappa L\ddot{\vec{q}}/\sigma. \tag{26}$$

We have introduced new parameters (B_{L1}, B_{L2}) which differ from the parameters for rotation by omission of the effects of the Magnus force. They are given by expressions analogous to (12) with x and y replaced by x_L and y_L. The differences between B_{L1} and B_1, and between B_{L2} and B_2, are less than 10% for $t < 2.0$ K.

Using a dimensional argument, Vinen related L to the average counterflow velocity v by

$$L^{1/2} = (\beta/\kappa)v, \tag{27}$$

where v is averaged over many cycles of second sound and β is a dimensionless parameter. (In Vinen's notation, $\beta = 2\pi\alpha = \pi B_{L1}\rho_n\chi_1/\chi_2$.) This expression is valid for large v, when turbulence is fully developed. The values of β determined from Vinen's experiments range from about 0.09 at 1.3 K to 0.25 at 2.0 K. By using (27) in the wave equation (26), we see that the second-sound attenuation and velocity reduction owing to a heat current are expected to have quadratic dependences on v:

$$\alpha' = \frac{\beta^2 B_{L1}}{6\kappa u_2}v^2, \tag{28}$$

$$\frac{\Delta u_2}{u_2} = -\frac{\beta^2 B_{L2}}{6\kappa\sigma}v^2. \tag{29}$$

Vinen (and others) generally found that the v^2 term in (28) had to be replaced by $(v - v_0)^2$ to agree with experiments, where v_0 is a small positive constant which tends to decrease with increasing channel size. Thus the theoretical expression (28) was expected to apply only in the limit of very wide channels. The physical meaning of the v_0 term is not fully understood, but it is probably related to the effects of the channel walls on vorticity production.[5]

B. Comparisons with experiments

Vidal *et al.*[10] have made simultaneous measurements of α' and Δu_2 in heat currents at 1.44 and 1.52 K. The direction of sound propagation was normal to the heat current to avoid entrainment effects.[26,27] Their results have been replotted in Fig. 2 in a form which facilitates comparison with (28) and (29). The attenuation data differ from similar data of Vinen in two ways: (i) A linear fit to $(\alpha')^{1/2}$ at high v extrapolates to a negative v_0, while all the published curves of Vinen show a positive v_0. (The magnitudes of the required values of v_0 are on the same order of magnitude as those found by Vinen.) (ii) The slope of the $(\alpha')^{1/2}$-vs-v curve at high v should be $(A\rho/2u_2)^{1/2}$ in terms of Vinen's parameter A; the actual slopes in Fig. 2 are about 55% of the values calculated using Vinen's values of A. Vinen found that his values of A were independent of channel size, but investigators in other laboratories find different values. For example, Brewer and Edwards[28] found that the values of A that they determined from measurements of temperature gradients in heat flow were independent of channel size but were about 50% larger than Vinen's values. Kramers[29] notes that the values of A determined in a variety of experiments are spread over a large range. Thus the slopes of the attenuation data in Fig. 2 are not inconsistent with other measurements, but we can expect difficulty in comparing (28) and (29) with experiments. The best procedure would seem to be to make direct comparisons of the attenuation- and velocity-reduction measurements from the same experiment. Such comparisons should eliminate the effect of vortex density. Regardless of the dependence of L on v, we would expect the ratio of α' to Δu_2 to be independent of v. This is clearly not the case. The velocity-reduction data appear to be consistent with the unmodified form of (29) at low v, rather than a form with a negative v_0 as required to fit the attenuation data. It is not clear whether the disagreements arise from a defect in the experiments or from a defect in the model. Meaningful, direct comparisons of α' and Δu_2 are, however, difficult. Instead the Δu_2 data have been compared directly with (29), using values of β calculated from Vinen's values of A. The solid lines in Fig. 2 are calculated from (29) in this way. If, alternately, values of β calculated from the slopes of the attenuation measurements in Fig. 2 are used, the slopes of the solid lines are smaller by a factor of about 65%.

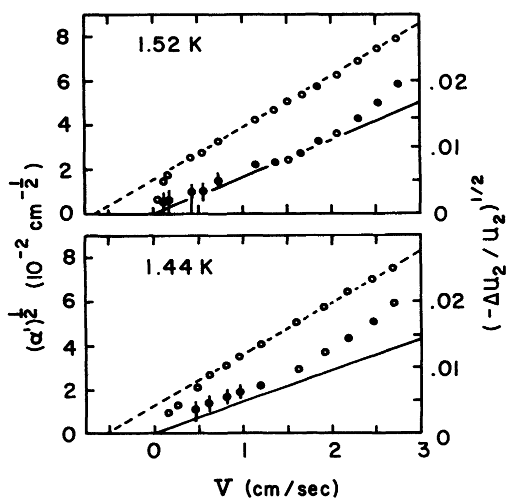

FIG. 2. Plots of $(\alpha')^{1/2}$ and $(-\Delta u_2/u_2)^{1/2}$ as functions of the counterflow velocity v, where α' is the increase in second-sound attenuation due to a heat current and Δu_2 is the change in the second-sound velocity due to a heat current. The data is from Ref. 10. Open circles $(\alpha')^{1/2}$ (left scale); solid circles, $(-\Delta u_2/u_2)^{1/2}$ (right scale). The vertical lines through the solid circles at low v show the error estimates from Ref. 10. The dashed lines are fit to the attenuation data. The solid lines are calculated from (29) as described in the text.

Consideration of Fig. 2 leads us to conclude that while the ideas discussed here fail to give a complete and accurate description of the data of Fig. 2, the second-sound velocity reduction calculated from (29) is in sufficiently good agreement with the data to support the general assumptions on which (29) is based. Perhaps more experimental work will clarify the situation, particularly with respect to the need for replacing v by $(v - v_0)$ in (27).

The second-sound velocity reduction described by (29) is due to random vorticity, hence the effect should be independent of the relative direction of the second-sound wave vector $\vec{k}$ and the counterflow velocity $\vec{v}$. In addition to this effect, the second-sound velocity will be decreased in a heat current by the entrainment effect calculated Khalatnikov[26] and Mikeska.[27] Unlike the isotropic effect predicted by (29), this effect vanishes for $\vec{k}\perp\vec{v}$, and hence should have no effect on the results of Vidal *et al.*

Bhagat and Davis[30] have measured the velocity of second sound in heat currents at temperatures within a few millikelvin of T_λ. With $\vec{k}\parallel\vec{v}$ they observe a decrease which is quadratic in v, with a slope $(-\Delta u_2)/(u_2 v^2) \approx 0.1\ \mathrm{cm}^{-2}\ \mathrm{sec}^2$. With $k\perp v$, on the other hand, they were unable to detect an effect. The sensitivity of their measurements places an upper limit of about $10^{-3}\ \mathrm{cm}^{-2}\ \mathrm{sec}^2$ on the slope. Bhagat and Davis attribute their results to the effect calculated by Khalatnikov and Mikeska.

The anisotropy of the data is consistent with this interpretation, although the magnitude of the observed effect is much larger than the calculated value.

It is worth considering the expected magnitude of u_2 predicted by (29) at temperatures near T_λ. One difficulty in estimating the size of the expected effect is finding a suitable value of β for use in (29). Since no measurements of α' have been made near T_λ, it is necessary to estimate β from measurements of temperature gradients in heat currents. Several groups have made such measurements near T_λ.[31-34] Ahlers[31] and Leiderer and Pobell[32] analyze and present their data in a form from which β can be estimated. Ahlers finds the parameter A is approximately 150 cm sec g^{-1} at $T_\lambda - T \approx 1$ mK; Leiderer and Pobell find $A \approx 6300$ cm sec g^{-1} at the same temperature. (The reader is referred to the original papers for the details of the difficulties which arise in the analysis.) If β is expressed in terms of the parameters A and B_{L1}, (29) becomes

$$\frac{\Delta u_2}{u_2} = \frac{A\rho}{2\sigma}\frac{B_{L2}}{B_{L1}} v^2 . \qquad (30)$$

The calculated value of the ratio (B_{L2}/B_{L1}) depends strongly on whether a mutual-force theory or Hall's 1970 theory is used in the calculation. If the ratio is calculated as described in Sec. IV A, it is found to approach zero rapidly as T approaches T_λ. However, if the ratio is calculated on the basis of Hall's 1970 theory, using a suitably modified form of (14), the ratio remains constant at approximately 0.1 near T_λ. (This is at least a factor of 10 higher than the mutual-force result.) If a value of σ appropriate for the experiment of Bhagat and Davis and $(B_{L2}/B_{L1}) = 0.1$ are used in (30), the slope $(-\Delta u_2)/(u_2 v^2)$ is predicted to be 0.003 cm^{-2} sec^2 using Ahler's value of A, and 0.13 cm^{-2} sec^2 using the value of Leiderer and Pobell. Since these values are larger than the experimental uncertainty, we conclude that the results of the experiment of Bhagat and Davis are inconsistent with a prediction based on Hall's 1970 theory, and consistent with the mutual-force theory. We note, however, that there is no direct experimental evidence supporting the use of (28) and (29) at temperatures near T_λ. Particularly in view of the difficulties in the interpretation of the temperature-gradient measurements,[31,34] it appears that useful information about superfluid turbulence could be gained from further experimental work. Simultaneous measurements of α' and Δu_2 in heat currents at temperatures extending from below 2.0 K to near T_λ would be particularly desirable as tests of (28) and (29).

V. SUMMARY

Two new terms in the expressions for the macroscopic forces describing the vortex second-sound interaction arise when the phase of the roton drag is taken into account. The new terms can be obtained by replacing the parameters (B, B') introduced by Hall and Vinen[4] by complex parameters $(B_1 + iB_2, B_1' + iB_2')$. Expressions for the complex parameters in terms of the phenomenological scattering parameters (D, D') have been derived. The parameters B_2 and B_2' were calculated using values of D and D' calculated from experimental values of B_1 and B_1'. Comparisons of the parameters B_2 and B_2' with experiments provide a test of the accuracy of the Hall-Vinen model of the vortex second-sound interaction.

The force term proportional to B_2 describes a decrease in the second-sound velocity which is proportional to the angular velocity and inversely proportional to the second-sound frequency, in agreement with the experiment of Lynall and Mehl.[8] Moreover, the agreement of the observed and calculated magnitudes of the effect provides new evidence in support of Hall and Vinen's theory of the vortex second-sound interaction.

The force term proportional to B_2' describes an additional contribution to the losses of certain modes of second-sound resonators. The effect is small, but may be observable in a carefully designed experiment.

The form of the force terms applicable to the case of the random vorticity in superfluid turbulence has also been discussed. The theory predicts the approximate magnitude and counterflow dependence of the second-sound velocity decrease which has been observed in heat currents.[10] Analysis of the results of experiments with heat currents suggests that experimental studies of second sound in heat currents at temperatures extending from below 2 K to near T_λ may yield useful information about superfluid turbulence, including the diverging mutual friction observed in temperature-gradient measurements near T_λ.

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[1]H. E. Hall and W. F. Vinen, Proc. R. Soc. Lond. A 238, 204 (1956).

[2]W. F. Vinen, Proc. R. Soc. Lond. A 240, 114 (1957);

A 240, 128 (1957).
[3]L. D. Landau, J. Phys. USSR 5, 71 (1941); 11, 91 (1947).
[4]H. E. Hall and W. F. Vinen, Proc. R. Soc. Lond. A 238, 215 (1956).
[5]W. F. Vinen, Proc. R. Soc. Lond A 242, 493 (1957); A 243, 400 (1957).
[6]A review of the properties of vortex lines in rotating helium and in superfluid turbulence is given by J. Wilks, *Properties of Liquid and Solid Helium* (Clarendon, Oxford, 1967), Chaps. 12 and 13.
[7]F. Vidal, M. leRay, and M. Francois, Phys. Lett. A 36, 401 (1971).
[8]I. H. Lynall and J. B. Mehl, Phys. Lett. A 46, 115 (1973).
[9]D. Lhuillier, F. Vidal, M. Francois, and M. leRay, Phys. Lett. A 38, 161 (1972).
[10]F. Vidal, M. leRay, M. Francois, and D. Lhuillier, *Low Temperature Physics LT 13* (Plenum, New York, 1974), Vol. 1; F. Vidal, C. R. Acad. Sci. B 275, 609 (1972).
[11]E. M. Lifshitz and L. P. Pitaevski, Zh. Eksp. Teor. Fiz. 33, 535 (1957) [Sov. Phys.—JETP 6, 418 (1958)].
[12]W. J. Titus, Phys. Rev. A 2, 206 (1970).
[13]H. E. Hall, *Liquid Helium: Proceedings of the Enrico Fermi International School of Physics, Course 21* (Academic, New York, 1963), pp. 326–335.
[14]H. E. Hall, J. Phys. C 3, 1166 (1970).
[15]A. J. Hillel, H. E. Hall, and P. Lucas (to be published).
[16]S. V. Iordanskii, Ann. Phys. (N.Y.) 29, 335 (1964); Zh. Eksp. Teor. Fiz. 49, 225 (1966) [Sov. Phys.—JETP 22, 160 (1966)].
[17]P. Lucas, J. Phys. C 6, 3372 (1973).
[18]The original version of this paper was based on Hall's 1970 theory. Although most of the results of this paper do not depend on whether the calculations are based on Hall's 1970 theory or a mutual-force theory, some differences exist near T_λ. These will be noted in the text.
[19]An experimental study of the effects of the angle between $\hat{\omega}$ and $\vec{v}_n - \vec{v}_s$ on second-sound attenuation is reported in H. A. Snyder and Z. Putney, Phys. Rev. 150, 110 (1966).
[20]H. A. Snyder and D. M. Linekin, Phys. Rev. 147, 131 (1966).
[21]P. Lucas, J. Phys. C 3, 1180 (1970).
[22]H. A. Snyder and P. J. Westervelt, Ann. Phys. (N.Y.) 43, 158 (1967).
[23]F. Vidal (private communication).
[24]P. Lucas, Phys. Rev. Lett. 15, 750 (1965).
[25]P. Lucas (private communication).
[26]I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 30, 617 (1956) [Sov. Phys.—JETP 3, 649 (1956)].
[27]H. J. Mikeska, Phys. Rev. 179, 166 (1969).
[28]D. F. Brewer and D. O. Edwards, Philos. Mag. 7, 721 (1962).
[29]H. C. Kramers, Physica 26, S81 (1960).
[30]S. M. Bhagat and R. S. Davis, J. Low Temp. Phys. 7, 157 (1972); see also S. M. Bhagat, R. S. Davis, and R. A. Lasken, *Low Temperature Physics LT 13* (Plenum, New York, 1974), Vol. 1.
[31]G. Ahlers, Phys. Rev. Lett. 22, 54 (1969).
[32]P. Leiderer and F. Pobell, J. Low Temp. Phys. 3, 577 (1970).
[33]M. J. Crooks and D. L. Johnson, Can. J. Phys. 49, 1035 (1971).
[34]S. M. Bhagat and R. A. Lasken, Phys. Rev. A 5, 2297 (1972).