PHYSICAL REVIEW A VOLUME 10, NUMBER 1 JULY 1974

Contribution to the Kapitza conductance from spin-spin interactions

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(Received 3 December 1973)

In this paper, we examine the theory of the rate at which spin-spin interactions produce energy transfer across an interface between a degenerate Fermi gas (liquid He^3) and a system of electron spins. We consider first the case where the electron spins are conduction-electron spins in a metal, and examine the rate of energy transfer produced by spin-spin interactions of dipolar form, and also by a short-range exchange interaction. In the former case (dipolar coupling), the Kapitza conductance exhibits a $T^3 \ln T$ dependence on temperature, but is very small in magnitude. In the second case (short-range interaction), the Kapitza conductance varies as T^3, with a magnitude that is also small. We conclude that neither dipolar interactions nor short-ranged interactions of the exchange type can produce a significant transfer of energy from the He^3 spins to conduction-electron spins in metals. We also calculate the rate at which the He^3 spins transfer energy to an array of paramagnetic local moments. We recover a result similar to that obtained earlier by Leggett and Vuorio. In contrast to Leggett and Vuorio, we use the full form of the dipolar interaction to couple the He^3 spins to the local moments, and in all the calculations reported here, the effect of the surface on the single-particle wave functions is included by treating the surface as a barrier of infinite height.

I. INTRODUCTION

Some time ago, Wheatley[1] suggested that the anomalously small thermal boundary resistance[2] between cerium magnesium nitrate (CMN) and liquid He^3 has its origin in the coupling between the electron spins in the CMN and the He^3 nuclear spins. More recently, Leggett and Vuorio[3] have carried out an explicit calculation of the contribution to the boundary conductance from this source, and they obtain a good account of the temperature dependence, magnetic field dependence and the order of magnitude of the anomalous thermal boundary conductance at the He^3-CMN interface. A recent experimental study[4] of the boundary resistance between He^3 and several metals which contain magnetic impurities in low concentration suggest that for these systems, a similar mechanism also plays a role.

In the present paper, we wish to explore some aspects of this phenomena further, from a theoretical point of view. Our investigation has been motivated by two considerations. First of all, Bishop, Mota, and Wheatley[5] have studied experimentally the Kapitza resistance R_K associated with the interface between He^3 and Pt metal. They find that the Kapitza conductance varies as T^2, a behavior distinctly different from the classical T^3 law, or that observed at the He^3-CMN interface. It has been suggested[5] that the contribution to the He^3-Pt interface conductance has its origin in the transfer of energy between the He^3 nuclear spins, and the conduction-electron spins in the Pt. We investigate the possibility that this may be so in the present paper by examining the theory of the Kapitza conductance between two degenerate Fermi gases for the case where the spin-spin interaction has the dipolar form, and also for the case where it has the form of a short-ranged exchange interaction. In the first case, we find that the Kapitza conductance exhibits a $T^3 \ln T$ variation with temperature, but it is estimated that it is several orders of magnitude smaller than typical values observed for the thermal boundary conductance, even if exchange enhancement effects in both the He^3 and the metal are strong. In the second case, we find a T^3 variation of the boundary conductance which may be larger than that which comes from the dipolar interaction, but which is still quite small, for reasonable values of the interaction strength. We conclude that energy transfer between He^3 spins and conduction-electron spins in metals is too small to make an observable contribution to the thermal boundary conductance.

We also wish to reexamine the problem considered by Leggett and Vuorio, which is the rate at which energy is transferred across an interface between a degenerate Fermi gas (He^3), and an array of paramagnetic local moments. There are two assumptions in their work we would like to remove. Their theory presumes the He^3 nuclei and the local moments interact via a short-ranged interaction of the contact form. The He^3 nuclei thus "see" only electron spins very close to the crystal surface, in their model. For quantitative purposes, Leggett and Vuorio adjust the strength of the contact interaction to mimic that of the dipolar interaction. It is not obvious that the dipolar interaction, which is long ranged, can be replaced by a contact interaction. Indeed, in our study of energy transfer between He^3 spins and conduction electrons, we find qualitatively different results for the behavior of the Kapitza conductance for the dipolar interaction and a short-ranged interaction, as one can see from the remarks of the preceding paragraph.

We have concern about one other approximation made by Leggett and Vuorio. They compute R_K from a Kubo formula (as we do also) which relates the energy transfer rate to certain integrals over the spectral densities associated with spin-spin correlation functions. For the He^3 spins, they use a form for the spectral density appropriate to the bulk region of the fluid. But if one accepts the fact that the dipolar interaction is equivalent to a short-ranged interaction of some strength, then in the Kubo formula it is important to use a form for the spectral density appropriate to the surface region of the Fermi fluid. If the interface is regarded as an infinite barrier to the He^3 atoms, the spectral density function must vanish identically right at the surface where the contact interaction acts, since the wave function of each He^3 atom vanishes there. Thus, the use of a proper form for the spectral density in combination with a contact form of the He^3–paramagnetic-spin interaction should greatly reduce the predicted rate of energy transfer.

Nonetheless, despite these concerns, the success of the Leggett-Vuorio theory is most impressive. In our calculation of the rate of energy transfer between He^3 nuclei and local moments, which uses both the full form of the dipolar interaction and spectral densities appropriate to a surface that behaves as an infinite barrier, we find a formula for the Kapitza conductance identical in form to that obtained by Leggett and Vuorio, except the numerical coefficient differs from theirs, as one would expect.

An investigation of the Leggett-Vuorio contribution to the Kapitza conductance which includes the dipolar interaction explicitly has also been reported by Guyer,[6] who also recovers results similar to those of Leggett and Vuorio. Our method avoids approximations introduced in his work. For example, along with Leggett and Vuorio, we need not assume the spin correlation functions are diagonal in coordinate space for the He^3 atoms. Indeed, for a Fermi fluid, we believe this approximation is poorly satisfied. Guyer has also not considered the effect of the surface on the wave functions in the Fermi liquid, and replaced the anisotropic dipole-dipole interaction by an isotropic exchange interaction.

In their paper, Leggett and Vuorio have not taken exchange enhancement effects in the He^3 into account explicitly, although they modify their final formula in a phenomenological manner before quantitative estimates of the Kapitza conductance are made. In the presence of the surface, inclusion of exchange enhancement effects for either the He^3 spins or the conduction electrons considered in the earlier portion of the paper is quite difficult at the present time,[7] and these effects are omitted from our work also, although for the case where He^3 spins are coupled to conduction electrons via dipolar interactions, it will be clear from physical considerations how the final result is to be modified.

This paper is organized as follows. In Sec. II we begin with a brief derivation of the Kubo formula for the Kapitza conductance similar to the one used by Leggett and Vuorio. In Sec. III we study the form of a spectral density function that enters the calculation. In Secs. IV–VI we derive the formulas for R_K for these cases described above (coupling to conduction electrons via dipolar and contact interactions, and coupling to local moments via dipolar interactions). Finally, a discussion of the results is presented in Sec. VII.

II. THEORETICAL FRAMEWORK

A. General considerations

The purpose of this section is to derive an expression for the Kapitza resistance for two systems coupled by spin-spin interactions, and to case this expression in a form suitable for use later in the paper. Consider a volume of He^3 bounded by a plane surface, and let $\vec{s}(\vec{x})$ be the spin density operator for the He^3. The coordinate system is oriented so that the x_3 axis is normal to the bounding plane, and the x_1 and x_2 axes lie in the plane. The He^3 lies in the region $x_3 > 0$. We attach the subscript $\parallel$ to any vector which lies in the $x_1 - x_2$ plane, or the projection of any vector onto this plane. Thus $\vec{x}_\parallel = x_1\hat{x}_1 + x_2\hat{x}_2$. By means of spin-spin interactions, the He^3 nuclear spins are in contact with the spins in a material with spin density $\vec{S}(\vec{x})$ which lies in the lower half-space with $x_3 < 0$. We write the interaction between the two sets of spins in the form

$$V=\sum_{\alpha\beta}\int d^3x\,d^3y\,d_{\alpha\beta}(\vec{x}-\vec{y})\,s_\alpha(\vec{x})S_\beta(\vec{y}), \tag{2.1}$$

where in Eq. (2.1) and the equations displayed below, α and β refer to the three Cartesian coordinate axes, the integration over $\vec{x}$ covers the region $x_3>0$ and that over $\vec{y}$ the region $y_3<0$.

For the case of dipolar coupling between nuclei with magnetic moment μ_n, and electron spins with moment μ_e, we have

$$d_{\alpha\beta}(\vec{r})=\mu_e\mu_n\left(\frac{r^2\delta_{\alpha\beta}-3r_\alpha r_\beta}{r^5}\right). \tag{2.2a}$$

We shall consider also an interaction with short range. For this case we choose $d_{\alpha\beta}(\vec{r})$ to be of the form

$$d_{\alpha\beta}(\vec{r})=d(\vec{r})\delta_{\alpha\beta}, \tag{2.2b}$$

where the function $d(\vec{r})$ is sharply peaked about $\vec{r}=0$.

In what follows, all quantities which refer to the He^3 will be described by lower-case letters, and the quantities which refer to the system of electron spins by upper-case letters. When $V=0$, the eigenstates of the entire system assume the product form $|M\rangle|m\rangle$ with energy $E_M+\epsilon_m$. The rate at which V induces transitions between these states is found from the golden rule of perturbation theory. With $\hbar=1$, we have

$$W(nN\to mM)=2\pi|\langle m|\langle M|V|N\rangle|n\rangle|^2 \times\delta(E_N+\epsilon_n-E_M-\epsilon_m). \tag{2.3}$$

The rate at which energy is transferred from the electron-spin system to the He^3 by this particular transition is just $(E_N-E_M)W(nN\to mM)$. If P_n and P_N are the probabilities of finding the He^3 in state n and the electrons in N, respectively, the total rate $\dot{Q}$ at which energy is transferred from the electron-spin system to the He^3 spins is

$$Q=\sum_{\substack{nN\\mM}}(E_N-E_M)W(nN\to mM)P_nP_N. \tag{2.4}$$

Now consider the term which corresponds to a particular set of states nN, mM and combine it with the term with nN, mM interchanged, i.e., we write Eq. (2.4) in the form

$$\dot{Q}=\frac{1}{2}\sum_{\substack{nN\\mM}}(E_N-E_M)[W(nN\to mM)P_nP_N -W(mM\to nN)P_mP_M] \tag{2.4a}$$

$$=\frac{1}{2}\sum_{\substack{nN\\mM}}(E_N-E_M)(P_nP_N-P_mP_M)W(mM\to nN). \tag{2.4b}$$

From Eq. (2.4b), and the energy conservation condition $E_N+\epsilon_n=E_M+\epsilon_m$, it is clear that when both systems are in thermal equilibrium at the same temperature, $P_nP_N=P_mP_M$ and $\dot{Q}=0$. We presume that the electron spins are at the temperature T_1 and the He^3 spins at temperature T_2, where T_1-T_2 is small. Then

$$P_n=e^{-\beta_2E_n}/Z_H, \qquad P_N=e^{-\beta_1E_N}/Z_e, \tag{2.5}$$

where $\beta_i=1/k_BT_i$ and Z_e, Z_H are the partition functions of the electron- and helium-spin systems, respectively. To first order in $\Delta T=T_1-T_2$, $\dot{Q}$ is proportional to ΔT. We write

$$Q=A(\Delta T/R_K), \tag{2.6}$$

where A is the area of the interface between the two systems, and R_K is the Kapitza resistance per unit area. Then upon using the explicit form of the transition probability $W(mM\to nN)$ we have

$$\frac{1}{R_K}=\frac{\pi}{Ak_BT^2}\sum_{\substack{\alpha\beta\\\alpha'\beta'}}\sum_{\substack{nN\\mM}}P_n^{(0)}P_N^{(0)}\int d^3x\,d^3x'\,d^3y\,d^3y'\,d_{\alpha\beta}(\vec{x}-\vec{y})\,d_{\alpha'\beta'}(\vec{x}'-\vec{y}')\langle n|s_{\alpha'}(\vec{x}')|m\rangle\langle m|s_\alpha(\vec{x})|n\rangle \times\langle N|S_{\beta'}(\vec{y}')|M\rangle\langle M|S_\beta(\vec{y})|N\rangle(E_N-E_M)^2\delta(\epsilon_n+E_N-\epsilon_m-E_M), \tag{2.7}$$

where in this expression $P_n^{(0)}=e^{-\beta E_n}/Z_H$.

We next proceed in the standard manner by reexpressing the δ function as an integral over time:

$$\delta(\mathcal{E})=\int_{-\infty}^{+\infty}\frac{dt}{2\pi}e^{i\mathcal{E}t}.$$

It will be convenient to use the energy-conservation condition to write $(E_N-E_M)^2=-(\epsilon_n-\epsilon_M)\times(E_N-E_M)$. Furthermore, if $s_\alpha(\vec{x},t)=\exp(iH_Ht)\times s_\alpha(\vec{x})\exp(-iH_Ht)$, with a similar relation for $S_\alpha(\vec{x},t)$, where H_H is the Hamiltonian of the He^3, then note

$$(\epsilon_n-\epsilon_m)\langle n|s_{\alpha'}(\vec{x}'t)|m\rangle\langle m|s_\alpha(\vec{x}0)|n\rangle =\frac{1}{i}\frac{d}{dt}\langle n|s_{\alpha'}(\vec{x}'t)|m\rangle\langle m|s_\alpha(\vec{x}0)|n\rangle.$$

By utilizing the expressions given above, the expression for R_K may be put into the form

$$\frac{1}{R_K}=\frac{1}{2Ak_BT^2}\sum_{\substack{\alpha\beta\\ \alpha'\beta'}}\int d^3x\,d^3x'\,d^3y\,d^3y'\,d_{\alpha\beta}\times(\vec{x}-\vec{y})\,d_{\alpha'\beta'}(\vec{x}'-\vec{y}')\times\int_{-\infty}^{+\infty}dt\frac{d}{dt}\times\langle s_{\alpha'}(\vec{x}'t')\,s_\alpha(\vec{x}\,0)\rangle\frac{d}{dt}\langle S_{\beta'}(\vec{y}'t)S_\beta(\vec{y}\,0)\rangle. \tag{2.8}$$

B. Application to coupling between paramagnetic systems

So far our discussion has been quite general. For convenience, we now restrict attention to the case where both spin systems are paramagnetic. The extension of the results below to more general situations is straightforward. In this case, the spin correlation functions satisfy the isotropy condition

$$\langle s_{\alpha'}(\vec{x}'t)\,s_\alpha(\vec{x}\,0)\rangle=\delta_{\alpha\alpha'}\langle s_z(\vec{x}'t)\,s_z(\vec{x}\,0)\rangle$$

so that for R_K we have

$$\frac{1}{R_K}=\frac{1}{2Ak_BT^2}\sum_{\alpha\beta}\int d^3x\,d^3x'\,d^3y\,d^3y'\,d_{\alpha\beta}\times(\vec{x}-\vec{y})\,d_{\alpha\beta}(\vec{x}'-\vec{y}')\int_{-\infty}^{+\infty}dt\,\frac{d}{dt}\langle s_z(\vec{x}'t)\times s_z(\vec{x}\,0)\rangle\frac{d}{dt}\langle S_z(\vec{y}'t)S_z(\vec{y}\,0)\rangle. \tag{2.9}$$

Now we introduce the spectral densities associated with the spin correlation functions by writing

$$\langle s_z(\vec{x}'t)\,s_z(\vec{x}\,0)\rangle=\int_{-\infty}^{+\infty}d\Omega\,e^{+i\Omega t}\,n(\Omega)\,\mathcal{A}(\vec{x}'\vec{x};\,\Omega), \tag{2.10a}$$

$$\langle S_z(\vec{y}'t)S_z(\vec{y}\,0)\rangle=\int_{-\infty}^{+\infty}d\Omega\,e^{+i\Omega t}\,n(\Omega)A(\vec{y}'\vec{y};\,\Omega), \tag{2.10b}$$

where $n(\Omega)=(e^{\beta\Omega}-1)^{-1}$ is the Bose-Einstein function. The spectral densities satisfy the condition

$$n(-\Omega)A(\vec{y}'\vec{y};\,-\Omega)=[1+n(\Omega)]A(\vec{y}'\vec{y};\,\Omega), \tag{2.10c}$$

with a similar relation for $\mathcal{A}(\vec{x}'\vec{x};\,\Omega)$. Then the expression for R_K becomes

$$\frac{1}{R_K}=\frac{\pi}{Ak_BT^2}\int d^3x\,d^3x'\,d^3y\,d^3y'\times\sum_{\alpha\beta}d_{\alpha\beta}(\vec{x}'-\vec{y}')\,d_{\alpha\beta}(\vec{x}-\vec{y})\int_{-\infty}^{+\infty}d\Omega\,\Omega^2n(\Omega)\times[1+n(\Omega)]\,\mathcal{A}(\vec{x}'\vec{x};\,\Omega)A(\vec{y}'\vec{y};\,\Omega). \tag{2.11}$$

Now we make use of the translational invariance of the model we consider. In the infinitely extended system the spectral densities in Eq. (2.11) are functions of $\vec{x}'-\vec{x}$ and $\vec{y}'-\vec{y}$ only. This is no longer true in the presence of a surface. However, since we choose the surface to be a plane, translational invariance in the two directions parallel to the surface still obtains. Thus $\mathcal{A}(\vec{x}'\vec{x};\,\Omega)$ is a function of $\vec{x}_\parallel'-\vec{x}_\parallel$ only, although it depends on x_3 and x_3' separately. We write

$$\mathcal{A}(\vec{x}'\vec{x};\,\Omega)=\int\frac{d^2k_\parallel}{(2\pi)^2}\,e^{i\vec{k}_\parallel\cdot(\vec{x}_\parallel'-\vec{x}_\parallel)}\mathcal{A}(x_3'x_3;\,\vec{k}_\parallel\Omega) \tag{2.12a}$$

and

$$A(\vec{y}'\vec{y};\,\Omega)=\int\frac{d^2k_\parallel}{(2\pi)^2}\,e^{i\vec{k}_\parallel\cdot(\vec{y}_\parallel'-\vec{y}_\parallel)}A(x_3'x_3;\,\vec{k}_\parallel\Omega). \tag{2.12b}$$

Then define a quantity

$$D_{\alpha\beta}(\vec{k}_\parallel;\,x_3)=\int d^2x_\parallel\,e^{i\vec{k}_\parallel\cdot\vec{x}_\parallel}d_{\alpha\beta}(\vec{x}). \tag{2.12c}$$

Then in terms of the quantities defined in Eqs. (2.12), we may write

$$\frac{1}{R_K}=\frac{1}{4\pi k_BT^2}\int dx_3\,dx_3'\,dy_3\,dy_3'\sum_{\alpha\beta}\int d^2k_\parallel\,D^*_{\alpha\beta}\times(\vec{k}_\parallel;\,x_3-y_3)D_{\alpha\beta}(\vec{k}_\parallel;\,x_3'-y_3')\times\int_{-\infty}^{+\infty}d\Omega\,\Omega^2n(\Omega)\times[1+n(\Omega)]\,\mathcal{A}(x_3'x_3;\,\vec{k}_\parallel\Omega)A(y_3'y_3;\,\vec{k}_\parallel\Omega). \tag{2.13}$$

In Eq. (2.13), the integration over x_3 and x_3' ranges from 0 to ∞, while that over y_3 and y_3^- from −∞ and 0.

C. Form of R_K for dipolar coupled spin systems

An important special case we wish to consider here is the case where the two spin systems are coupled by dipolar interactions. Then the form of $d_{\alpha\beta}(\vec{r})$ has the form given in Eq. (2.2), and one has after some straightforward algebra,

$$\sum_{\alpha\beta}D^*_{\alpha\beta}(\vec{k}_\parallel,z)D_{\alpha\beta}(\vec{k}_\parallel,z)=16\pi^2\mu_e^2\mu_n^2k_\parallel^2\,e^{-k_\parallel(z+z')}\,. \tag{2.14}$$

We then introduce the quantities

$$\mathcal{F}(\vec{k}_\parallel,\Omega)=\int_0^\infty dx\,dx'\,e^{-k_\parallel(x+x')}\,\mathcal{A}(x,x';\,\vec{k}_\parallel\Omega) \tag{2.15a}$$

and

$$F(\vec{k}_\parallel,\Omega)=\int_{-\infty}^0 dy\,dy'\,e^{k_\parallel(y+y')}A(y'y;\,\vec{k}_\parallel\Omega). \tag{2.15b}$$

Then for the case of dipolar coupled spin systems,

$$\frac{1}{R_K}=\frac{4\pi\mu_e^2\mu_n^2}{k_BT^2}\int d^2k_\parallel k_\parallel^2\int_{-\infty}^{+\infty}d\Omega\Omega^2n(\Omega)\times[1+n(\Omega)]\mathfrak{F}(\vec{k}_\parallel,\Omega)F(\vec{k}_\parallel,\Omega). \tag{2.16a}$$

Since the integrand is an even function of Ω, and depends only on $|\vec{k}_\parallel|$ for the systems we consider, Eq. (2.16a) becomes

$$\frac{1}{R_K}=\frac{16\pi^2\mu_e^2\mu_n^2}{k_BT^2}\int_0^\infty dk_\parallel k_\parallel^3\int_0^\infty d\Omega\Omega^2n(\Omega)\times[1+n(\Omega)]\mathfrak{F}(\vec{k}_\parallel,\Omega)F(\vec{k}_\parallel,\Omega). \tag{2.16b}$$

In Sec. III we study the form of the functions $F(k_\parallel,\Omega)$ and the spectral density for a set of fermions bounded by a rigid wall.

III. CONSTRUCTION OF SPECTRAL DENSITY AND FUNCTION $F(k_\parallel,\Omega)$ FOR A DEGENERATE FERMI GAS

A. General remarks on spectral density

We first begin with a few general remarks. Consider a gas of noninteracting fermions, where $\varphi_{i\sigma}(\vec{x})$ denotes the orbital part of the wave function for an electron of spin σ, and the index i refers to all quantum numbers excluding the spin. With $\hbar=1$, in second-quantized notation, the spin density operator $S_z(\vec{x})$ is given by (with $\mu=+1$ for spin up and -1 for spin down)

$$S_z(\vec{x})=\frac{1}{2}\sum_{ij\sigma}\mu\varphi_{i\sigma}^*(\vec{x})\varphi_{j\sigma}(\vec{x})C_{i\sigma}^\dagger C_{j\sigma}, \tag{3.1}$$

where $C_{i\sigma}^\dagger$ and $C_{j\sigma}$ are the appropriate fermion annihilation and creation operators. From the definition of the spectral density $A(\vec{x}\vec{x}';\Omega)$ in Eq. (2.10), one has

$$n(\Omega)A(\vec{x}'\vec{x};\Omega)=\frac{1}{4}\sum_{ij\sigma}\varphi_{i\sigma}^*(\vec{x}')\varphi_{i\sigma}(\vec{x})\varphi_{j\sigma}^*(\vec{x})\times\varphi_{j\sigma}(\vec{x}')f_{i\sigma}(1-f_{j\sigma})\delta(\Omega+\epsilon_{j\sigma}-\epsilon_{i\sigma}),$$

where $\epsilon_{i\sigma}$ is the energy of the state $i\sigma$, and $f_{i\sigma}$ is the Fermi Dirac occupation function for this level. A bit of rearrangement yields, for the case where the single-particle energies and wave functions are spin independent,

$$A(\vec{x}'\vec{x};\Omega)=\frac{1}{2}\sum_{ij}\varphi_i^*(\vec{x}')\varphi_i(\vec{x})\varphi_j^*(\vec{x})\varphi_j(\vec{x}')\times(f_j-f_i)\delta(\Omega+\epsilon_j-\epsilon_i). \tag{3.2}$$

We apply this form to a gas of free fermions, free to move unconstrained in the x_1-x_2 directions, but confined in the x_3 direction to a well of width L and infinitely steep sides. Periodic boundary conditions are applied in the x_1 and x_2 directions to a region with cross-sectional area A. In this geometry, the single-particle wave functions are characterized by the quantum numbers $\vec{Q}_\parallel$ and Q_z, and have the form

$$\varphi_{\vec{Q}_\parallel Q_z}(\vec{x})=e^{i\vec{Q}_\parallel\cdot\vec{x}_\parallel}\sin(Q_zx_3)(2/AL)^{1/2}, \tag{3.3}$$

where

$$Q_z=n\pi/L,\quad n=1,2,3,\ldots.$$

Notice that $Q_z>0$, necessarily. The energy of the state $\vec{Q}_\parallel$, Q_z is

$$E(\vec{Q}_\parallel,Q_z)=(1/2M)(Q_\parallel^2+Q_z^2). \tag{3.4}$$

We then have

$$A(\vec{x}'\vec{x};\Omega)=\frac{2}{A^2L^2}\sum_{\substack{\vec{Q}_\parallel\vec{Q}_\parallel'\\ Q_zQ_z'}}\exp[i(\vec{Q}_\parallel-\vec{Q}_\parallel')\cdot(\vec{x}_\parallel'-\vec{x}_\parallel)]\sin(Q_zx_3')\sin(Q_zx_3)\sin(Q_z'x_3)\sin(Q_z'x_3')\times[f(E(\vec{Q}_\parallel',Q_z'))-f(E(\vec{Q}_\parallel,Q_z))]\delta(\Omega+E(\vec{Q}_\parallel'Q_z')-E(\vec{Q}_\parallel Q_z)) \tag{3.5}$$

We next recall the definition of $A(x_3'x_3;\vec{k}_\parallel\Omega)$ implicit in Eqs. (2.12). If the sums over $\vec{Q}_\parallel$ and $\vec{Q}_\parallel'$ in Eq. (3.5) are converted to integrations in the standard manner, then Eqs. (2.12) may be inverted to yield

$$A(x_3'x_3;\vec{k}_\parallel\Omega)=\frac{1}{L^2}\sum_{Q_zQ_z'}\int\frac{d^2Q_\parallel}{2\pi^2}\sin(Q_zx_3')\sin(Q_zx_3)\sin(Q_z'x_3)\sin(Q_z'x_3')\times[f(E(\vec{Q}_\parallel+\vec{k}_\parallel,Q_z'))-f(E(\vec{Q}_\parallel,Q_z))]\delta(\Omega+E(\vec{Q}_\parallel+\vec{k}_\parallel,Q_z')-E(\vec{Q}_\parallel,Q_z)). \tag{3.6}$$

B. The function $F(k_\parallel,\Omega)$

To compute the Kapitza resistance for dipolar-coupled spin systems, we saw in Sec. II that we require the function

$$F(\vec{k}_\parallel,\Omega)=\int_0^\infty dx_3dx_3'e^{-k_\parallel(x_3+x_3')}A(x_3'x_3;\vec{k}_\parallel\Omega).$$

Upon noting that

$$\int_0^\infty dx\, e^{-k_\parallel x}\sin(Q_z x)\sin(Q_z' x)$$
$$=\frac{2k_\parallel Q_z Q_z'}{[k_\parallel^2+(Q_z-Q_z')^2][k_\parallel^2+(Q_z+Q_z')^2]}\,,$$

one finds that $F(\vec{k}_\parallel \Omega)$ is given by

$$F(\vec{k}_\parallel,\Omega)=\frac{2k_\parallel^2}{\pi^2L^2}\sum_{Q_zQ_z'}\int\frac{d^2Q_\parallel Q_z^2Q_z'^2}{[k_\parallel^2+(Q_z-Q_z')^2]^2}\times\frac{[f(E(\vec{Q}_\parallel+\vec{k}_\parallel,Q_z'))-f(E(\vec{Q}_\parallel,Q_z))]}{[k_\parallel^2+(Q_z+Q_z')^2]^2}\times\delta(\Omega+E(\vec{Q}_\parallel+\vec{k}_\parallel,Q_z')-E(\vec{Q}_\parallel,Q_z)).\qquad(3.7)$$

To evaluate the function $F(k_\parallel,\Omega)$ in closed form even for a degenerate Fermi gas at $T=0$ is a formidable task, for general values of $k_\parallel$ and Ω. However, we shall be able to extract from the general form in Eq. (3.7) sufficient information to allow computation of R_K at temperatures T small compared to the Fermi temperature T_F. At such low temperatures, we only need to consider the behavior of $F(\vec{k}_\parallel,\Omega)$ for frequencies $\Omega \ll E_F$. We first consider the case $\Omega\to 0$ for $k_\parallel$ fixed and arbitrary, and then consider the region where both $k_\parallel$ and Ω are small, but the ratio $\Omega/V_Fk_\parallel$ is arbitrary.

a. Behavior of $F(\vec{k}_\parallel,\Omega)$ for $k_\parallel$ fixed, $\Omega\to 0$. To obtain the behavior of $F(\vec{k}_\parallel,\Omega)$ in this limit, we make the following replacement in Eq. (3.7):

$$\begin{aligned}f(E(\vec{Q}_\parallel+\vec{k}_\parallel,Q_z'))-f(E(\vec{Q}_\parallel,Q_z))&\\ &\equiv f(E(\vec{Q}_\parallel,Q_z)-\Omega)-f(E(\vec{Q}_\parallel,Q_z))\\ &\approx-\Omega\frac{\partial f}{\partial E(\vec{Q}_\parallel,Q_z)}\\ &=+\Omega\,\delta(E_F-E(\vec{Q}_\parallel,Q_z)).\qquad(3.8)\end{aligned}$$

Elsewhere in Eq. (3.7), we set $\Omega=0$. The last statement in Eq. (3.8) assumes $T\ll T_F$. Then as $\Omega\to 0$, we have

$$F(\vec{k}_\parallel,\Omega)=\frac{2\Omega k_\parallel^2}{\pi^2L^2}\sum_{Q_zQ_z'}\int\frac{d^2Q_\parallel Q_z^2Q_z'^2}{[k_\parallel^2+(Q_z-Q_z')^2]^2}\times\frac{\delta(E_F-E(\vec{Q}_\parallel,Q_z))\,\delta(E_F-E(\vec{Q}_\parallel+\vec{k}_\parallel,Q_z'))}{[k_\parallel^2+(Q_z+Q_z')^2]^2}\,.\qquad(3.9)$$

Now replace the sums over Q_z and Q_z' by integrations according to the prescription

$$\sum_{Q_z}\to\frac{L}{\pi}\int_0^\infty dQ_z\,.$$

Furthermore, let $E_F=k_F^2/2M$ so that we have

$$F(\vec{k}_\parallel,\Omega)=\frac{8\Omega M^2k_\parallel^2}{\pi^4}\int d^2Q_\parallel\int_0^\infty dQ_z\,dQ_z'\,\frac{Q_z^2Q_z'^2\delta(k_F^2-(\vec{k}_\parallel+\vec{Q}_\parallel)^2-Q_z'^2)\,\delta(k_F^2-Q_\parallel^2-Q_z^2)}{[k_\parallel^2+(Q_z-Q_z')^2]^2[k_\parallel^2+(Q_z+Q_z')^2]^2}\,.\qquad(3.10)$$

Now define the function

$$q_\perp^2(\vec{Q}_\parallel)=k_F^2-Q_\parallel^2\qquad(3.11)$$

and notice that, if $\theta(x)=1$ for $x>0$ and 0 otherwise,

$$\int_0^\infty dQ_z\,g(Q_z)\delta(Q_z^2-q_0^2)=\frac{\theta(k_F-Q_\parallel)}{2q_0}\,g(q_0).$$

By the use of this identity, the integrations over Q_z and Q_z' which appear in Eq. (3.10) may be evaluated to give

$$F(\vec{k}_\parallel,\Omega)=\frac{2\Omega k_\parallel^2M^2}{\pi^2}\int d^2Q_\parallel\times\frac{q_0(\vec{Q}_\parallel)q_0(\vec{Q}_\parallel+\vec{k}_\parallel)}{\{k_\parallel^2+[q_0(\vec{Q}_\parallel)-q_0(\vec{Q}_\parallel+\vec{k}_\parallel)]^2\}^2}\times\frac{\theta(k_F-|Q_\parallel|)\theta(k_F-|\vec{Q}_\parallel+\vec{k}_\parallel|)}{\{k_\parallel^2+[q_0(Q_\parallel)+q_0(Q_\parallel+k_\parallel)]^2\}^2}\,.\qquad(3.12)$$

The expression in Eq. (3.12) can be simplified further. If θ is the angle between $\vec{Q}_\parallel$ and $\vec{k}_\parallel$, then

$$q_0^2(\vec{k}_\parallel+\vec{Q}_\parallel)=k_F^2-Q_\parallel^2-2k_\parallel Q_\parallel\cos\theta.$$

After some algebra, one may demonstrate that

$$\{k_\parallel^2+[q_0(\vec{Q}_\parallel)-q_0(\vec{Q}_\parallel+\vec{k}_\parallel)]^2\}^2\times\{k_\parallel^2+[q_0(\vec{Q}_\parallel)+q_0(\vec{Q}_\parallel+\vec{k}_\parallel)]^2\}^2=16k_\parallel^4(k_F^2-Q_\parallel^2\sin^2\theta)^2.$$

Then we have

$$F(\vec{k}_\parallel,\Omega)=\frac{M^2\Omega}{8\pi^2k_\parallel^2}\int_0^\infty dQ_\parallel Q_\parallel\int_0^{2\pi}d\theta\times\frac{q_0(\vec{Q}_\parallel)q_0(\vec{k}_\parallel+\vec{Q}_\parallel)}{(k_F^2-Q_\parallel^2\sin^2\theta)^2}\,\theta(k_F-Q_\parallel)\times\theta(k_F-|\vec{k}_\parallel+\vec{Q}_\parallel|).$$

In the integration over $Q_\parallel$, introduce the dimensionless variable x by setting

$$Q_\parallel=k_Fx.$$

Then if we define a function $g(\xi)$ by the statement

$$g(\xi)=\frac{3}{4\pi}\int_0^1 dx\,x(1-x^2)^{1/2}\times\int_0^{2\pi}\frac{d\theta(1-\xi^2-x^2-2\xi x\cos\theta)^{1/2}}{(1-x^2\sin^2\theta)^2}\times\theta(1-\xi^2-x^2-2\xi x\cos\theta),\qquad(3.13)$$

the expression for $F(\vec{k}_\parallel,\Omega)$ becomes

$$F(k_\parallel,\Omega)=\frac{M^2\Omega}{6\pi^3k_\parallel^2}\,g\left(\frac{k_\parallel}{k_F}\right),\qquad(3.14)$$

where we now replace $\vec{k}_\parallel$ by its magnitude in the argument of F, because F is independent of the direction of $\vec{k}_\parallel$.

It is straightforward to establish that $g(\xi)\equiv 0$ for $\xi>2$, since the argument of the θ function is then never positive. This means that for $k_\parallel>2k_F$, $F(k_\parallel,\Omega)$ vanishes identically. Furthermore, in the limit $\xi\to 0$ one has

$$\lim_{\xi\to 0} g(\xi)=\frac{3}{4\pi}\int_0^1 dx\,x(1-x^2)\times\int_0^{2\pi}\frac{d\theta}{(1-x^2\sin^2\theta)^2}=1.\qquad(3.15)$$

For values of ξ which lie between 0 and 2, where $g(\xi)$ is nonzero, we have calculated $g(\xi)$ numerically. The results are displayed in Fig. 1.[8] While $g(\xi)$ indeed approaches unity as $\xi\to 0$, as stated in Eq. (3.15), notice that $g(\xi)$ decreases linearly with ξ for small ξ, and in fact $g(\xi)$ rapidly becomes small as ξ increases toward unity.

The qualitative behavior of the small Ω limit of $F(k_\parallel,\Omega)$ may be deduced from the remarks in the preceding two paragraphs. For fixed Ω, as $k_\parallel\to 0$, from Eq. (3.14) we see that $F(k_\parallel,\Omega)$ diverges as $k_\parallel^{-2}$. Because of the rapid falloff of the function $g(\xi)$ with increasing ξ, as $k_\parallel$ increases toward $2k_F$ where $F(k_\parallel,\Omega)$ vanishes, $F(k_\parallel,\Omega)$ decreases more rapidly than $k_\parallel^{-2}$.

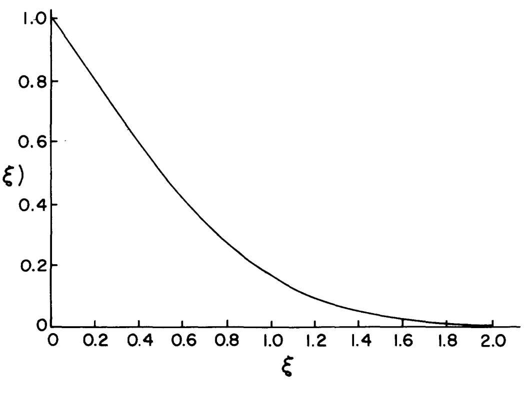

FIG. 1. The function $g(\xi)$ defined in Eq. (3.11) of the text, for $0<\xi<2$.

The form of $F(k_\parallel,\Omega)$ for small Ω may now be inserted into the expression for the Kapitza resistance given in Eq. (2.16b), if we wish to compute the boundary resistance between two dipolar coupled Fermi gases, or between one Fermi gas, and a system of local moments, so long as we are interested in temperatures small compared to T_F. However, one may readily appreciate the fact that the form displayed in Eq. (3.14) will lead to difficulties, if used as it stands. To see this, consider the calculation of the rate of energy transfer between two degenerate gases of noninteracting fermions. For small $k_\parallel$, the product $\mathcal{F}(k_\parallel,\Omega)\times F(k_\parallel,\Omega)$ is proportional to $k_\parallel^{-4}$, and the integral over $k_\parallel$ diverges logarithmically when this factor is multiplied by the explicit factor of $k_\parallel^3$ which appears in the integrand. The divergence at small $k_\parallel$ means that spin fluctuations of very long wavelength make an anomalous contribution to the Kapitza conductance. The dipolar character of the spin-spin interaction plays an important role in bringing about this divergence, as one can see from the factor $\exp[-k_\parallel(x_3+x_3')]$ that appears in the definition of $F(k_\parallel,\Omega)$ [see Eq. (2.15a)]. The physical origin of this divergence is that a spin fluctuation of long wavelength sets up a macroscopic magnetic field and the coupling between two such long wavelength spin fluctuations (one on each side of the surface) becomes very large.

b. Behavior of $F(k_\parallel,\Omega)$ *for* $\Omega\ll E_F$, $k_\parallel\ll k_F$, *but arbitrary* $\Omega/V_Fk_\parallel$. A closer examination of the general expression for $F(k_\parallel,\Omega)$ shows that the form displayed in Eq. (3.14) requires for its validity not only the criterion $\Omega\ll E_F$ stated earlier, but in addition one needs the condition $V_Fk_\parallel>\Omega$ fulfilled, where V_F is the Fermi velocity. When $V_Fk_\parallel$ is comparable to or smaller than Ω, the form of $F(k_\parallel,\Omega)$ is modified in a manner which eliminates the divergence in the Kapitza resistance calculation. Recall that in the bulk of the material, for small Ω, the spectral density is a function of the ratio Ω/V_Fk.[9] We thus find here a similar behavior for the function $F(k_\parallel,\Omega)$. We conclude this section by deriving an expression for $F(k_\parallel,\Omega)$ valid in the region $k_\parallel\ll k_F$, but for arbitrary values of the ratio $V_Fk_\parallel/\Omega$.

If we consider the form given in Eq. (3.12) for $F(k_\parallel,\Omega)$ for the case where $k_\parallel\ll k_F$, the factor of $\{k_\parallel^2+[q_0(\vec{Q}_\parallel)-q_0(\vec{Q}_\parallel+\vec{k}_\parallel)]^2\}^2$ becomes proportional to $k_\parallel^4$, while the remaining factors in Eq. (3.12) remain insensitive to $k_\parallel$, and smoothly approach their limiting values as $k_\parallel\to 0$. Thus, for $k_\parallel\ll k_F$ and $\Omega\ll E_F$, we may obtain an expression for $F(k_\parallel,\Omega)$ valid for arbitrary values of the ratio $V_Fk_\parallel/\Omega$ provided we use an improved approximation for this one sensitive factor; we can then set $k_\parallel=0$ in the remaining quantities in the integrand.

We begin by writing Eq. (3.7) in a slightly different form:

$$F(k_\parallel,\Omega)=\frac{2k_\parallel^2}{\pi^4}\int_0^\infty dQ_z\,dQ'_z\int d^2Q_\parallel\,\frac{Q_z^2Q_z'^2}{[k_\parallel^2+(Q_z-Q'_z)^2]^2}\times\frac{[f(E(\vec{Q}_\parallel Q_z)-\Omega)-f(E(\vec{Q}_\parallel,Q_z))]}{[k_\parallel^2+(Q_z+Q'_z)^2]^2}\times\delta(\Omega+E(\vec{Q}_\parallel+\vec{k}_\parallel,Q'_z)-E(\vec{Q}_\parallel Q_z)).\tag{3.16}$$

Now perform the integration over Q'_z. The argument of the δ function vanishes when

$$Q'_z=+[2M\Omega+Q_z^2+2\vec{Q}_\parallel\cdot\vec{k}_\parallel-\vec{k}_\parallel^2]^{1/2},\tag{3.17a}$$

$$Q'_z=+(2M)^{1/2}[\Omega+E(\vec{Q}_\parallel Q_z)-E(\vec{Q}_\parallel-\vec{k}_\parallel,0)]^{1/2},\tag{3.17b}$$

where for the argument of the δ function in Eq. (3.16) to vanish, it is necessary that the quantity inside the square brackets in Eq. (3.17b) be positive. The expression for $F(k_\parallel,\Omega)$ may then be written in the form

$$F(k_\parallel,\Omega)=\frac{2Mk_\parallel^2}{\pi^4}\int_{Q_z>0}d^3Q\,\frac{Q_z^2Q'_z[f(E(\vec{Q})-\Omega)-f(E(\vec{Q}))]}{[k_\parallel^2+(Q_z-Q'_z)^2]^2[k_\parallel^2+(Q_z+Q'_z)^2]}\,\theta(E(\vec{Q})-E(\vec{Q}_\parallel-\vec{k}_\parallel)+\Omega),\tag{3.18}$$

where in Eq. (3.18), Q'_z is the quantity defined in Eq. (3.17b), $E(\vec{Q})\equiv E(\vec{Q}_\parallel,Q_z)$, and the θ function in the integrand ensures that the quantity inside the square root in Eq. (3.17b) is always positive.

Now for $\Omega \ll E_F$, we make the replacement (as before)

$$\begin{aligned}[f(E(\vec{Q})-\Omega)-f(E(\vec{Q}))]\,\theta(E(\vec{Q})-E(\vec{Q}_\parallel-\vec{k}_\parallel)+\Omega)&\\ \cong+\Omega\delta(E(\vec{Q})-E_F)\,\theta(E(\vec{Q})-E(\vec{Q}_\parallel-\vec{k}_\parallel)+\Omega)&\\ =\Omega\delta\,(E(Q)-E_F)\,\theta(E_F+\Omega-E(Q_\parallel-k_\parallel)).&\end{aligned}$$

The quantity Q'_z now assumes the form

$$Q'_z=(2M)^{1/2}[E_F+\Omega-E(\vec{Q}_\parallel-\vec{k}_\parallel)]^{1/2}.$$

Now the integral over Q_z may be performed exactly as before, with the use of the identity exhibited just after Eq. (3.11). If in the following expression we let

$$Q_z=(2M)^{1/2}(E_F-Q_\parallel^2/2M)^{1/2},$$

[this quantity was called $q_0(\vec{Q}_\parallel)$ earlier; a slightly different notation appears useful here], then

$$F(k_\parallel,\Omega)=\frac{2M^2\Omega k_\parallel^2}{\pi^4}\int\frac{d^2Q_\parallel Q_zQ'_z\,\theta(E_F-E(\vec{Q}_\parallel))\,\theta(E_F+\Omega-E(\vec{Q}_\parallel-\vec{k}_\parallel))}{[k_\parallel^2+(Q_z-Q'_z)^2]^2[k_\parallel^2+(Q_z+Q'_z)^2]^2}.\tag{3.19}$$

We now suppose that $k_\parallel$ is small, but we make no assumption about the relative order of magnitude of $V_Fk_\parallel$ and Ω. Then $E(\vec{Q}_\parallel-\vec{k}_\parallel)\cong E(\vec{Q}_\parallel)-(1/M)\times\vec{Q}_\parallel\cdot\vec{k}_\parallel$,

$$\begin{aligned}Q'_z&=(2M)^{1/2}\{E_F-E(Q_\parallel)+[\Omega+(1/M)k_\parallel\cdot Q_\parallel]\}^{1/2}\cong(2M)^{1/2}[E_F-E(Q_\parallel)]^{1/2}\left[1+\frac{1}{2}\left(\frac{\Omega+(1/M)\vec{k}_\parallel\cdot\vec{Q}_\parallel}{E_F-E(\vec{Q}_\parallel)}\right)\right]\\&=Q_z+\frac{(M\Omega+\vec{Q}_\parallel\cdot\vec{k}_\parallel)}{Q_z}.\end{aligned}$$

Hence, for both Ω and $\vec{k}_\parallel$ small, we have

$$Q'_z-Q_z\cong(1/Q_z)(M\Omega+\vec{Q}_\parallel\cdot\vec{k}_\parallel),$$

while at the same time

$$Q'_z+Q_z\cong 2Q_z$$

and

$$k_\parallel^2+(Q_z+Q'_z)^2\approx 4Q_z^2.$$

When these approximations are put together, and the results rearranged, we have for $k_\parallel \ll k_F$ and $\Omega \ll E_F$,

$$F(k_\parallel,\Omega)=\frac{M^2\Omega}{8\pi^4k_\parallel^2}\times\int_0^{2\pi}\int_0^1\frac{d\theta\,d\xi\,\xi(1-\xi^2)}{[(1-\xi^2)+(\xi\cos\theta+\Omega/V_Fk_\parallel)^2]^2}.\tag{3.20}$$

We have not been able to evaluate the integrals in Eq. (3.20) for arbitrary values of $\Omega/V_Fk_\parallel$. However, this form serves to illustrate the behavior of $F(k_\parallel, \Omega)$ for fixed Ω, as $k_\parallel \to 0$. When $V_F k_\parallel >> \Omega$, note that the expression in Eq. (3.20) reduces to that in Eq. (3.14), with $g(k_\parallel/k_F)$ equal to unity, the limit appropriate to the region $k_\parallel << k_F$. However, the behavior is qualitatively different in the opposite limit $V_F k_\parallel << \Omega$. There Eq. (3.20) becomes

$$F(k_\parallel, \Omega) = \left(\frac{M^2 V_F^4}{16\pi^3}\right) \frac{k_\parallel^2}{\Omega^3} \; . \qquad (3.21)$$

Quite clearly, this result shows that when the proper form for $F(k_\parallel, \Omega)$ is employed in the Kapitza resistance calculation, no divergence will occur because of the $k_\parallel$ integration.

To summarize with the conclusions reached in the rather lengthy analysis presented in this section, we find the following behavior for the function $F(k_\parallel, \Omega)$ introduced in Sec. II, for a degenerate gas of noninteracting fermions, provided $\Omega << E_F$:

$$F(k_\parallel, \Omega) = \begin{cases} \left(\dfrac{M^2 V_F^4}{16\pi^3}\right) \dfrac{k_\parallel^2}{\Omega^3}, & k_\parallel << \dfrac{\Omega}{V_F} \qquad (3.22\text{a}) \\[2ex] \left(\dfrac{M^2 \Omega}{6\pi^3 k_\parallel^2}\right) g\!\left(\dfrac{k_\parallel}{k_F}\right), & \dfrac{\Omega}{V_F} << k_\parallel \qquad (3.22\text{b}) \end{cases}$$

where a plot of the function $g(\xi)$ is presented in Fig. 1, and a more complete expression for $F(k_\parallel, \Omega)$ valid for arbitrary $V_F k_\parallel/\Omega$ is given in Eq. (3.20), provided we have both $k_\parallel << k_F$ and $\Omega << E_F$.

These results will provide ingredients for the considerations in sections which follow.

IV. RATE OF ENERGY TRANSFER BETWEEN LIQUID He^3 AND CONDUCTION-ELECTRON SPINS: CASE OF DIPOLAR COUPLING

In this section we apply the formalism developed in Secs. II and III to the computation of the rate at which energy is transferred from He^3 nuclear spins to conduction-electron spins in a metal for the case where the spin-spin coupling has the dipolar form. We shall proceed by inserting the expression for $F(k_\parallel, \Omega)$ developed in Sec. III into Eq. (2.16b).

When we do this, we are using an extremely oversimplified picture of the nature of liquid He^3, since this amounts to treating it as a noninteracting collection of fermions. Actually, it is by now well established that liquid He^3 is strongly exchange enhanced. This is clear when the order of magnitude and the temperature dependence of the nuclear-spin susceptibility are examined.[10] We shall proceed for the moment as if the He^3 is a fluid of noninteracting fermions, and at the end of the discussion, physical considerations will lead us to modify the final formula we obtain to correct for exchange enhancement effects. In this particular case, we shall see that the nature of the modification we make is well justified from physical considerations.

In what follows, the mass of the He^3 atom will be denoted by M_H, its Fermi wave vector and Fermi velocity by $k_F(H)$ and $V_F(H)$, respectively, and the corresponding properties of the conduction electron will be denoted by M_e, $k_F(e)$, and $V_F(e)$.

Consider the expression for R_K given in Eq. (2.16b). We first perform the integration over $k_\parallel$, then that over Ω. In the region where both $V_F(H)k_\parallel > \Omega$ and also $V_F(e)k_\parallel > \Omega$, both $\mathfrak{F}(k_\parallel, \Omega)$ and $F(k_\parallel, \Omega)$ may be replaced by the forms given in Eq. (3.22b). Upon noting that $V_F(e) >> V_F(H)$, in the intermediate region where $\Omega/V_F(e) << k_\parallel << \Omega/V_F(H)$, we need to replace $\mathfrak{F}(k_\parallel, \Omega)$ (which describes the He^3) by the form in Eq. (3.22a) while $F(k_\parallel, \Omega)$ may still be approximated by Eq. (3.22b). Finally, when $k_\parallel << \Omega/V_F(e)$, both functions $\mathfrak{F}(k_\parallel, \Omega)$ and $F(k_\parallel, \Omega)$ may be approximated by Eq. (3.22a). Actually, if one examines the nature of the integrand, the third region makes a negligible contribution to the integration. This is because once $\mathfrak{F}(k_\parallel, \Omega)$ is replaced by Eq. (3.22a), the resulting integrand no longer diverges as $k_\parallel \to 0$ (in fact it approaches zero as $k_\parallel^3$), and the region near $k_\parallel = 0$ makes a small contribution to the integral. Thus, we shall be able to approximate $F(k_\parallel, \Omega)$ by Eq. (3.22b) throughout the entire region of the integration. We then have from Eq. (2.16b) that

$$\frac{1}{R_K} = \frac{8}{3\pi} \frac{\mu_e^2 \mu_n^2 M_e^2}{k_B T^2} \int_0^\infty d\Omega\, \Omega^3 n(\Omega)[1 + n(\Omega)] \times \int_0^\infty \frac{dk_\parallel}{k_\parallel} g\!\left(\frac{k_\parallel}{k_F(e)}\right) \mathfrak{F}(k_\parallel, \Omega). \qquad (4.1)$$

Now, as we have seen in Sec. III, $\mathfrak{F}(k_\parallel, \Omega)$ vanishes for $k_\parallel > 2k_F(H)$. Since $k_F(H)$ is small compared to $k_F(e)$, we can replace $g(k_\parallel/k_F(e))$ by its value at $k_\parallel = 0$ throughout the integration with little error. Then Eq. (4.1) becomes

$$\frac{1}{R_K} = \frac{8}{3\pi} \frac{\mu_e^2 \mu_n^2 M_e^2}{k_B T^2} \int_0^\infty d\Omega\, \Omega^3 n(\Omega)[1 + n(\Omega)] \times \int_0^{2k_F(H)} dk_\parallel\, k_\parallel\, \mathfrak{F}(k_\parallel, \Omega). \qquad (4.2)$$

We lack a simple closed formula for $\mathfrak{F}(k_\parallel, \Omega)$ valid for all values of $\Omega/V_F k_\parallel$. Our procedure will be to use Eq. (3.22a) for $k_\parallel \leqslant \Omega/V_F(H)$, and Eq. (3.22b) for $k_\parallel \geqslant \Omega/V_F(H)$. If we equate these two expressions in the small $k_\parallel$ region, we find they become equal when $\Omega = V_F(H) k_\parallel (\frac{3}{8})^{1/4} = 0.78 V_F(H) k_\parallel$. This procedure thus extrapolates the form of $\mathfrak{F}(k_\parallel, \Omega)$ valid for small $k_\parallel$ up to larger values, and that valid for large $k_\parallel$ down until the two extrapolated functions become equal. This procedure leads to the correct qualitative behavior of the integral, and should not lead to serious quantitative errors.

Thus, we have

$$\int_0^{2k_F(H)} dk_\parallel\, k_\parallel\, \mathfrak{F}(k_\parallel, \Omega) \equiv \frac{M_H^2 V_F(H)^4}{16\pi^3\Omega^3} \int_0^{\Omega/V_F(H)} dk_\parallel\, k_\parallel^3 + \frac{M_H^2\Omega}{6\pi^3} \int_{\Omega/V_F(H)}^{2k_F(H)} \frac{dk_\parallel}{k_\parallel}\, g\left(\frac{k_\parallel}{k_F(H)}\right)$$

$$= \frac{\Omega M_H^2}{64\pi^3} + \frac{M_H^2\Omega}{6\pi^3} \int_{\Omega/V_F(H)}^{2k_F(H)} \frac{dk_\parallel}{k_\parallel} + \frac{M_H^2\Omega}{6\pi^3} \int_{\Omega/V_F(H)}^{2k_F(H)} \frac{dk_\parallel}{k_\parallel} \left[g\left(\frac{k_\parallel}{k_F(H)}\right) - 1\right] .$$

The integrand in the third integral is nonsingular in the limit $k_\parallel \to 0$ since, as we have seen, $g(k_\parallel/k_F(H)) = 1 + O(k_\parallel/k_F(H))$ as $k_\parallel$ approaches zero. Thus, we may set the lower limit of this integral equal to zero. Then we find

$$\int_0^{2k_F(H)} dk_\parallel\, k_\parallel\, \mathfrak{F}(k_\parallel, \Omega) = \frac{M_H^2\Omega}{6\pi^3} \ln\left(\frac{4E_F(H)}{\Omega}\right) + \frac{M_H^2\Omega}{64\pi^3}\left(1 + \frac{32}{3}\int_0^2 \frac{d\xi}{\xi}[g(\xi) - 1]\right) . \tag{4.3}$$

A numerical integration based on the numerical evaluation of $g(\xi)$ presented in Fig. 1 yields

$$\int_0^2 \frac{d\xi}{\xi}[g(\xi) - 1] \cong -1.66, \tag{4.4}$$

so that our approximation for the integral which appears in Eq. (4.3) is

$$\int_0^{2k_F(H)} dk_\parallel\, k_\parallel\, \mathfrak{F}(k_\parallel, \Omega) = \frac{M_H^2\Omega}{6\pi^3}\left[\ln\left(\frac{4E_F(H)}{\Omega}\right) - 1.55\right] . \tag{4.5}$$

Thus, for the Kapitza resistance we have

$$\frac{1}{R_K} = \frac{4}{9\pi^4} \frac{\mu_e^2 \mu_n^2 M_e^2 M_H^2}{k_B T^2} \int_0^\infty d\Omega\, \Omega^4 n(\Omega)$$

$$\times [1 + n(\Omega)]\, [\ln(4E_F(H)/\Omega) - 1.55] . \tag{4.6}$$

In the integration over frequency, let $\Omega = k_B T \xi$, where ξ is a dimensionless variable. Then

$$\int_0^\infty d\Omega\, \Omega^4 n(1+n)[\ln(4E_F(H)/\Omega) - 1.55] = (k_B T)^5 \left\{ \int_0^\infty d\xi\, \xi^4 n(1+n)\left[\ln\left(\frac{4E_F(H)}{k_B T \xi}\right) - 1.55\right]\right\}$$

$$= (k_B T)^5 \ln\left(\frac{4E_F(H)}{k_B T}\right) \int_0^\infty d\xi\, \xi^4 n(1+n) + (k_B T)^5 \int_0^\infty d\xi\, \xi^4 n(1+n)\left[\ln\left(\frac{1}{\xi}\right) - 1.55\right] .$$

Now we have

$$\int_0^\infty d\xi\, \xi^4 n(1+n) = 24\zeta(4),$$

where $\zeta(4)$ is the Riemann ζ function with argument 4, so that we have

$$\frac{1}{R_K} = \frac{32\zeta(4)}{3\pi^4} k_B \mu_e^2 \mu_n^2 M_e^2 M_H^2 (k_B T)^3$$

$$\times\left[\ln\left(\frac{4E_F(H)}{k_B T}\right) - 1.55 - \int_0^\infty \frac{d\xi\, \xi^4 n(1+n)}{24\zeta(4)} \ln\xi\right] . \tag{4.7}$$

We have calculated numerically the integral which remains in Eq. (4.6), and we find

$$\int_0^\infty \frac{d\xi\, \xi^4 n(1+n)\ln\xi}{24\zeta(4)} = 1.37. \tag{4.8}$$

Thus, our final result for the Kapitza conductance assumes the form (with $\hbar$ inserted explicitly)

$$\frac{1}{R_K} = \frac{32\zeta(4)}{3\pi^4\hbar^9} k_B \mu_e^2 \mu_n^2 M_e^2 M_H^2 (k_B T)^3$$

$$\times\left[\ln\left(\frac{4E_F(H)}{k_B T}\right) - 2.92\right] . \tag{4.9}$$

From Eq. (4.8), we see that as $T \to 0$, the contribution to the Kapitza coupling from dipolar coupling between the He^3 nuclear spins and the conduction-electron spins in the metal exhibits a $T^3 \ln T$ variation with temperature. The $\ln T$ portion of the temperature dependence has its origin in the near divergence in the integration over $k_\parallel$ that occurs because in the region $V_F(H)k_\parallel > \Omega$, the function $\mathfrak{F}(k_\parallel, \Omega)$ increases as $k_\parallel^{-2}$ as $k_\parallel$ decreases.

One major approximation in the result displayed in Eq. (4.9) is that the liquid He^3 was treated as a noninteracting Fermi fluid, while it is known that exchange enhancement effects are very large in this system. It is extremely difficult to modify our calculation in a manner that takes proper account of exchange enhancement effects, largely because the description of the magnitude of the exchange enhancement near a surface is quite difficult. The theoretical investigations of this question which have appeared to date[7] have all confined their attention to static properties of the surface region, and the extension of these descriptions to the construction of the spectral density for nonzero frequencies near the surface is nontrivial. Indeed, even in the static limit, the physics is still not completely clear.

However, in the region where the $T^3 \ln T$ behavior characterizes the Kapitza conductance, physical considerations suggest a simple modification of the result in Eq. (4.8) should be adequate. In the $T^3 \ln T$ region, the dominant contribution to the Kapitza conductance comes from values of $k_\parallel$ the order of $k_B T/V_F(H)$, as we see from the divergent behavior of the $k_\parallel$ integration in Eq. (2.16b), if careful account is not taken of the behavior of $\mathcal{F}(k_\parallel, \Omega)$ for values of $k_\parallel$ the order of $\Omega/V_F(H)$. Now when we construct $\mathcal{F}(k_\parallel, \Omega)$ from the spectral density $\mathcal{A}(x'x;\, \vec{k}_\parallel \Omega)$[see Eq. (2.12a)], we average the spectral density function over a spatial distance the order of $k_\parallel^{-1}$. Thus, to compute $\mathcal{F}(k_\parallel, \Omega)$ for $k_\parallel$ the order of $k_B T/V_F(H)$, we take an average of the spectral density function over a rather extended region from the surface, a distance the order of $V_F(H)/k_B T$. Thus, we should be able to calculate $\mathcal{F}(k_\parallel, \Omega)$ from the bulk paramagnon propagator to obtain the Kapitza conductance in the $T^3 \ln T$ temperature range, since the effect of the surface on the spectral density must be small over much of the volume involved in the calculation of $\mathcal{F}(k_\parallel, \Omega)$, for these small values of $k_\parallel$. For long wavelengths, in bulk-paramagnon theory, the spectral density is enhanced over the value appropriate to the noninteracting Fermi gas result by the factor $(1-\bar{I})^{-2}$, where $\bar{I}$ is the Stoner exchange enhancement factor. As long as the wave vector of the paramagnon is small compared to the Fermi wave vector, the enhancement of the spectral density is wave-vector independent. Thus, if $(1-\bar{I}_H)^{-1}$ denotes the exchange enhancement factor for the He^3, and $(1-\bar{I}_e)^{-1}$ that for the conduction electrons in the metal, then in the $T^3 \ln T$ regime, we may presume that

$$\frac{1}{R_K} = \frac{32\zeta(4)}{3\pi^4 \hbar^9} \frac{k_B \mu_e^2 \mu_n^2 M_e^2 M_H^2}{(1-\bar{I}_H)^2 (1-\bar{I}_e)^2} (k_B T)^3 \ln\left(\frac{4E_F(H)}{k_B T}\right). \tag{4.10}$$

Two additional comments should be made. Since $(1-\bar{I}_H)^{-1}$ varies with pressure,[10] the contribution to the Kapitza conductance from this contribution should vary with pressure. Second, since the interaction between spin fluctuations across the interface involves coupling between regions rather far from the surface, at least at low temperatures, this contribution should be insensitive to a layer or two of He^4 or solid He^3 deposited on the interface.

While previous authors[5] have suggested that the anomalous temperature dependence of the Kapitza resistance of the boundary between Pt metal and He^3 may be explained by transfer of energy between He^3 nuclear spins and the conduction electrons of the Pt, the present results show that the contribution to the Kapitza conductance from this mechanism is too small by many orders of magnitude to explain the order of magnitude of the observed boundary resistance, if the coupling proceeds through the dipolar interaction. From Eq. (4.9) with $\bar{I}_e = 0$ and $\bar{I}_H = 0.9$, we estimate that the product

$$\begin{aligned} R_K T^3 &\approx \left(\frac{2\times 10^5}{\ln(4E_F(H)/k_B T)}\right) \frac{\mathrm{cm}^2\,{}^\circ\mathrm{K}^4\,\mathrm{sec}}{\mathrm{erg}} \\ &= \left(\frac{2\times 10^{12}}{\ln(4E_F(H)/k_B T)}\right) \frac{\mathrm{cm}^2\,{}^\circ\mathrm{K}^4}{\mathrm{W}}\,. \end{aligned}$$

At the boundary between He^3 and metals at low temperatures, the Kapitza resistance per unit area is smaller than this estimate by some ten orders of magnitude. This discrepancy is so large that even with strong enhancement in the host matrix, ($\bar{I}_e$ near 1), the calculated and observed rate of energy transfer cannot be reconciled. Our conclusion is therefore that the contribution to the interface conduction from the present mechanism is so tiny that it plays no role in any current experimental configuration.

V. RATE OF ENERGY TRANSFER BETWEEN LIQUID He^3 AND CONDUCTION-ELECTRON SPINS: SHORT-RANGED EXCHANGE INTERACTION

In this section we consider the energy transfer between the He^3 nuclei and conduction electrons which results from a short-range spin-dependent interaction of the exchange type. The formalism of Sec. II applies quite directly to this case, of course. We presume the interaction is an isotropic exchange interaction, so the quantity $d_{\alpha\beta}(\vec{x}-\vec{y})$ which appears in Eq. (2.1) has the form given in Eq. (2.2b):

$$d_{\alpha\beta}(\vec{x}-\vec{y}) = \delta_{\alpha\beta}\, d(\vec{x}-\vec{y}). \tag{5.1}$$

We cannot take the interaction to be a contact interaction strictly speaking, since both the He^3 and conduction-electron wave functions vanish at $x_3 = y_3 = 0$ for our model. We adopt the following form for $d(\vec{x})$:

$$d(\vec{x}) = (\tfrac{1}{2} g\sigma)\delta(\vec{x}_{\parallel})e^{-\sigma|x_3|}. \tag{5.2}$$

We shall assume the interaction length $l=\sigma^{-1}$ is smaller than all other lengths in the problem (i.e., the Fermi wavelengths) for convenience in performing the algebra. In the end, we shall rewrite the result obtained on that basis in a form we feel is valid more generally.

If we begin with Eq. (2.13), and follow steps quite analogous to those which lead to the form in Eq. (2.16), for the present case we find we may write

$$\frac{1}{R_K} = \frac{3g^2\sigma^2}{4k_B T^2} \int_0^\infty d\Omega\, \Omega^2 n(\Omega)[1+n(\Omega)] \times \int_0^\infty dk_{\parallel} k_{\parallel} g(k_{\parallel},\Omega)\, G(k_{\parallel},\Omega), \tag{5.3}$$

where

$$g(k_{\parallel},\Omega) = \int_0^\infty dx\, dx'\, e^{-\sigma(x+x')}\, \mathfrak{A}(x'x;\, k_{\parallel}\Omega). \tag{5.4}$$

By using the explicit form of $\mathfrak{A}(x'x;\, k_{\parallel}\Omega)$ obtained in Sec. III, and proceeding in the same way as the derivation which led to Eq. (3.7), we have

$$g(k_{\parallel},\Omega) = \frac{2\sigma^2}{\pi^4} \int_0^\infty dQ_z\, dQ_z' \int d^2Q_{\parallel} \frac{Q_z^2 Q_z'^2}{[\sigma^2+(Q_z-Q_z')^2]^2} \times \frac{[f(E(\vec{Q}_{\parallel}+\vec{k}_{\parallel}, Q_z')) - f(E(\vec{Q}_{\parallel}, Q_z))]}{[\sigma^2+(Q_z+Q_z')]^2} \times \delta(\Omega + E(\vec{Q}_{\parallel}+\vec{k}_{\parallel}, Q_z) - E(\vec{Q}_{\parallel}, Q_z)). \tag{5.5}$$

As stated above, we consider the limit that $l=\sigma^{-1}$ is very small. Thus, the factors of $\sigma^2 + (Q_z \pm Q_z')^2$ in Eq. (5.5) may be replaced by σ^2 to yield

$$g(k_{\parallel},\Omega) = \frac{2}{\pi^4\sigma^6} \int_0^\infty dQ_z\, dQ_z'\, Q_z^2 Q_z'^2 \int d^2Q_{\parallel} \times [f(E(\vec{Q}_{\parallel}+\vec{k}_{\parallel}, Q_z')) - f(E(\vec{Q}_{\parallel} Q_z))] \times \delta(\Omega + E(\vec{Q}_{\parallel}+\vec{k}_{\parallel}, Q_z') - E(\vec{Q}_{\parallel}, Q_z)). \tag{5.6}$$

As before, we need the form of $g(k_{\parallel},\Omega)$ for small Ω. Furthermore, we shall not encounter difficulties at small $k_{\parallel}$ similar to those that proved difficult in the dipolar case. Thus, we may keep the term linear in Ω in $g(k_{\parallel},\Omega)$, and presume $V_F k_{\parallel}$ is large compared to Ω. Upon expanding the Fermi functions as we did in the discussion which precedes Eq. (3.17), then keeping the term linear in Ω, we have

$$g(k_{\parallel},\Omega) = \frac{2\Omega}{\pi^4\sigma^6} \int_0^\infty dQ_z\, dQ_z'\, Q_z^2 Q_z'^2 \int d^2Q_{\parallel} \times \delta(E_F - E(\vec{Q}_{\parallel} Q_z))\, \delta(\Omega - E(\vec{Q}_{\parallel}+\vec{k}_{\parallel}, Q_z')). \tag{5.7}$$

Since $E(\vec{Q}_{\parallel}, Q_z) = (1/2M)(Q_{\parallel}^2 + Q_z^2)$, the integration over Q_z and Q_z' is readily performed to yield

$$G(k_{\parallel},\Omega) = \frac{2\Omega M^2}{\pi^4\sigma^6} \int_0^{k_F} dQ_{\parallel}\, Q_{\parallel}(k_F^2 - Q_{\parallel}^2)^{1/2} \times \int_0^{2\pi} d\theta [k_F^2 - (\vec{Q}_{\parallel}+\vec{k}_{\parallel})^2]^{1/2} \theta(k_F^2 - (\vec{Q}_{\parallel}+\vec{k}_{\parallel})^2), \tag{5.8}$$

where again, $\theta(x)$ is unity for $x \geqslant 0$ and 0 for $x<0$.

The integral in Eq. (5.8) cannot be evaluated in closed form. However, quite clearly at $k_{\parallel}=0$, $G(k_{\parallel},\Omega)$ remains finite, with no divergence as in the case of the dipolar interaction, where we found that $F(k_{\parallel},\Omega)$ varied as $k_{\parallel}^{-2}$ for $k_{\parallel} \ll k_F$, and $V_F k_{\parallel} > \Omega$. The absence of the divergence means that the region of small $k_{\parallel}$ need not be treated with care, and we shall find R_K^{-1} will vary as T^3. Since the integral cannot be evaluated in closed form, we shall explore two special limits, and then use an interpolation formula to extrapolate between these limits. The use of an approximate form for $G(k_{\parallel},\Omega)$ in the calculation of R_K will affect the numerical coefficient of the result, but not its temperature dependence.

(i) The limit $k_{\parallel} \to 0$. A straightforward integration shows that

$$G(0,\Omega) = \Omega M^2 k_F^4/\pi^3\sigma^6. \tag{5.9a}$$

(ii) The limit $k_{\parallel} \to 2k_F$. The function $G(k_{\parallel},\Omega)$ vanishes for $k_{\parallel} > 2k_F$. We study the manner in which $G(k_{\parallel},\Omega)$ approaches zero as $k_{\parallel} \to 2k_F$.

If the structure of the integral in Eq. (5.8) is examined for $k_{\parallel}$ near $2k_F$, then only values of $Q_{\parallel}$ near k_F enter the integration, along with values of θ near π. Thus, if we let

$$k_{\parallel} = k_F(2-\epsilon),$$

$$|\vec{Q}_{\parallel}| = k_F(1-\eta),$$

and

$$\theta = \pi + \varphi,$$

then for $\epsilon \ll 1$ the integral may be transformed to read

$$G(k_{\parallel},\Omega) = \frac{4\Omega M^2 k_F^4}{\pi^4\sigma^6} \int_0^\epsilon d\eta\, \eta^{1/2} \int_{-(\epsilon-\eta)^{1/2}}^{+(\epsilon-\eta)^{1/2}} \times d\varphi[\epsilon - \eta - \varphi^2]^{1/2} = \frac{8\Omega M^2 k_F^4}{15\pi^3\sigma^6}\epsilon^{5/2}$$

or

$$G(k_{\parallel},\Omega)=\frac{8\Omega M^2k_F^4}{15\pi^3\sigma^6}\left(\frac{2k_F-k_{\parallel}}{k_F}\right)^{5/2},\quad k_{\parallel}\text{ near }2k_F. \tag{5.9b}$$

The general form of $G(k_{\parallel},\Omega)$ is evident from the results displayed in Eqs. (5.9a) and (5.9b). The function is finite at $k_{\parallel}=0$, and decreases monotonically to zero at $k_{\parallel}=2k_F$. For $k_{\parallel}$ near $2k_F$, the manner in which $G(k_{\parallel},\Omega)$ approaches zero is described by Eq. (5.9b).

In our calculation of the Kapitza resistance, the region near $k_{\parallel}=0$ is relatively unimportant, because of the explicit factor of $k_{\parallel}$ that appears in Eq. (5.3). Thus we shall proceed by presuming $G(k_{\parallel},\Omega)$ may be described by the form in Eq. (5.9b) for the important values of $k_{\parallel}$. This approximation seems reasonable, since, if the result in Eq. (5.9b) is extrapolated back to zero, it gives a result of the same form as that in Eq. (5.9a), but with a numerical coefficient that differs by a numerical factor of roughly 3. Thus we have

$$\frac{1}{R_K}=\frac{16g^2M_H^2M_e^2}{75\pi^6\sigma^{10}k_BT^2}k_F^4(H)k_F^4(M)$$
$$\times\int_0^{\infty}d\Omega\,\Omega^4n(\Omega)[1+n(\Omega)]\int_0^{2k_F^{(H)}}dk_{\parallel}\,k_{\parallel}$$
$$\times\left(\frac{2k_F(H)-k_{\parallel}}{k_F(H)}\right)^{5/2}\left(\frac{2k_F(M)-k_{\parallel}}{k_F(M)}\right)^{5/2}, \tag{5.10}$$

where $k_F(H)$ and $k_F(M)$ are the magnitudes of the Fermi wave vector of the liquid helium and the metal, respectively. The integral over frequency is carried out to give

$$\frac{1}{R_K}=k_B\frac{64g^2M_H^2M_e^2}{1125\pi^2\sigma^{10}}k_F^4(H)k_F^4(M)(k_BT)^3$$
$$\times\int_0^{2k_F^{(H)}}dk_{\parallel}\,k_{\parallel}\left(\frac{2k_F(H)-k_{\parallel}}{k_F(H)}\right)^{5/2}\left(\frac{2k_F(M)-k_{\parallel}}{k_F(M)}\right)^{5/2}.$$

For convenience, we next presume that $k_F(M) >> k_F(H)$, so the last factor in the integral may be replaced by $2^{5/2}$. The integral over $k_{\parallel}$ is then readily performed to yield

$$\frac{1}{R_K}=0.45\frac{g^2M_H^2M_e^2k_F^4(H)k_F^4(M)}{\hbar^9\pi^2\sigma^{10}}(k_BT)^3. \tag{5.11}$$

We rewrite this result in a form which makes the factors which appear in it more transparent. First we introduce the length $l=\sigma^{-1}$ which measures the range of the interaction. Then notice that for $k_Fl<<1$,

$$\tfrac{1}{3}k_F^2l^2=\int_0^l\sin^2k_Fz\,dz\Big/\int_0^l dz\equiv\langle\Psi_F^2\rangle. \tag{5.12}$$

The quantity $\langle\Psi_F^2\rangle$ measures the overlap between the probability density of an electron at the Fermi surface and the interaction region near the surface. In terms of this quantity we have (with $\hbar$ inserted)

$$\frac{1}{R_K}\cong 37k_B\frac{g^2M_H^2M_e^2[lk_F(H)]^2}{\pi^2\hbar^9}$$
$$\times|\langle\Psi_F^2(H)\rangle|^2|\langle\Psi_F^2(M)\rangle|^2(k_BT)^3. \tag{5.13}$$

We expect this result to be valid even when l is the order of the Fermi wavelength, since each factor in it has a simple physical interpretation; the factor of $k_F^2(H)$ measures the amount of phase space available to the fluctuations with wave vector parallel to the surface [$k_{\parallel}$ is limited to the range between 0 and $2k_F(H)$ when $k_F(H)<k_F(M)$], and the factors $|\langle\Psi_F^2\rangle|$ measure the strength of the spin fluctuations that lie within the interaction range.

We conclude this section with an estimate of the magnitude of the contribution to the Kapitza conductance in Eq. (5.13). If the interaction length l is the order of 1 Å, then we expect the dimensionless factors $\langle\Psi_F^2(H)\rangle$, $\langle\Psi_F^2(M)\rangle$, and $k_F(H)l$ to be close to unity in value. The major uncertainty is the value of g, which has the dimensions of energy times volume. Now the parameter I, which measures the spin-dependent interaction between two He^3 atoms in the liquid in paramagnon theory (through the term $I\sum_i n_{i\uparrow}n_{i\downarrow}$ in the Hamiltonian) is ≈ 5°K. If we assume the spin-dependent interaction between the conduction electron and the He^3 atom to be similar in strength, then g will be a number the order of an atomic volume (10^{-24} cm^3) multiplied by an energy of 10^{-15} erg, i.e., $g\approx 10^{-39}$ erg cm^3. With this number we estimate

$$\frac{1}{R_KT^3}\approx 50\frac{\text{cm sec °K}^4}{\text{erg}}$$
$$=5\times 10^8\frac{\text{cm}^2\,\text{°K}^4}{\text{W}}. \tag{5.14}$$

Thus, this estimate indicates that while the contact interaction is far more efficient than the dipole-dipole interaction at transferring energy from the He^3 nuclear spins to the conduction-electron spins (presuming that our estimate of g is reasonable), nonetheless, the contribution from this source is also many orders of magnitude smaller than the observed thermal boundary conductance. While our estimate of g may have led us to assign a value to this quantity that is too small, g would have to be some three orders of magnitude larger than our choice to fit the data. Such a value for g seems quite unphysical. We thus conclude that neither dipolar coupling between the He^3 spins and the conduction electrons nor short-range interactions of the exchange type can transfer a physically significant amount of energy across the surface.

VI. ENERGY TRANSFER FROM He^3 NUCLEAR SPINS TO AN ARRAY OF LOCAL MOMENTS

In this section we examine the contribution to the Kapitza conductance from the dipolar interaction between He^3 nuclear spins, and an array of paramagnetic local moments. We begin by examining the spin correlation function that appears on the left-hand side of Eq. (2.10b). We have

$$S_z(\vec{y}t) = \sum_{\vec{l}} S_z(\vec{l}t)\delta(\vec{y} - \vec{l})$$

so that

$$\langle S_z(\vec{y}'t)S_z(\vec{y}0)\rangle = \sum_{\vec{l}\vec{l}'}\delta(\vec{y} - \vec{l}) \times \delta(\vec{y}' - \vec{l}')\langle S_z(\vec{l}'t)S_z(\vec{l}0)\rangle .$$

If we assume that the local moments are paramagnetic, and the spins are dynamically independent, then

$$\langle S_z(\vec{l}'t)S_z(\vec{l}0)\rangle = \delta_{\vec{l}\vec{l}'}\langle S_z(\vec{l}t)S_z(\vec{l}0)\rangle .$$

It then follows that the spectral density $A(\vec{y}'\vec{y};\, \Omega)$ defined in Eq. (2.10b) has the form

$$A(\vec{y}'\vec{y};\, \Omega) = \delta(\vec{y} - \vec{y}')\sum_{\vec{l}}\delta(\vec{y} - \vec{l})A_l(\Omega), \tag{6.1}$$

where

$$\langle S_z(\vec{l}t)S_z(\vec{l}0)\rangle = \int_{-\infty}^{+\infty} d\Omega\, e^{+i\Omega t}\, n(\Omega)A_l(\Omega).$$

If this form for the spectral density is inserted into Eq. (2.11), then we have

$$\frac{1}{R_K} = \frac{\pi}{Ak_BT^2}\sum_{\alpha\beta}\sum_{\vec{l}} \int d^3x\, d^3x'\, d_{\alpha\beta}(\vec{x}' - \vec{l}) \times d_{\alpha\beta}(\vec{x} - \vec{l})\int_{-\infty}^{+\infty} d\Omega\, \Omega^2 n(\Omega) \times [1 + n(\Omega)]\,\mathcal{A}(\vec{x}'\vec{x};\, \Omega)A_{\vec{l}}(\Omega). \tag{6.2}$$

If we now insert the expression for $\mathcal{A}(\vec{x}'\vec{x};\, \Omega)$ given in Eq. (2.12a) into Eq. (6.2), then note the definition of $D_{\alpha\beta}(\vec{k}_\parallel;\, x_3)$ in Eq. (2.12c) along with the identity in Eq. (2.14), we obtain

$$\frac{1}{R_K} = \frac{8\pi^2\mu_e^2\mu_n^2}{Ak_BT^2}\sum_{\vec{l}_\parallel l_3}\int_0^\infty dk_\parallel\, k_\parallel^3\, e^{-2k_\parallel l_3} \times \int_{-\infty}^{+\infty} d\Omega\, \Omega^2 n(1 + n)\, \mathcal{F}(k_\parallel, \Omega)A_{\vec{l}}(\Omega). \tag{6.3}$$

In this expression, we have written $\vec{l} = \vec{l}_\parallel + \hat{z}l_3$, and we have assumed that the spins are distributed over a set of layers, each of which is parallel to the surface.

Now we can expect that the spectral density $A_{\vec{l}}(\Omega)$ is independent of $\vec{l}_\parallel$ and depends only on l_3, if all spins in a given layer are equivalent (or if we perform an average over a dilute array of local moments). The sum over $\vec{l}_\parallel$ then gives a factor of N_s, the number of spins in each layer. We let

$$n_s = N_s/A,$$

the number of local moments per unit area. Also, we shall not encounter divergences in the integration over $k_\parallel$ for small $k_\parallel$, so we may use the form in Eq. (3.20b) for $\mathcal{F}(k_\parallel, \Omega)$. We then have

$$\frac{1}{R_K} = \frac{4M_H^2\mu_e^2\mu_n^2 n_s}{3\pi k_BT^2}\sum_{l_3}\int_0^\infty dk_\parallel\, k_\parallel\, g\left(\frac{k_\parallel}{k_F(H)}\right) \times e^{-2k_\parallel l_3}\int_{-\infty}^{+\infty} d\Omega\, \Omega^3 n(1 + n)A_{l_3}(\Omega). \tag{6.4}$$

The integral over $k_\parallel$ in Eq. (6.4) has a distinctly different structure than the integration encountered in Sec. IV. If we set $l_3 = 0$, then the dominant contribution to the sum comes from values of $k_\parallel$ the order of $k_F(H)$, the Fermi wave vector of the He^3. It then follows that if a_0 is the spacing between successive layers of spins, and $2k_F(H)a_0 \gg 1$, then the dominant contribution to the sum comes from the spins in the surface layer $l_3 = 0$. This condition is well satisfied in CMN, but may be satisfied less well in other solids, where the distance between near-neighbor spins is far smaller than CMN. If we keep only the term $l_3 = 0$ in the sum in Eq. (6.4), then we can regard the dipolar interaction as formally equivalent to a contact interaction between the He^3 spins, and the local moment. It is important to stress that this equivalence obtains only if $2k_F(H)a_0 \gg 1$, and one may encounter systems for which the approximation of the dipolar interaction by a contact interaction is less good. Then if we keep only the term $l_3 = 0$, we have

$$\frac{1}{R_K} = \frac{4M_H^2 k_F^2(H)\mu_e^2\mu_n^2 n_s}{3\pi k_BT^2}\, \mathcal{G}\int_{-\infty}^{+\infty} d\Omega\, \Omega^3 n(1 + n)A_0(\Omega), \tag{6.5}$$

where

$$\mathcal{G} = \int_0^2 d\xi\, \xi g(\xi) = 0.26, \tag{6.6}$$

where the numerical value of $\mathcal{G}$ has been obtained from a numerical integration.

This result is equivalent to that obtained by Leggett and Vuorio, except for an over-all numerical factor. To make the comparison between our work and their earlier work explicit, we display their result in their notation ignoring for the moment the exchange enhancement factor they insert in a phenomenological fashion:

$$\frac{1}{(R_K)_{\text{L-V}}}=\frac{\pi k_B^2 g_0^2}{\hbar}\left(\frac{dn}{d\epsilon}\right)^2 n_a T Q_s(T,H)\ , \tag{6.7}$$

where n_a is the number of spins per unit area, and their factor g_0 for the case where the local moments have spin $\frac{1}{2}$ is denoted by $\frac{1}{4}\mu_e\mu_n$ in our work. (Their spin variables $\vec{\sigma}$ and $\vec{S}$ are the Pauli matrices, while our $\vec{s}$ and $\vec{S}$ are spin operators.) The function $dn/d\epsilon$ is the density of states of the fluid at the Fermi surface, so that

$$\frac{dn}{d\epsilon}=\frac{M_H k_F(H)}{\pi^2}\ ,$$

so that in our notation, Eq. (6.7) becomes (with $\hbar = 1$)

$$\frac{1}{(R_K)_{\text{L-V}}}=\frac{k_B M_H^2 k_F^2(H)\mu_e^2\mu_n^2 n_s}{16\pi^3}\,k_B T Q_s(T,H)\ . \tag{6.8}$$

After a slight modification of Eq. (21) in Ref. 3, their function Q_s may be written

$$Q_s=\frac{1}{2(k_BT)^3}\sum_\alpha\int_{-\infty}^{\infty}d\Omega\,\Omega^3 n(\Omega)[1+n(\Omega)]\chi_{s\alpha}(\Omega)\ ,$$

where

$$\chi_{s\alpha}=\frac{1}{Z}\sum_{nm}e^{-\beta E_m}|\langle n|S_\alpha|m\rangle|^2$$
$$\times[\delta(E_n-E_m-\Omega)-\delta(E_n-E_m+\Omega)]\ .$$

In zero magnetic field, the only case where our final result applies [we have used the assumption that $H=0$ which precedes Eq. (2.9); this assumption is readily relaxed] $\chi_{s\alpha}$ becomes independent of α, so that

$$Q_s=\frac{3}{2(k_BT)^3}\int_{-\infty}^{+\infty}d\Omega\,\Omega^3 n(\Omega)[1+n(\Omega)]\chi_{s_z}(\Omega)\ . \tag{6.9}$$

It is a straightforward exercise to show that when the local moments have spin $\frac{1}{2}$, their function $\chi_{s_z}(\Omega)$ is $4A_0(\Omega)$ in our notation, where the factor of 4 has its origin once again in the fact that their $\vec{S}$ is a Pauli matrix, while ours is the spin angular momentum of the local moment. After this identification is made, and Eq. (6.9) is substituted into Eq. (6.8), in our notation Leggett and Vuorio's expression for the Kapitza resistance becomes

$$\frac{1}{(R_K)_{\text{L-V}}}=\frac{3M_H^2 k_F^2(H)\mu_e^2\mu_n^2 n_s}{8\pi^3 k_B T^2}$$
$$\times\int_{-\infty}^{+\infty}d\Omega\,\Omega^3 n(\Omega)[1+n(\Omega)]A_0(\Omega)\ . \tag{6.10}$$

If Leggett and Vuorio's expression for the Kapitza resistance is compared with ours, one has

$$\frac{(R_K)_{\text{L-V}}}{(R_K)_{\text{Present}}}=\frac{32\pi^2}{9}\mathscr{g}\cong 9\ . \tag{6.11}$$

Thus, while the two calculations produce the same final formula for R_K to within an over-all constant, there is an appreciable quantitative difference between the two. Our result suggests that Leggett and Vuorio's approximation procedures lead them to underestimate the contribution to the Kapitza conductance form dipolar spin-spin interactions. If we accept Leggett and Vuorio's phenomenological modification of the result for R_K to account for exchange enhancement effects, then we find a value for the Kapitza conductance larger than that observed experimentally at the He^3-CMN interface. In view of the uncertainty of the exchange enhancement correction, along with differences between the ideal surface that forms the basis for this calculation and the real interface, we do not regard this discrepancy as significant. In experimental-grade He^3, a tightly bound monolayer of solid He^4 may be present on the solid surface, unless the He^3 is extremely pure. This will increase the distance of closest approach between a He^3 atom and the local moments in the surface, and decrease the rate of direct energy transfer between the two systems. (The rate of transfer between the He^3 nuclei in the liquid, and those in any adsorbed layers may be small compared to the transfer to the local moment system, because $\mu_n \ll \mu_e$). For reasons such as this, one can regard the value of R_K provided by our calculation to be a lower limit on the value this quantity may assume in practice.

We also see from our work that the notion that the dipolar interaction can be replaced by a contact interaction is valid as long as $2k_F(H)a_0 \gg 1$. This criterion seems valid both for CMN, where $a_0 \approx 15$ Å, and also in the alloy systems studied recently.[4]

VII. GENERAL DISCUSSION

There are two general conclusions which follow from the present work. First of all, we have rederived the result of Leggett and Vuorio in a manner which removes two important assumptions which were introduced by them. Second, we have examined the contribution to the Kapitza conductance which arises from spin-spin coupling between He^3 nuclei and conduction electrons at He^3-metal interfaces, and we find that this contribution is quantitatively very small. We begin this section with a comparison between the two cases considered here, the case where the He^3 spins couple to local moments, and the case where they couple to conduction electrons.

First, consider the case where the He^3 nuclei couple to conduction electrons. The results for the short-ranged exchange interaction and for the dipolar interaction are quite similar in form, except for the $\ln T$ factor, which owes its existence to the peculiar long-ranged nature of the dipolar interaction. For simplicity, we confine our general comments to the first case, which gives a T^3 variation of the Kapitza conductance.

The T^3 variation of the Kapitza conductance in this case is identical to the result for direct transmission of phonons across the interface. In fact, one can see that the physics in the two cases is quite similar. The fermion systems on each side of the interface experience thermodynamic fluctuations in spin density. These fluctuations obey Bose statistics, as one can see from the Bose factors $n(\Omega)$ which appear in the expression for $n(\Omega)$. The exchange interaction couples spin fluctuations in the He^3 with spin fluctuations in the metal, and allows for the possibility of energy transfer. If each system is regarded as a degenerate Fermi gas, then the spectrum of spin fluctuations extends from zero frequency to $E_F/\hbar$, a frequency very large compared to $k_B T$, for the conditions examined in this paper. Thus, the transfer of energy between the two spin systems may be described by a language very similar to that used in the discussion of the phonon contribution to the Kapitza conductance, for temperatures low compared to the Debye temperature Θ_D of the material on each side of the boundary; in fact in each case, a T^3 law obtains.

In the presence of exchange enhancement, the long wavelength fluctuations in spin density with wave vector $k \ll k_F$ are characterized by the spin-fluctuation frequency $\Omega_{\mathrm{sf}} = (1 - \bar{I})E_F/\hbar$, while the short wavelength fluctuations are still characterized by a maximum frequency the order of $E_F/\hbar$. As long as $k_B T \ll \hbar\Omega_{\mathrm{sf}}$, the T^3 term will be enhanced by a factor which is an appropriately weighted average between the enhancement factor appropriate to long wavelength fluctuations (near the surface) and to short wavelength fluctuations in the vicinity of the surface. As remarked earlier, this enhancement factor is very hard to calculate, given the present state of our knowledge of the effect of a surface on enhanced spin fluctuations (paramagnons).

The situation in CMN is quite different, at least for the temperature range explored so far. The experiments to date probe the region $T > T_c$. The characteristic fluctuation frequencies of the CMN spins lie in the range from zero to roughly kT_c, so we have $k_B T$ large compared to the characteristic frequency, instead of small compared to it, as we have when the He^3 spins interact with metallic conduction electrons. To pursue the analogy with the phonon contribution to the Kapitza conductance further, the He^3-CMN system is rather similar to the case where for one component (the He^3), $k_B T$ is small compared to the characteristic frequency Θ_D but for the second, $kT \gg \Theta_D$. In the phonon case, when $\Theta_D^{(a)}$ and $\Theta_D^{(b)}$ are the Debye temperatures for each constituent, and when $\Theta_D^{(a)} \ll k_B T \ll \Theta_D^{(b)}$, the Kapitza conductance is proportional to

$$\frac{1}{(k_B T)^2} \int_0^{\Theta_D^{(a)}} d\Omega\,\Omega^4 n(\Omega)\left[1 + n(\Omega)\right] = \tfrac{1}{3}(\Theta_D^{(a)})^3, \qquad k_B T \ll \Theta_D^{(a)} .$$

Thus, in this limit, the Kapitza conductance is independent of temperature. In the He^3-CMN case, since the entities are spins each with $2S+1$ levels instead of the infinite ladder of levels associated with each phonon in an harmonic solid, one finds $R_K^{-1} \sim T^{-1}$ in the high-temperature limit, and not the temperature-independent result appropriate to the phonon problem.

One may now appreciate a fundamental difference between the He^3-CMN case and the He^3-metal case for the range of temperatures explored so far. In the former, all of the spin degrees of freedom are thermally excited, since kT is large compared to $k_B T_c$. Thus, all of the degrees of freedom may participate in the energy transfer process. For the metal, quite the opposite is true. Since $k_B T \ll E_F$, only a very small fraction of the spin-fluctuation degrees of freedom are thermally excited, and the number of modes available for participation in the energy transfer process are very severely reduced. The Fermi-Dirac statistics obeyed by the conduction electrons thus play an important role in suppressing this contribution to the Kapitza conductance.

The recent discussions of the transfer of energy from liquid He^3 to solids via spin-dependent interactions have focused primarily on the role of these interactions in enhancing energy transfer between liquid He^3 and magnetic solids, i.e., on their role in the Kapitza resistance problem. However, there are other phenomena closely related to the Kapitza-resistance problem where these same interactions play a role. An area we feel particularly fruitful for further experimental and theoretical study is the study of the longitudinal relaxation time T_1 of the He^3 nuclei, when the liquid is in contact with a magnetic salt. It is well known that in practice, the intrinsic He^3 relaxation time is so long that experimental values of T_1 have their origin in the interaction of the He^3 nuclei with the container walls.[11] One should be able to use T_1 as a means of studying the spin-dependent part of the He^3-wall interaction in detail. If one is interested in the study of this interaction, and the physical information that may be obtained by this means, T_1 measurements may be more useful than the study of the Kapitza resistance, because the T_1 measurement singles out the spin-dependent part of the interaction, while the Kapitza conductance studies include the sum of all processes which contribute to the energy flow between the liquid and the solid.

There are two examples of the use of T_1 to study spin-dependent interactions between He^3 and magnetic salts that strike us as particularly interesting, from a physical point of view. The interaction between He^3 nuclei and the CMN surface may be studied at higher temperatures (say around 1^0K), where the behavior of the spin system remains simple, but the liquid He^3 exhibits large deviations from Fermi-liquid behavior, and behaves more like a classical fluid.[12] It is presumably difficult to study this regime via the Kapitza resistance, since the boundary conductance is dominated by the phonon contribution, and the contribution from spin-dependent interactions is a small fraction of the total. One may turn the situation around, and study T_1 at low temperatures (say below 100 mK) where the He^3 behaves like a good Fermi liquid, and use T_1 as a probe of spin dynamics very near the surface of a magnetic salt. There are a number of salts which order magnetically near or below 100 mK, in a range convenient for experimental study. The measurement of T_1 as a function of temperature for He^3 in contact with such a salt would be a possible method of gaining information about the behavior of spins very near the surface of the crystal, as the temperature passes through the ordering temperatures. We presently have a theoretical investigation of these phenomena under way.

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*Work supported by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. A. F., under Grant No. AFOSR 71-2018.

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