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Formulation of many-body theory for composite particles*

A. Goldberg
Lawrence Livermore Laboratory, University of California, Livermore, California 94550

R. D. Puff
University of Washington, Seattle, Washington 98106

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We present a formalism for the description of many-particle systems when the particles are themselves composites of several elementary constituents. We investigate specifically the case of composites made up of two fermions, the method being easily extended to more general composites. The formalism, although its development is guided by the knowledge that there are fermion constituents in the composites, is given entirely in the composite language and requires no reference to these underlying fermions. Thus, the procedure maintains all required symmetries at every step. Composite field operators satisfying all symmetries are defined, these fields obeying neither Bose nor Fermi commutation rules, but more complicated ones. The many-composite dynamics is studied via the Green's functions formed from composite operator products. A Hartree-Fock-like approximation is introduced and leads to the generation of composite wave functions describing the distortion of the composite internal states by the presence of other composites in the system.

I. INTRODUCTION

The theoretical construction of composite particles from "elementary" constituents has within it a number of formidable problems, especially when formulated in the context of many-body (in this case many-composite) methodology. If one assumes the constituents to be either bosons or fermions, it is intuitively clear that the composites themselves can be treated as elementary bosons or fermions when the average intercomposite distance is large compared to the range of the particle-particle interaction and when the average wavelength associated with the composite center-of-mass motion is also large compared with this range.[1] For example, traditional treatments of liquid helium, at laboratory densities and temperatures, completely ignore the internal structure of the helium atoms.[2] However, in many systems of interest it has become necessary to consider the nonelementary nature of the composites, their spacial extent, and their excitations. The effect in nuclear matter of the first nucleon excited state (at 300 MeV) has been shown to manifest itself through an effective three-body force which contributes a sizable fraction (1 MeV) of the binding energy per nucleon.[3] Moreover, in many systems where two-body bound states are known to be important, e.g., Cooper pairs in superconductors,[4] it is very useful to treat the medium in terms of composites rather than in terms of the electron constituents. Another type of problem in which composite effects have been studied and shown to be important lies in the coupling of atomic gases and a radiation field.[5]

The primary conceptual and practical difficulty in attempting to formulate a theory of interacting composites, and the chief question to which we address ourselves below, arises from the requirement of symmetrization between members of different composites. That is, the fact that the full Hamiltonian is symmetric under interchange of constituents implies symmetry under (i) interchange of constituents within a composite (internal parity), (ii) interchange of two composites, and (iii) interchange of a constituent in one composite with that of another. In the case of elementary fermions, where full antisymmetry of the states under constituent interchange is required, the equivalent requirement in composite language is (i) the composites have internal negative parity, (ii) the states are symmetric under composite interchange, and (iii) the states are antisymmetric under interchange of constituents from different composites.

If one works with the constituents themselves, the proper symmetrization is easily achieved through the standard method of second quantization, with the constituent field operators satisfying elementary commutation relations. Within the framework of these constituents, however, it has proven extremely difficult to describe even the clustering of tightly bound two-body composites in the low-density limit in problems where it is physically obvious that two-body bound clusters are important. In such a system with strongly bound two-body composites (composed of fermions) one expects that in the low-density limit ($n \to 0$) the energy per fermion $\mathcal{E} \to \frac{1}{2}\epsilon_0$, with ϵ_0 the lowest-energy eigenvalue of the two-body system. This physical solution, condensation into two-body bound states, is quite different from the free-particle result, $\mathcal{E} \propto n^{2/3}$. Yet a perturbative solution built up from the free-fermion system which satisfies all the required symmetries and which yields that latter result for $\mathcal{E}$, can never achieve a physical condensation into bound composites.

Several different approaches to this symmetrization question have been explored. These approaches have utilized the second-quantization procedure in some form. The essential element of each lies in the definition of the composite-particle field operators and in their resultant commutation relations. The hitherto most extensive development has been that of Girardeau.[6] In that procedure the state vector for an N composite state with given internal properties is constructed, and field operators acting on these state vectors are then defined *ad hoc*. These field operators are assumed to satisfy elementary (Bose or Fermi) commutation rules. The additional symmetry (interchange of constituents in different composites) is then imposed as a subsidiary condition, via a projection operator which retains from the set of solutions of the dynamics only those symmetric (or antisymmetric depending on the assumed nature of the composites) under this additional symmetry. The net result of this projection operator, applied back on the Hamiltonian, is to generate a modified Hamiltonian in the form of an infinite series, the nth term corresponding to n-composite correlations arising from this symmetry. As yet, the projection operator and the modified Hamiltonian have been explicitly constructed for $n=2$, i.e., to include two-composite symmetry correlations, and are applicable in the regime of low density where three- or more-composite exchange effects are small. This procedure thus utilizes simple commutation rules and places the burden of the additional symmetry on the dynamics of the system.[7]

An alternative procedure, espoused by Sahlin and Schwartz,[8] defines the composite field operators, for a given internal state of the composite, directly as a product of the constituent fields weighted over the internal wave functions. This is intuitively attractive since it clearly maintains all required symmetry at the start and since it begins with the composites in their correct noninteracting physical states, and would hence form a logical basis for a perturbation expansion. The interactions are thus separated into two types, those which give rise to the formation of the composites and those which give correlations between composites. The field commutation rules are then interaction dependent (in part placing the burden of the dynamics on the symmetry), and further these commutation rules and also the Hamiltonian are not written in terms of the composites only, but explicitly involve the constituents.

In the procedure to be described below, a third method of defining composite field operators is given. With this definition it is clear that here also all symmetry requirements are satisfied at the outset. However, this definition is independent of any interactions and the commutation rules, while not elementary, involve no dynamics. Secondly, these commutation rules and the Hamiltonian can be given entirely in terms of the composites, with no reference to the underlying constituents. It is in this spirit that we wish to study the many-composite system.

In this paper we consider as a model composites constructed of two spin-$\frac{1}{2}$ fermions. Although the specific results obtained here pertain only to this model, their generalization to n-fermion composites is straightforward albeit lengthy. Section II contains the exact mathematical formulation of the composite field operators in terms of their fermion constituents.[9] Further, the dynamical equations of motion are rewritten in terms of the composites only, allowing then disregard of their underlying structure, while still maintaining all symmetries. Although this procedure when applied to one-composite states recovers the two-body problem, the operators defined differ from those of Ref. 8 in that they may be defined with no reference to the interaction potentials. By defining the Green's functions analogous to those used in standard many-body theory, it is then shown how all observable quantities of a many-composite system may be calculated within this formulation. As an example of the techniques, and as a guide to appropriate approximation methods this formalism is applied to the case of no interaction, recovering from the composite theory the Fermi distribution for free fermions.

In Sec. III we give a factorization approximation for the Green's function, similar to that leading to the Hartree-Fock equations in many-fermion systems, and correct in the limit of low density. Within this approximation the composite Green's function contains both a "condensed-mode" term, which embodies the physical composite formation, and a self-consistent "T-matrix" term similar to that obtained in conventional treatments of the fermion many-body dynamics.

II. MATHEMATICAL FORMULATION

We develop here the formalism appropriate to a system of N composites, each composed of two spin-$\frac{1}{2}$ Fermions. The fermions interact through two-body central potentials. In terms of the Fermion coordinates, the Hamiltonian is

$$H=\sum_{i=1}^{2N}\frac{\vec{p}_i^{\,2}}{2m}+\frac{1}{2}\sum_{i\neq j}^{2N}V_{ij}\ . \tag{2.1}$$

In order to confine the eigenstates of (2.1) to the Hilbert space totally antisymmetric under interchange of fermions, we rewrite (2.1) in second-quantized form, in momentum space,

$$H=\sum_{1}\frac{\vec{p}_1^{\,2}}{2m}a_1^\dagger a_1+\frac{1}{2}\sum_{1234}\langle 12|V|34\rangle\, a_1^\dagger a_2^\dagger a_4 a_3\,, \tag{2.2}$$

where the subscript i on a_i refers to both momentum $\vec{p}_i$ and spin projection $s_i(s_i=\pm\frac{1}{2})$, and

$$2N=\sum_{1}a_1^\dagger a_1\,. \tag{2.3}$$

Consider the kinetic energy only. From Eq. (2.1), this can be written in the form

$$T=\sum_{i=1}^{2N}\frac{\vec{p}_i^{\,2}}{2m}=\sum_{j=1}^{N}\left(\frac{\vec{P}_j^{\,2}}{4m}+\frac{\vec{q}_j^{\,2}}{m}\right)\ , \tag{2.4}$$

where $\vec{P}_j$ and $\vec{q}_j$ are the momenta conjugate to $\vec{R}_j=\frac{1}{2}(\vec{r}_j+\vec{r}_{N+j})$ and $\vec{\rho}_j=\vec{r}_j-\vec{r}_{N+j}$. In this transcription a particular method of pairing has been used, but Eq. (2.4) follows for any choice of pairing. We wish to second quantize the kinetic energy in terms of pair creation and destruction operators $B^\dagger_{\vec{P}\vec{q}s_1s_2}$ and $B_{\vec{P}\vec{q}s_1s_2}$,

$$T=\sum_{\vec{P}\vec{q}s_1s_2}\left(\frac{\vec{P}^{\,2}}{4m}+\frac{\vec{q}^{\,2}}{m}\right)B^\dagger_{\vec{P}\vec{q}s_1s_2}B_{\vec{P}\vec{q}s_1s_2}\ , \tag{2.5}$$

$$N=\sum_{\vec{P}\vec{q}s_1s_2}B^\dagger_{\vec{P}\vec{q}s_1s_2}\,B_{\vec{P}\vec{q}s_1s_2}\ . \tag{2.6}$$

If B (and $B^\dagger$) satisfy Bose commutation rules, and if $B_{\vec{P}\vec{q}s_1s_2}=-B_{\vec{P},-\vec{q},s_2,s_1}$ (internal antisymmetry), the space of allowed states would be larger than that required by the full set of symmetries, i.e., only (i) and (ii) are satisfied, and one must then attempt a projection as proposed by Girardeau.[6]

Instead, to motivate a choice of pair creation operator, we write

$$T=\sum_{1}\frac{\vec{p}_1^{\,2}}{2m}a_1^\dagger a_1=\frac{1}{2}\left(\sum_{1}\frac{\vec{p}_1^{\,2}}{2m}a_1^\dagger a_1+\sum_{2}\frac{\vec{p}_2^{\,2}}{2m}a_2^\dagger a_2\right). \tag{2.7}$$

Noting from (2.3) that

$$1=\frac{1}{(2N)^{1/2}}\sum_{1}a_1^\dagger a_1\frac{1}{(2N)^{1/2}}\ ,$$

we have

$$T=\sum_{1,2}\frac{1}{2m}(\vec{p}_1^{\,2}+\vec{p}_2^{\,2})\left(a_1^\dagger\frac{1}{2\sqrt{N}}a_2^\dagger\right)\left(a_2\frac{1}{2\sqrt{N}}a_1\right) \tag{2.8}$$

and

$$N=\sum_{1,2}\left(a_1^\dagger\frac{1}{2\sqrt{N}}a_2^\dagger\right)\left(a_2\frac{1}{2\sqrt{N}}a_1\right)\ . \tag{2.9}$$

Thus, if we write

$$T=\sum_{1,2}\frac{1}{2m}(\vec{p}_1^{\,2}+\vec{p}_2^{\,2})c_{12}^\dagger c_{12}\,, \tag{2.10}$$

$$N=\sum_{1,2}c_{12}^\dagger c_{12}\,, \tag{2.11}$$

we should identify a composite operator

$$c_{12}^\dagger=a_1^\dagger\frac{1}{2\sqrt{N}}a_2^\dagger\ . \tag{2.12}$$

From this fermion-constituent identification the commutation rules follow immediately

$$[c_{12},c_{1'2'}]=[c_{12}^\dagger,c_{1'2'}^\dagger]=0\,, \tag{2.13}$$

$$[c_{12},c_{1'2'}^\dagger]=\frac{1}{2(2N+1)}\{[\delta_{11'}\delta_{22'}-\delta_{11'}\rho_{2'2}-\delta_{22'}\rho_{1'1}]-[\delta_{12'}\delta_{21'}-\delta_{12'}\rho_{1'2}-\delta_{21'}\rho_{2'1}]-4c_{1'2'}^\dagger c_{12}\}, \tag{2.14}$$

where the operator ρ_{12} is

$$\rho_{12}\equiv 2\sum_{3}c_{13}^\dagger c_{23}\,. \tag{2.15}$$

According to (2.12), the operator ρ_{11} is to be interpreted, in fermion language, as the number operator for fermions in state $(\vec{p}_1s_1)$, i.e., $n(1)=\rho_{11}$. Moreover, any average of ρ_{12}, taken over eigenstates of the Hamiltonian, has the diagonal form

$$\langle\rho_{12}\rangle=\langle n(1)\rangle\delta_{12} \tag{2.16}$$

if that Hamiltonian conserves both total spin and total linear momentum. In this case, the full Hamiltonian is given by

$$H=\sum_{12}(\epsilon_1+\epsilon_2)c_{12}^\dagger c_{12}+\sum_{1234}\langle 12|V|34\rangle c_{12}^\dagger(2N+1)c_{34}\,, \tag{2.17}$$

where $\epsilon_1=\vec{p}_1^{\,2}/2m$ and

$$\langle 12|V|34\rangle = \frac{1}{\Omega}\delta_{\vec{p}_1+\vec{p}_2,\,\vec{p}_3+\vec{p}_4}\sum_{S,\,M_s}(\tfrac{1}{2}\tfrac{1}{2}s_1s_2|SM_s)(\tfrac{1}{2}\tfrac{1}{2}s_3s_4|SM_s)\langle\tfrac{1}{2}(\vec{p}_1-\vec{p}_2)|v_S|\tfrac{1}{2}(\vec{p}_3-\vec{p}_4)\rangle\,, \tag{2.18}$$

with $\langle\vec{q}\,|v_S|\vec{q}\,'\rangle$ the Fourier transform $v_S(\vec{q}-\vec{q}\,') = \int d^3r\, e^{-i(\vec{q}-\vec{q}\,')\cdot\vec{r}}v_S(\vec{r})$, the local two-body potential in spin state S.

It might be imagined that the commutation rules and the Hamiltonian, Eqs. (2.13)–(2.15), (2.17), and (2.18), form a complete theory, equivalent to the original constituent formulation. This is not the case, as is seen from the dynamics of the c operators. If we evaluate $[c_{12}, H]$, using only these equations above, we must impose the additional subsidiary requirements,

$$\begin{aligned} &c_{12} = -c_{21} \text{ (internal antisymmetry)},\\ &c_{12}c_{34} = c_{13}c_{42}, \end{aligned} \tag{2.19}$$

in order to obtain the same result as that produced by the fermion identification [Eq. (2.12)] directly.

Equations (2.13)–(2.19) form a complete theory, in the sense that there is a complete isometry between the solutions of (2.13)–(2.19) and those generated from the conventional formulation in terms of the constituent fermions. Further, the formulation requires no reference at all to those "elementary" fermions. In fact, the properties of the operators c_{12} and $c_{12}^\dagger$ could be deduced solely from the requirement of complete antisymmetry of the states $c_{12}^\dagger c_{34}^\dagger \cdots |0\rangle$, again without reference to the underlying fermions.

It is clear how to extend this composite-operator definition to the case where the composites contain more than two constituents. Equation (2.12) is generalized by writing as many $a^\dagger$ operators as there are constituents, each separated by the factor $N^{-1/2}$ (N is the number of composites), and in addition there is a normalization factor fixed by the equivalent of Eq. (2.11) (for the case above, this factor is $\frac{1}{2}$). To rewrite the Hamiltonian in terms of these c's, one applies the identity corresponding to Eq. (2.3) as many times as necessary to obtain the composite operator. The number of subsidiary conditions [corresponding to Eq. (2.19)] increases, of course, with increasing composite complexity.

If one could solve the equations of motion generated from (2.13)–(2.19), and if one further recognized the fermiology underlying the composites, one could deduce from the composite formulation all properties of the fermions. For example, the operator representing the number of fermions with momentum $\vec{p}_1$ and spin projection s_1 is given by

$$n(\vec{p}_1 s_1) = \rho_{11} = 2\sum_2 c_{12}^\dagger c_{12}. \tag{2.20}$$

The commutation rule for $[c, c^\dagger]$, Eq. (2.14), is of course, quite complex while the Hamiltonian is rather simple. In spite of the complexity of (2.14), the equations of motion are also rather simple;

$$i\frac{\partial}{\partial t}c_{12} = [c_{12}, H] \tag{2.21}$$

in the Heisenberg picture, with

$$[c_{12}, H] = \sum_{34}\langle 12|h|34\rangle c_{34} - \sum_{345}[\langle 15|V|34\rangle\rho_{52}c_{34} + \langle 52|V|34\rangle\rho_{51}c_{34}] \tag{2.22}$$

and

$$\langle 12|h|34\rangle = (\epsilon_1+\epsilon_2)\delta_{13}\delta_{24} + \langle 12|V|34\rangle\,. \tag{2.23}$$

To obtain (2.22) without the underlying "Fermiology" one must use the condition

$$\rho_{32}c_{41} = \rho_{34}c_{12}\,, \tag{2.24}$$

which follows from (2.19).

If we first consider one-composite states, $N = 1$, the Hamiltonian is trivially diagonalizable by the transformation

$$\begin{aligned} c_{12} &= \frac{1}{\sqrt{\Omega}}\sum_{\lambda,S,M_s}(\tfrac{1}{2}\tfrac{1}{2}s_1s_2|SM_s)\psi_{\lambda S}(\vec{q})c_{\vec{P}\lambda SM_s}\,,\\ c_{\vec{P}\lambda SM_s} &= \frac{1}{\sqrt{\Omega}}\sum_{s_1s_2}\sum_{\vec{q}}(\tfrac{1}{2}\tfrac{1}{2}s_1s_2|SM_s)\psi^*_{\lambda S}(\vec{q})c_{12}\,, \end{aligned} \tag{2.25}$$

where $\psi_{\lambda S}$ refers to the set of functions satisfying

$$\frac{q^2}{m}\psi_{\lambda S}(\vec{q}) + \frac{1}{\Omega}\sum_{q'}\langle\vec{q}\,|v_S|\vec{q}\,'\rangle\psi_{\lambda S}(\vec{q}\,') = \omega_{\lambda S}\psi_{\lambda S}(\vec{q})\,, \tag{2.26}$$

together with the orthogonality and completeness relations

$$\begin{aligned} \delta_{\lambda\lambda'} &= \frac{1}{\Omega}\sum_{\vec{q}}\psi^*_{\lambda' S}(\vec{q})\psi_{\lambda S}(\vec{q})\,,\\ \delta_{\vec{q}\vec{q}\,'} &= \frac{1}{\Omega}\sum_{\lambda}\psi^*_{\lambda S}(\vec{q}\,')\psi_{\lambda S}(\vec{q})\,. \end{aligned} \tag{2.27}$$

This, of course, is simply the solution of the two-body problem. One may see here the relationship between the c operators and the operators defined by Sahlin and Schwartz,[8] which may be labeled $\Pi_{SM_s}(\vec{P},\lambda)$. When operating on a state of N_0 composites, $|N_0\rangle$, we have

$$c_{\vec{P}\lambda SM_s}|N_0\rangle=[2(2N_0-1)]^{-1/2}\Pi_{SM_s}(\vec{P},\lambda)|N_0\rangle,$$

and the resultant states differ only by a normalization factor. There is, however, no simple relation between the operators themselves; when *two* c's operate, the resultant states have entirely different normalization factors.

If we now use the exact one-composite states as a basis for the N-composite system, we obtain the Hamiltonian

$$H=\sum_{\substack{\vec{P}\lambda\\ SM_s}}\left(\frac{\vec{P}^2}{4m}+\omega_{\lambda s}\right)c^\dagger_{\vec{P}\lambda SM_s}c_{\vec{P}\lambda SM_s}$$

$$+2\sum_{\substack{P\lambda\lambda'\\ SM_s}}\langle\lambda|v_S|\lambda'\rangle c^\dagger_{\vec{P}\lambda SM_s}Nc_{\vec{P}\lambda' SM_s} \tag{2.28}$$

and

$$\langle\lambda|v_S|\lambda'\rangle=\frac{1}{\Omega^2}\sum_{qq'}\psi^*_{\lambda S}(\vec{q})\langle\vec{q}|v_S|\vec{q}'\rangle\psi_{\lambda' S}(\vec{q}'). \tag{2.29}$$

It would appear from (2.28) that the interaction between composites is independent of their momenta, and depends only on their internal states. Although the Hamiltonian is simple, the commutation rules of the c's produce in themselves a dynamics which contains the proper pair interaction [see Eq. (2.22)].

In order to utilize the dynamics [Eqs. (2.21) and (2.22)] to evaluate quantities of interest, we wish to introduce Green's functions formed from averages of time-ordered operator products. The time derivative of such time-ordered products $[A(t)B(t')]_+$ will contain a term $\delta(t-t')[A(t),B(t')]$, and we would like to define the appropriate function in such a way as to make this commutator term simple. Although we do not require a c-number commutator, at least we can seek a formulation which produces an easily interpreted average for this commutator.

To this end, consider the operator

$$B^\dagger_{1'2'}\equiv 2(2N-1)c^\dagger_{1'2'}. \tag{2.30}$$

This quantity, which can be noted from (2.12) to have the fermion representation $a^\dagger_{1'}2\sqrt{N}a^\dagger_{2'}$ satisfies a simple commutation rule with c_{12}. Thus from (2.14), we find the very simple result

$$[c_{12},B^\dagger_{1'2'}]=(\delta_{11'}\delta_{22'}-\delta_{11'}\rho_{2'2}-\delta_{22'}\rho_{1'1})$$
$$-(\delta_{12'}\delta_{21'}-\delta_{12'}\rho_{1'2}-\delta_{21'}\rho_{2'1}) \tag{2.31}$$

while any average, taken over states which conserve both total spin and total momentum, will be given by

$$\langle[c_{12},B^\dagger_{1'2'}]\rangle=[\delta_{11'}\delta_{22'}-\delta_{12'}\delta_{21'}]$$
$$[1-\langle n(1)\rangle-\langle n(2)\rangle], \tag{2.32}$$

where we have used the interpretation of $\langle\rho_{12}\rangle$ provided by Eq. (2.16).

The averages of interest here are thermodynamic averages with respect to a density matrix $e^{-\beta(H-\bar{\mu}N)}/\mathrm{Tr}\,e^{-\beta(H-\bar{\mu}N)}$. In this case, $\bar{\mu}$ is the Gibbs free energy per composite and serves to fix the average number of composites in exactly the way as is done in conventional statistical mechanics. The average $\langle\rho_{11}\rangle$ can be simply related to $\sum_2\langle B^\dagger_{12}c_{12}\rangle$ by differentiating $\langle\rho_{11}\rangle$ with respect to $\bar{\mu}$, holding temperature ($\beta^{-1}=k_BT$) and volume fixed. Recognizing that $\langle n(1)\rangle=\langle\rho_{11}\rangle$, we find

$$\langle n(1)\rangle=\frac{1}{2\langle N\rangle}\left(\sum_2\langle B^\dagger_{12}c_{12}\rangle+\langle n(1)\rangle-\frac{2}{\beta}\frac{\partial}{\partial\bar{\mu}}\langle n(1)\rangle\right). \tag{2.33}$$

This result couples the quantity to be calculated by Green's-function methods (i.e., $\langle B^\dagger c\rangle$) to the single-constituent momentum distribution $\langle n(1)\rangle$. Again, we emphasize that, as a result of Eq. (2.32), the dynamics will supply another equation for $\langle B^\dagger c\rangle$ which will involve $\langle n(1)\rangle$, and that a self-consistent solution becomes necessary. *Without a knowledge of the underlying "Fermiology", $\langle n(1)\rangle$, or more properly $\langle\rho_{11}\rangle$, is merely a parameter in the theory which must be determined self-consistently in order to specify equilibrium properties.* [In many ways, our dynamical equations will resemble those for the spin-operator Green's functions in magnetic systems,[10] where the $\delta(t-t')$ term contains the magnetization. Equation (2.33) is then the analog of the thermodynamic magnetization relation which must be coupled to the Green's function to find the actual degree of magnetization.]

Because of the particular average taken here, we choose to modify the time dependence of all operators by introducing the additional phase factor $e^{i\bar{\mu}t}$ in $c_{12}(t)$ [and $e^{-i\bar{\mu}t'}$ in $c^\dagger_{1'2'}(t')$]. Thus, instead of Eq. (2.21), we now have

$$\left(i\frac{\partial}{\partial t}+\bar{\mu}\right)c_{12}(t)=[c_{12},H], \tag{2.34}$$

where the commutator is given by Eq. (2.22).

Consider now the one-composite Green's operator

$$g_1(12t;1'2't')\equiv(c_{12}(t)B^\dagger_{1'2'}(t'))_+, \tag{2.35}$$

where $(\cdots)_+$ indicates time ordering of the operators (earliest time to the right). The dynamical equation, according to (2.34) is then

$$\left(i\frac{\partial}{\partial t}+\bar{\mu}\right)g_1(12t;1'2'\,t')=i\delta(t-t')[c_{12}(t),B^\dagger_{1'2'}(t)]+([c_{12}(t),H]B^\dagger_{1'2'}(t'))_+\,. \tag{2.36}$$

The thermodynamic average, or the Green's function

$$\begin{aligned}G_1(12;1'2';t-t')&\equiv\langle g_1(12t;1'2't')\rangle\\&=\Theta(t-t')G_1^>+\Theta(t'-t)G_1^<\end{aligned} \tag{2.37}$$

depends only on $t-t'$, and in addition $\vec{p}_1+\vec{p}_2=\vec{p}_1{}'+\vec{p}_2{}'$. These properties follow from momentum conservation and the cyclic property of the trace.

The boundary condition in time on G_1 can be derived in the usual manner.[11] We find the periodic (Bose) condition for complex temperature and time

$$G_1^>(-i\beta-t')=G_1^<(0-t')\,, \tag{2.38}$$

where the time $t-t'$ lies along a line in the complex plane such that $(t-t')/(-i\beta)$ is real and $\mathrm{Re}\beta>0$. The solution for real β and $(t-t')$ is then obtained by analytic continuation. At no point in the development of this Bose boundary condition are the commutation relations used. The commutation relation enters only in the condition

$$\begin{aligned}[G^>(12;1'2';0)-G^<(12;1'2';0)]&=\langle[c_{12},B^\dagger_{1'2'}]\rangle\\&=[\delta_{11'}\delta_{22'}-\delta_{12'}\delta_{21'}][1-\langle n(1)\rangle-\langle n(2)\rangle]\,.\end{aligned} \tag{2.39}$$

From Eqs. (2.36), (2.37), (2.32), and (2.22), we have

$$\begin{aligned}\sum_{34}\left[\left(i\frac{\partial}{\partial t}+\bar{\mu}\right)\delta_{13}\delta_{24}-\langle 12|h|34\rangle\right]G_1(34;1'2';t-t')&=i\delta(t-t')[\delta_{11'}\delta_{22'}-\delta_{12'}\delta_{21'}][1-\langle n(1)\rangle-\langle n(2)\rangle]\\&\quad-\sum_{3456}[\langle 15|V|34\rangle\delta_{26}+\langle 52|V|34\rangle\delta_{16}]\langle(\rho_{56}(t)c_{34}(t)B^\dagger_{1'2'}(t'))_+\rangle\,.\end{aligned} \tag{2.40}$$

Higher-order equations in the hierarchy, relating $(\rho cB^\dagger)_+$ to a three-composite Green's function, can be obtained in the usual way. Approximate truncations of the hierarchy, in particular the "composite Hartree-Fock" truncation, will be explored in Sec. III.

Finally, we note that $\langle n(1)\rangle$ is coupled to $G_1^<$ by Eq. (2.33). Thus, from (2.33) and the definition of G_1, we see that

$$\langle n(1)\rangle=\frac{1}{2\langle N\rangle}\left(\sum_2 G_1^<(12;12;0)+\langle n(1)\rangle-\frac{2}{\beta}\frac{\partial}{\partial\bar{\mu}}\langle n(1)\rangle\right)\,. \tag{2.41}$$

It is instructive to solve our "fermion-free" equations for the case of no interaction. The solution to Eq. (2.40), with $V=0$ is

$$\begin{aligned}G_1^{>,<}(12;1'2';t-t')&=G_1^{>,<}(12;1'2';0)\\&\quad\times\exp[-i(\epsilon_1+\epsilon_2-\bar{\mu})(t-t')]\,.\end{aligned} \tag{2.42}$$

The boundary condition, Eq. (2.38), then implies $G_1^>(0)\,e^{-\beta(\epsilon_1+\epsilon_2-\bar{\mu})}=G_1^<(0)$, and Eq. (2.39) demands

$$G_1^<(12;1'2';0)=\langle[c_{12},B^\dagger_{1'2'}]\rangle f_{\mathrm{B}}(\epsilon_1+\epsilon_2-\bar{\mu})\,, \tag{2.43}$$

where $f_{\mathrm{B}}(x)$ is the Bose function $(e^{\beta x}-1)^{-1}$. The relation for the commutator average (see 2.39), together with Eq. (2.41), then gives

$$\begin{aligned}G^<(12;12;0)&=[1-\langle n(1)\rangle-\langle n(2)\rangle]\\&\quad\times(1-\delta_{12})f_{\mathrm{B}}(\epsilon_1+\epsilon_2-\bar{\mu})\end{aligned} \tag{2.44}$$

and

$$\begin{aligned}\langle n(1)\rangle&=\frac{1}{2\langle N\rangle}\sum_2 f_{\mathrm{B}}(\epsilon_1+\epsilon_2-\bar{\mu})\\&\quad\times[1-\langle n(1)\rangle-\langle n(2)\rangle][1-\delta_{12}]\\&\quad+\frac{1}{2\langle N\rangle}\left(\langle n(1)\rangle-\frac{2}{\beta}\frac{\partial}{\partial\bar{\mu}}\langle n(1)\rangle\right)\,.\end{aligned} \tag{2.45}$$

Recognizing that $\sum_2\to\Omega\int[d^3p_2/(2\pi)^3]\sum_{s_2}$, we see that there are terms of order $\Omega/\langle N\rangle$ and of order $\langle N\rangle^{-1}$ in (2.45). Equating coefficients, we have the two equations:

$$\langle n(1)\rangle = \frac{1}{2n}\sum_{s_2}\int\frac{d^3p_2}{(2\pi)^3}f_B(\epsilon_1+\epsilon_2-\overline{\mu})$$
$$\times[1-\langle n(1)\rangle-\langle n(2)\rangle]\,,\quad n\equiv\langle N\rangle/\Omega \qquad (2.46)$$

and

$$\langle n(1)\rangle-\frac{2}{\beta}\frac{\partial}{\partial\overline{\mu}}\langle n(1)\rangle=[1-2\langle n(1)\rangle]f_B(2\epsilon_1-\overline{\mu})\,. \qquad (2.47)$$

Either (2.46) or (2.47) is sufficient to determine $\langle n(1)\rangle$. In (2.46), we note that

$$f_B(\epsilon_1+\epsilon_2-\overline{\mu})=f(\epsilon_1-\tfrac{1}{2}\overline{\mu})f(\epsilon_2-\tfrac{1}{2}\overline{\mu})$$
$$\times[1-f(\epsilon_1-\tfrac{1}{2}\overline{\mu})-f(\epsilon_2-\tfrac{1}{2}\overline{\mu})]^{-1},$$

where $f(x)=(e^{\beta x}+1)^{-1}$ is the Fermi function. Then, we have

$$\langle n(1)\rangle=\frac{1}{2n}\sum_{s_2}\int\frac{d^3p_2}{(2\pi)^3}f(\epsilon_2-\tfrac{1}{2}\overline{\mu})$$
$$\times\left(\frac{1-\langle n(1)\rangle-\langle n(2)\rangle}{1-f(\epsilon_1-\tfrac{1}{2}\overline{\mu})-f(\epsilon_2-\tfrac{1}{2}\overline{\mu})}\right)f(\epsilon_1-\tfrac{1}{2}\overline{\mu})\,.$$

Recognizing the normalization $2n=\sum_{s_2}\int[d^3p_2/(2\pi)^3]\langle n(2)\rangle$, we see that

$$\langle n(\vec{p}_1s_1)\rangle=f(\epsilon_1-\tfrac{1}{2}\overline{\mu})\,. \qquad (2.48)$$

Thus, $\langle n(1)\rangle$ is the familiar distribution for free fermions.

Equation (2.47) is a first-order differential equation in $\overline{\mu}$ and a boundary condition in $\overline{\mu}$ is required for a unique solution. The appropriate condition is the known classical limit for large negative chemical potential, or the limit of small fugacity

$$\lim_{\overline{\mu}\to-\infty}\langle n(1)\rangle=e^{-\beta\epsilon_1}e^{\beta\overline{\mu}/2}\,. \qquad (2.49)$$

With this boundary condition, the solution of (2.47) is then just Eq. (2.48). Note that $\frac{1}{2}\overline{\mu}$ plays the role of the ordinary chemical potential μ in these equations.

It is interesting to observe the sort of error one makes if c and $c^\dagger$ are assumed to have ordinary Bose commutation rules. If $[c_{12},\,c^\dagger_{1'2'}]=\delta_{11'}\delta_{22'}$, then $[c_{12},B^\dagger_{1'2'}]=2(2N+1)\delta_{11'}\delta_{22'}+4c^\dagger_{1'2'}c_{12}$, and the noninteracting solution is, according to (2.43),

$$G^<(12;12;0)=[2\langle 2N+1\rangle+4\langle c^\dagger_{12}c_{12}\rangle]f_B(\epsilon_1+\epsilon_2-\overline{\mu}) \qquad (2.50)$$

instead of the correct result in Eq. (2.44). Clearly, the use of ordinary Bose commutation rules allows an enormous overcount of allowed states to appear in the thermodynamic average, and this overcount results in errors of order $\langle N\rangle$.

The zero interaction problem is, of course, much simpler to solve with conventional methods. Formulation in terms of our basic Eqs. (2.38), (2.40), and (2.41) certainly obscures the simplicity of the free-fermion result in Eq. (2.48). Such a formulation is appropriate only when there are interactions such that formation of true composites is physically realizable.

Finally, we should note that the total energy can be expressed quite simply in terms of our basic Green's function and $\langle n(1)\rangle$. Equations (2.15) and (2.30) allow us to rewrite the Hamiltonian of Eq. (2.17) in the form

$$H=\frac{1}{2}\sum_{1}\rho_{11}\epsilon_1+\frac{1}{2}\sum_{2}\rho_{22}\epsilon_2$$
$$+\frac{1}{2}\sum_{1234}\langle 12|V|34\rangle B^\dagger_{12}c_{34}, \qquad (2.51)$$

so that the total energy can be expressed in terms of $\langle n(1)\rangle=\langle\rho_{11}\rangle$ and $G_1^<(34;12;0)$. On the other hand, we would like to have an expression for the average energy which depends only on $G_1^<$, so that the coupling to ρ_{11} does not appear explicitly. Such an expression can be obtained by noting from (2.34) and (2.22) that

$$H=\frac{1}{2}\sum_{1,2}\lim_{t'\to t}\sum_{3,4}\left[\left(i\frac{\partial}{\partial t}+\overline{\mu}+\epsilon_1+\epsilon_2\right)\delta_{13}\delta_{24}\right.$$
$$\left.+\langle 12|V|34\rangle\right]c^\dagger_{12}(t')c_{34}(t)\,. \qquad (2.52)$$

It should be observed that this is quite analogous to the expression

$$H=\frac{1}{2}\sum_{1}\left(i\frac{\partial}{\partial t}+\mu+\epsilon_1\right)a_1^\dagger(t')a_1(t),$$

where $a_1(t)$ is a fermion annihilation operator with an added phase factor so that $[i(\partial/\partial t)+\mu]a_1(t)=[a_1,H]$. This familiar fermion operator expression is used, for example, in conventional many-body theory to express $\langle H\rangle$ in terms of the single-fermion Green's function.

By differentiating $\langle H\rangle$ with respect to $\overline{\mu}$ at constant volume and temperature, using (2.52), (2.30), and the definition (2.37), we find

$$\frac{\langle H\rangle}{\Omega}=\frac{1}{2\langle N\rangle}\left\{\frac{1}{4\Omega}\sum_{12}\lim_{t'\to t^+}\sum_{3,4}\left[\left(i\frac{\partial}{\partial t}+\overline{\mu}+\epsilon_1+\epsilon_2\right)\delta_{13}\delta_{24}+\langle 12|V|34\rangle\right]G_1^<(34;12;t-t')+\frac{\langle H\rangle}{\Omega}-\frac{2}{\beta}\frac{\partial}{\partial\overline{\mu}}\frac{\langle H\rangle}{\Omega}\right\}, \qquad (2.53)$$

a result which is formally quite similar to (2.41). Analysis of (2.53) for the case of no interaction between constituents is then no more difficult (and no simpler) than the analysis following (2.41). The correct free-fermion result is obtained, of course.

The physical interpretation of the diagonal element $G^<(12;12;0)$ is clear. Since

$$\frac{1}{2}\sum_{12} B_{12}^\dagger c_{12} = \frac{2N(2N-1)}{2} \equiv \eta \qquad (2.54)$$

is the number of independent constituent pairs, or the operator representing the total number of ways of constructing composites from pairs of elementary constituents, we see that $\langle\eta\rangle = \frac{1}{2}\sum_{12} G^<(12;12;0)$ represents the expectation value of this number. Thus $n_{12} = \frac{1}{2} G_1^<(12;12;0)$ represents the average number of independent composites with constituent momenta and spin $\vec{p}_1 s_1$, $\vec{p}_2 s_2$.

III. FACTORIZATION APPROXIMATION AND STATISTICAL CORRELATIONS BETWEEN COMPOSITES

The dynamical equations for higher-order Green's functions, analogous to Eq. (2.40), are lengthy but simple to construct. Without constructing any of these in detail, we anticipate that the lowest-order approximation is one which maintains only statistical correlations between composites and that there is a simple factorization for the higher-order functions in this approximation. The specific function appearing on the right-hand side of (2.40) is $\langle(\rho_{56}(t) c_{34}(t) B_{1'2'}^\dagger(t'))_+\rangle$. This function can be evaluated exactly for the case of no interaction via methods similar to those used to obtain the noninteracting G_1 in Sec. II. The result is

$$\begin{aligned}\langle(\rho_{56}(t) c_{34}(t) B_{1'2'}^\dagger(t'))_+\rangle = {} & \langle\rho_{56}(t)\rangle G_1(34;1'2';t-t') \\ & + \langle\rho_{54}(t)\rangle G_1(63;1'2';t-t') \\ & + \langle\rho_{53}(t)\rangle G_1(46;1'2';t-t') .\end{aligned} \qquad (3.1)$$

This expression maintains manifestly the full set of necessary symmetries and is exact for the noninteracting case. We take Eq. (3.1) as an approximate factorization for the problem with interactions, where $\langle\rho_{ij}(t)\rangle$ and G_1 are not the free-particle solutions, but are determined by solving the coupled equations (3.1) and (2.40). We emphasize that precisely this factorization procedure appears in ordinary Hartree-Fock theory for boson or fermion constituents. In that case, the Green's function formed from two a's and two $a^\dagger$'s (the conventional two-particle Green's function) factors into a sum of two single-particle function products, again maintaining proper symmetry. The factorization (3.1) contains three terms, but the rationale is identical. We will see that this analogy with ordinary *particle* Hartree-Fock theory can be carried further. That is, a self-consistent *composite* wave function, analogous to the self-consistent single-*constituent* wave function of traditional Hartree-Fock theory, plays an important role.

Inserting the approximation (3.1) into Eq. (2.40), and using Eq. (2.16), we find

$$\begin{aligned}\sum_{3,4}\left[\left(i\frac{\partial}{\partial t} + \bar{\mu} - \epsilon_1^{\mathrm{HF}} - \epsilon_2^{\mathrm{HF}}\right)\delta_{13}\delta_{24} - [1 - \langle n(1)\rangle - \langle n(2)\rangle]\langle 12|V|34\rangle\right] G_1(34;1'2';t-t') \\ = i[\delta_{11'}\delta_{22'} - \delta_{12'}\delta_{21'}][1 - \langle n(1)\rangle - \langle n(2)\rangle]\delta(t-t') ,\end{aligned} \qquad (3.2)$$

with

$$\epsilon_1^{\mathrm{HF}} = \epsilon_1 + \sum_3 [\langle 13|V|13\rangle - \langle 13|V|31\rangle]\langle n(3)\rangle . \qquad (3.3)$$

We recognize that ϵ_1^{HF} is the ordinary Hartree-Fock energy for a constituent fermion with momentum $\vec{p}_1$ and spin projection s_1.

We define total and relative momenta $\vec{P} = \vec{p}_1 + \vec{p}_2$, $\vec{P}' = \vec{p}_1' + \vec{p}_2'$, $\vec{k} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$, $\vec{k}' = \frac{1}{2}(\vec{p}_1' - \vec{p}_2')$. The function $G_1(12;1'2';t-t')$ conserves total pair momentum $\vec{P} = \vec{P}'$ as well as total spin projection, $s_1 + s_2 = s_1' + s_2'$, and total spin. We therefore decompose G_1 into its spin components

$$\begin{aligned}G_1(12;1'2';t-t') \\ = \delta_{PP'} \sum_{SM_S} (\tfrac{1}{2}\tfrac{1}{2}\, s_1 s_2 | S M_s) \\ \times (\tfrac{1}{2}\tfrac{1}{2}\, s_1' s_2' | S M_s) \langle\vec{k}|\mathcal{G}_S(\vec{P}, t-t')|k'\rangle .\end{aligned} \qquad (3.4)$$

As required by rotational invariance, $\langle\rho_{11}\rangle = \langle n(\vec{p}_1 s_1)\rangle$ is independent of s_1 and will be denoted by $n(\vec{p}_1)$. Its normalization is then

$$\langle N\rangle = \sum_{P_1} n(\vec{p}_1) . \qquad (3.5)$$

Furthermore, the single-constituent Hartree-Fock energy is also independent of s_1 and, according to (3.3) is given by

$$\epsilon^{\mathrm{HF}}(\vec{p}_1)=\epsilon_1+\frac{1}{\Omega}\sum_{\vec{p}_2 S}\left(\tfrac{1}{2}\delta_{S,0}+\tfrac{3}{2}\delta_{S,1}\right)\langle\tfrac{1}{2}(\vec{p}_1-\vec{p}_2)|v_S+(-1)^S v_S^{\mathrm{ex}}|\tfrac{1}{2}(\vec{p}_1-\vec{p}_2)\rangle\, n(\vec{p}_2)$$

$$=\epsilon_1+\frac{1}{\Omega}\sum_{\vec{p}_2}\langle\tfrac{1}{2}(\vec{p}_1-\vec{p}_2)|U|\tfrac{1}{2}(\vec{p}_1-\vec{p}_2)\rangle\, n(\vec{p}_2)\,, \tag{3.6}$$

where $\langle\vec{k}|v_S|\vec{k}'\rangle=v_S(\vec{k}-\vec{k}')$ is the Fourier transform of the spin-singlet ($S=0$) or spin-triplet ($S=1$) two-body interaction, and $\langle\vec{k}|v_S^{\mathrm{ex}}|\vec{k}'\rangle=\langle\vec{k}|v_S|-\vec{k}'\rangle$ is the corresponding exchange potential. The spin sum in Eq. (3.6) defines the function

$$\langle\vec{k}|U|\vec{k}'\rangle\equiv\sum_{S}\left(\tfrac{1}{2}\delta_{S,0}+\tfrac{3}{2}\delta_{S,1}\right)\langle\vec{k}|v_S+(-1)^S v_S^{\mathrm{ex}}|\vec{k}'\rangle\,. \tag{3.7}$$

Equation (3.2) can then be written as an expression for $\mathcal{G}_S$. Using Eq. (3.4), we have

$$\sum_{q}\left[\left(i\frac{\partial}{\partial t}+\bar{\mu}-\epsilon^{\mathrm{HF}}(\tfrac{1}{2}\vec{P}+\vec{k})-\epsilon^{\mathrm{HF}}(\tfrac{1}{2}\vec{P}-\vec{k})\right)\delta_{\vec{k}\,\vec{q}}-\frac{1}{\Omega}[1-n(\tfrac{1}{2}\vec{P}+\vec{k})-n(\tfrac{1}{2}\vec{P}-\vec{k})]\langle\vec{k}|v_S|\vec{q}\rangle\right]\langle\vec{q}|\mathcal{G}_S(\vec{P},t-t')|\vec{k}'\rangle$$

$$=i\,[1-n(\tfrac{1}{2}\vec{P}+\vec{k})-n(\tfrac{1}{2}\vec{P}-\vec{k}]\,[\delta_{\vec{k}\,\vec{k}'}+(-1)^S\delta_{\vec{k},-\vec{k}'}]\,\delta(t-t')\,, \tag{3.8}$$

and the coupling of G_1 to $n(\vec{p}_1)$ given by Eq. (2.41) can be written in terms of $\mathcal{G}_S$ in the form

$$n(\vec{p}_1)=\frac{1}{2\langle N\rangle}\left(\sum_{S\,\vec{p}_2}\left(\tfrac{1}{2}\delta_{S,0}+\tfrac{3}{2}\delta_{S,1}\right)\langle\tfrac{1}{2}(\vec{p}_1-\vec{p}_2)|\mathcal{G}_S^{<}(\vec{p}_1+\vec{p}_2,0)|\tfrac{1}{2}(\vec{p}_1-\vec{p}_2)\rangle+n(\vec{p}_1)-\frac{2}{\beta}\frac{\partial}{\partial\bar{\mu}}n(\vec{p}_1)\right), \tag{3.9}$$

while the analogous expression for the total energy [Eq. (2.53)] becomes

$$\frac{\langle H\rangle}{\Omega}=\frac{1}{2\langle N\rangle}\left(\frac{1}{4\Omega}\sum_{\vec{P},\vec{k},S}(\delta_{S,0}+3\delta_{S,1})\lim_{t'\to t^+}\left\{\sum_{\vec{q}}\left[\left(i\frac{\partial}{\partial t}+\bar{\mu}+\frac{\vec{P}^2}{4m}+\frac{\vec{k}^2}{m}\right)\delta_{\vec{k},\vec{q}}\right.\right.\right.$$

$$\left.\left.\left.+\frac{1}{\Omega}\langle\vec{k}|v_S|\vec{q}\rangle\right]\langle\vec{q}|\mathcal{G}_S^{<}(\vec{P},t-t')|\vec{k}\rangle\right\}+\frac{\langle H\rangle}{\Omega}-\frac{2}{\beta}\frac{\partial}{\partial\bar{\mu}}\frac{\langle H\rangle}{\Omega}\right). \tag{3.10}$$

Equations (3.6), (3.8), and (3.9) now form a coupled set to be solved self consistently, using the boundary condition (2.38). The solution to Eq. (3.8) for no interactions is then

$$\langle\vec{k}|\mathcal{G}_S^{>,<}(\vec{P},t-t')|\vec{k}'\rangle=\mp[\delta_{\vec{k}\,\vec{k}'}+(-1)^S\delta_{\vec{k},-\vec{k}'}]f_{\mathrm{B}}\left[\mp\left(\frac{\vec{P}^2}{4m}+\frac{\vec{k}^2}{m}-\bar{\mu}\right)\right][1-n(\tfrac{1}{2}\vec{P}+\vec{k})-n(\tfrac{1}{2}\vec{P}-\vec{k})]$$

$$\times\exp\left[-i\left(\frac{\vec{P}^2}{4m}+\frac{\vec{k}^2}{m}-\bar{\mu}\right)(t-t')\right], \tag{3.11}$$

where the upper and lower signs refer to $\mathcal{G}^{>}$ and $\mathcal{G}^{<}$, respectively. Simultaneous solution of (3.9) and (3.11) gives the full free-particle solution for $n(\vec{p}_1)$ described in Sec. II. We note again that simultaneous solution of (3.9) and (3.11) is by no means the simplest method for obtaining the free-fermion solution for $n(\vec{p}_1)$; the method involving Eq. (3.9) and a solution for $\mathcal{G}_S$ is helpful only when actual composites can form; for free constituents we merely complicate the algebra.

In order to investigate the structure of Eq. (3.8), we follow a procedure familiar from conventional Green's-function analyses. We extend the time $t-t'$ and inverse temperature β into the complex plane, restricting $\mathrm{Re}\beta>0$, and $T=i(t-t')/\beta$ real in the interval $(-1,1)$. The solution obtained within this region is then analytically continued to the physical region of real time and temperature.

We expand $\mathcal{G}_S$ in a Fourier series in this complex time domain

$$\langle\vec{k}|\mathcal{G}_S(\vec{P},t-t')|\vec{k}'\rangle=\langle\vec{k}|\tilde{\mathcal{G}}_S(\vec{P},t-t')|\vec{k}'\rangle+\langle\vec{k}|F_S(\vec{P})|\vec{k}'\rangle\,, \tag{3.12}$$

with

$$\langle\vec{k}|\tilde{\mathcal{G}}_S(\vec{P},t-t')|\vec{k}'\rangle=\frac{i}{\beta}\sum_{n_{\mathrm{even}}}\langle\vec{k}|\hat{\mathcal{G}}_S(\vec{P},i\pi n/\beta)|\vec{k}'\rangle\, e^{-i\pi nT}\,. \tag{3.13}$$

The periodic boundary condition, Eq. (2.38), is satisfied by this Fourier series for any value of $\langle \vec{k} | F_S(\vec{P}) | \vec{k}' \rangle$. Such a constant term, for ordinary boson Green's functions, signals the onset of Bose condensation. It will play a similar role here. From Eq. (3.8), we see that F_S and $\hat{\mathcal{G}}_S$ satisfy

$$\langle \vec{k} | \hat{\mathcal{G}}_S(\vec{P}, z) | \vec{k}' \rangle = i \mathcal{K}(\vec{P}, \vec{k}, z)[\delta_{\vec{k}\vec{k}'} + (-1)^S \delta_{\vec{k}, -\vec{k}'}] + \mathcal{K}(\vec{P}, \vec{k}, z) \frac{1}{\Omega} \sum_{\vec{q}} \langle \vec{k} | v_S | \vec{q} \rangle \langle \vec{q} | \hat{\mathcal{G}}_S(\vec{P}, z) | \vec{k}' \rangle , \tag{3.14}$$

with

$$\mathcal{K}(\vec{P}, \vec{k}, z) \equiv \frac{[1 - n(\frac{1}{2}\vec{P} + \vec{k}) - n(\frac{1}{2}\vec{P} - \vec{k})]}{z + \bar{\mu} - \epsilon^{\mathrm{HF}}(\frac{1}{2}\vec{P} + \vec{k}) - \epsilon^{\mathrm{HF}}(\frac{1}{2}\vec{P} - \vec{k})} \tag{3.15}$$

and

$$\langle \vec{k} | F_S(\vec{P}) | \vec{k}' \rangle = \mathcal{K}(\vec{P}, \vec{k}, 0) \frac{1}{\Omega} \sum_{\vec{q}} \langle \vec{k} | v_S | \vec{q} \rangle \langle \vec{q} | F_S(\vec{P}) | \vec{k}' \rangle . \tag{3.16}$$

Equation (3.16) is an eigenvalue equation for $\bar{\mu}$, and we shall see later that a nonvanishing F_S implies a Bose condensation into self-consistent composites. $\hat{\mathcal{G}}_S$ is then the "noncondensate" part of the Green's function.

Equation (3.14) for $\hat{\mathcal{G}}_S$ is, in fact, equivalent to a T-matrix equation

$$\langle \vec{k} | T_S(\vec{P}, z) | \vec{k}' \rangle = \langle \vec{k} | v_S + (-1)^S v_S^{\mathrm{ex}} | \vec{k}' \rangle + \frac{1}{\Omega} \sum_{\vec{q}} \langle \vec{k} | v_S | \vec{q} \rangle \mathcal{K}(\vec{P}, \vec{q}, z) \langle \vec{q} | T_S(\vec{P}, z) | \vec{k}' \rangle , \tag{3.17}$$

with

$$\langle \vec{k} | \hat{\mathcal{G}}_S(\vec{P}, z) | \vec{k}' \rangle = i \mathcal{K}(\vec{P}, \vec{k}, z)[\delta_{\vec{k}\vec{k}'} + (-1)^S \delta_{\vec{k}, -\vec{k}'}] + (i/\Omega) \mathcal{K}(\vec{P}, \vec{k}, z) \langle \vec{k} | T_S(\vec{P}, z) | \vec{k}' \rangle \mathcal{K}(\vec{P}, \vec{k}', z) . \tag{3.18}$$

The numerator in $\mathcal{K}(\vec{P}, \vec{k}, z)$ plays the role of a projection operator common to T-matrix theories, and the energy denominator in this case contains only the single-particle Hartree-Fock energies.

In ordinary-particle Hartree-Fock theory, it is advantageous to discuss the Hartree-Fock wave functions. Analogous self-consistent wave functions are useful for the composite problem. We see that the wave functions which diagonalize the T-matrix satisfy

$$[\epsilon^{\mathrm{HF}}(\tfrac{1}{2}\vec{P} + \vec{q}) + \epsilon^{\mathrm{HF}}(\tfrac{1}{2}\vec{P} - \vec{q})] \phi_{\vec{P}S\nu}(\vec{q}) + [1 - n(\tfrac{1}{2}\vec{P} + \vec{q}) - n(\tfrac{1}{2}\vec{P} - \vec{q})] \frac{1}{\Omega} \sum_{\vec{q}'} \langle \vec{q} | v_S | \vec{q}' \rangle \phi_{\vec{P}S\nu}(\vec{q}') = E_{\vec{P}S\nu} \phi_{\vec{P}S\nu}(\vec{q}) , \tag{3.19}$$

together with orthogonality and completeness,

$$\delta_{\nu\nu'} = \frac{1}{\Omega} \sum_{q} \phi^*_{\vec{P}S\nu}(\vec{q}) \phi_{\vec{P}S\nu'}(\vec{q}) ,$$
$$\delta_{\vec{q}\vec{q}'} = \frac{1}{\Omega} \sum_{\nu} \phi^*_{\vec{P}S\nu}(\vec{q}) \phi_{PS\nu}(\vec{q}') . \tag{3.20}$$

Equation (3.19) follows directly from the factorization, Eq. (3.1), and clarifies the physical picture corresponding to this approximation. Each member of a pair travels in its own Hartree-Fock field and interacts with the other member of the pair, not through the two-body potential, but through this potential modified by over-all symmetry requirements. The internal composite states are not the free composite wave functions but are affected by the presence of other composites. For example, although ϕ is an internal wave function for the composite, it depends on the composite center-of-mass momentum $\vec{P}$. As ordinary-particle Hartree-Fock theory neglects dynamical correlations between *particles*, this factorization neglects dynamical correlations between composites.

The T-matrix and $\hat{\mathcal{G}}_S$ can be constructed immediately from these wave functions. From Eqs. (3.14)–(3.20) we have

$$\langle\vec{k}\,|T_S(\vec{P},z)\,|\vec{k}'\rangle=\frac{z+\bar{\mu}-\epsilon^{\rm HF}(\frac{1}{2}\vec{P}+\vec{k})-\epsilon^{\rm HF}(\frac{1}{2}\vec{P}-\vec{k})}{1-n(\frac{1}{2}\vec{P}+\vec{k})-n(\frac{1}{2}\vec{P}-\vec{k})}$$

$$\times\sum_{\nu}\phi_{\vec{P}S\nu}(\vec{k})\,\frac{E_{\vec{P}S\nu}-\epsilon^{\rm HF}(\frac{1}{2}\vec{P}+\vec{k}')-\epsilon^{\rm HF}(\frac{1}{2}\vec{P}-\vec{k}')}{z+\bar{\mu}-E_{\vec{P}S\nu}}[1+(-1)^S\pi]\phi^*_{\vec{P}S\nu}(\vec{k}') \tag{3.21}$$

and

$$\langle\vec{k}\,|\hat{\mathcal{G}}_S(\vec{P},z)\,|\vec{k}'\rangle=\frac{i}{\Omega}\sum_{\nu}\frac{\phi_{\vec{P}S\nu}(\vec{k})[1+(-1)^S\pi]\phi^*_{\vec{P}S\nu}(\vec{k}')}{z+\bar{\mu}-E_{\vec{P}S\nu}}[1-n(\tfrac{1}{2}\vec{P}+\vec{k}')-n(\tfrac{1}{2}\vec{P}-\vec{k}')]\,, \tag{3.22}$$

and from (3.13)

$$\langle\vec{k}\,|\mathcal{G}_S^{>,<}(\vec{P},t)|\vec{k}'\rangle=\mp\frac{1}{\Omega}\sum_{\nu}\frac{\phi_{\vec{P}S\nu}(\vec{k})[1+(-1)^S\pi]\phi^*_{\vec{P}S\nu}(\vec{k}')}{e^{\mp\beta(E_{\vec{P}S\nu}-\bar{\mu})}-1}[1-n(\tfrac{1}{2}\vec{P}+\vec{k}')-n(\tfrac{1}{2}\vec{P}-\vec{k}')]\,e^{-i(E_{\vec{P}S\nu}-\bar{\mu})t}\,, \tag{3.23}$$

where π is the reflection operator, $\pi\phi(\vec{k})=\phi(-\vec{k})$. Thus only the odd-parity states appear for the triplet ($S=1$) and even-parity states for the singlet ($S=0$). Further, from Eq. (3.16) it follows that $\langle\vec{k}\,|F_S(\vec{P})\,|\vec{k}'\rangle$ vanishes unless $\bar{\mu}$ is one of the eigenvalues $E_{\vec{P}S\nu}$ of Eq. (3.19). Thus,

$$\langle\vec{k}\,|F_S(\vec{P})\,|\vec{k}'\rangle=(\text{const})\sum_{\nu}\delta_{\bar{\mu},E_{\vec{P}S\nu}}\phi_{\vec{P}S\nu}(\vec{k})$$

$$\times\ \tfrac{1}{2}[1+(-1)^S\pi]\phi^*_{\vec{P}S\nu}(\vec{k}')\,, \tag{3.24}$$

where the factor $\frac{1}{2}[1+(-1)^S\pi]$ is needed to preserve the original symmetry requirements on G_1. $\bar{\mu}$ is a thermodynamic quantity, a chemical potential used to fix the average number of composites. At high temperature $\bar{\mu}$ will be large and negative (classical limit). As temperature is decreased, for fixed density, $\bar{\mu}$ will increase until it approaches the lowest self-consistent pair eigenvalue $E^{\rm min}_{PS\nu}$. If we assume that the pair energy increases with center-of-mass momentum $\vec{P}$, this minimum occurs at $\vec{P}=0$. Only when $\bar{\mu}$ reaches the value $E^{\rm min}_{0S\nu}$ will there be a contribution to $\mathcal{G}$ from F. For this value of $\bar{\mu}$, we then have

$$\langle\vec{k}\,|F_S(\vec{P})\,|\vec{k}'\rangle=(\text{const})\delta_{\vec{P},0}\sum_{\nu}\delta_{\bar{\mu},E_{0S\nu}}\phi_{0S\nu}(\vec{k})$$

$$\times\tfrac{1}{2}[1+(-1)^S\pi]\phi^*_{0S\nu}(\vec{k}')\,, \tag{3.25}$$

when

$$\bar{\mu}=E^{\rm min}_{0S\nu}\,. \tag{3.26}$$

The above F term then produces a contribution to $n(\vec{p}_1)$ [Eq. (3.9)] which is given by

$$n^c(\vec{p}_1)=\frac{(\text{const})}{4\langle N\rangle}\sum_{S,\nu}(\delta_{S,0}+3\delta_{S,1})\phi_{0S\nu}(\vec{p}_1)\tfrac{1}{2}[1+(-1)^S\pi]\phi^*_{0S\nu}(\vec{p}_1)\delta_{\bar{\mu},E_{0S\nu}}\,, \tag{3.27}$$

and a similar contribution to the total energy $\langle H\rangle/\,\Omega$ [Eq. (3.10)]

$$\frac{\langle H\rangle^c}{\Omega}=\frac{(\text{const})}{4\langle N\rangle}\frac{1}{2\Omega}\sum_{\vec{k},S,\nu}(\delta_{S,0}+3\delta_{S,1})\tfrac{1}{2}[1+(-1)^S\pi_\nu]\,\delta_{\bar{\mu},E_{0S\nu}}\phi^*_{0S\nu}(\vec{k})$$

$$\times\sum_{\vec{q}}\{(\bar{\mu}+k^2/m)\delta_{\vec{k},\vec{q}}+(1/\Omega)\langle\vec{k}\,|v_S\,|\vec{q}\rangle\}\,\phi_{0S\nu}(\vec{q})\,. \tag{3.28}$$

We have denoted this contribution by the superscript c because it arises physically from Bose condensation into the self-consistent composite ground state. Likewise, $F_S(\vec{P})$ gives a contribution to the average of η, the operator representing the total number of independent constituent pairs. From the definition (2.54), Eqs. (3.4), (3.12), and (3.20), we see that the expression (3.25) for F yields the contribution to $\langle\eta\rangle$,

$$\langle\eta\rangle^c=(\text{const})\frac{\Omega}{2}\sum_{S,\nu}(\delta_{S,0}+3\delta_{S,1})\tfrac{1}{2}[1+(-1)^S\pi_\nu]\,\delta_{\bar{\mu},E_{0S\nu}}\,, \tag{3.29}$$

where the (const) factor is that appearing in (3.25). Note that $(\delta_{S,0}+3\delta_{S,1})\frac{1}{2}[1+(-1)^S\pi_\nu]\equiv D_{S\nu}$ is the full degeneracy factor for the state $\phi_{0S\nu}$ including the fact that only even-parity states contribute for $S=1$ and odd parity for $S=0$. Note also that this same factor appears in (3.27)–(3.29). Combining (3.27) and (3.29) so as to eliminate the (const) factor, we have

$$n^c(\vec{p}_1)=\frac{\langle\eta\rangle^c}{2\langle N\rangle\Omega}\,\frac{\sum_{S,\nu}D_{S\nu}\delta_{\bar{\mu},E_{0S\nu}}|\phi_{0S\nu}(\vec{p}_1)|^2}{D},\tag{3.30}$$

where $D=\sum_{S\nu}D_{S\nu}\delta_{\bar{\mu},E_{0S\nu}}$ is the total degeneracy of the minimum-energy composite state. The quantity $\langle\eta\rangle^c/2\langle N\rangle\Omega$ has dimensions of a density, and we can write

$$\frac{\langle\eta\rangle^c}{2\langle N\rangle\Omega}=\gamma n$$

or

$$\langle\eta\rangle^c=2\gamma\langle N\rangle^2\approx\gamma\tfrac{1}{2}[2\langle N\rangle(2\langle N\rangle-1)]\,.\tag{3.31}$$

Thus, we may write the contribution of F to $n(\vec{p}_1)$ in the form

$$n^c(\vec{p}_1)=\frac{\gamma n}{D}\sum_{S\nu}D_{S\nu}\delta_{\bar{\mu},E_{0S\nu}}|\phi_{0S\nu}(\vec{p}_1)|^2,$$

$$D=\sum_{S\nu}D_{S\nu}\delta_{\bar{\mu},E_{0S\nu}},$$

$$\int\frac{d^3p_1}{(2\pi)^3}n^c(\vec{p}_1)=\gamma n,\quad\bar{\mu}=E_{0S\nu}^{\min}\,.\tag{3.32}$$

Similarly, Eq. (3.28) can be written

$$\frac{\langle H\rangle^c}{\Omega}=\frac{\gamma n}{D}\sum_{S,\nu}D_{S\nu}\delta_{\bar{\mu},E_{0S\nu}}\,\frac{1}{2\Omega}\sum_{\vec{k}}\phi^*_{0S\nu}(\vec{k})\times\sum_{\vec{q}}\left[\left(\bar{\mu}+\frac{k^2}{m}\right)\delta_{\vec{k}\,\vec{q}}+\frac{1}{\Omega}(\vec{k}|v_S|\vec{q})\right]\phi_{0S\nu}(\vec{q}).\tag{3.33}$$

The parameter γ is to be interpreted, according to (3.31), as that fraction of the total number of independent composites which are distributed with equal weight among the D self-consistent composite states with energy $\bar{\mu}=E_{0S\nu}^{\min}$. The value of γ, which plays the role of a "condensate fraction" or order parameter in the theory, is yet to be determined. However, if temperature and density are such that $\bar{\mu}(\beta,n)$ lies *below* the minimum eigenvalue $E_{0S\nu}^{\min}$, it is clear that $\gamma=0$.

When $\bar{\mu}$ lies below $E_{0S\nu}^{\min}$ and $\gamma=0$, we must solve Eq. (3.9), using $\tilde{\mathcal{G}}_S^{<}$ in place of $\mathcal{G}_S^{<}$. $\tilde{\mathcal{G}}_S^{<}$ is in turn given by Eq. (3.23). This "noncondensate" contribution to $n(\vec{p}_1)$ will be denoted by $n^{\mathrm{nc}}(\vec{p}_1)$; it is, of course, the full contribution when $\bar{\mu}<E_{0S\nu}^{\min}$ and $\gamma=0$. Thus, we have

$$n^{\mathrm{nc}}(\vec{p}_1)=\frac{1}{2\langle N\rangle}\left(\frac{1}{\Omega}\sum_{p_2S\nu}\frac{D_{S\nu}|\phi_{\vec{P}S\nu}(\vec{k})|^2}{e^{\beta(E_{\vec{P}S\nu}-\bar{\mu})}-1}\,[1-n(\vec{p}_1)-n(\vec{p}_2)]+n^{\mathrm{nc}}(\vec{p}_1)-\frac{2}{\beta}\,\frac{\partial}{\partial\bar{\mu}}n^{\mathrm{nc}}(\vec{p}_1)\right)\tag{3.34}$$

[note: $\vec{P}=\vec{p}_1+\vec{p}_2,\ \vec{k}=\frac{1}{2}(\vec{p}_1-\vec{p}_2)$].

At zero temperature ($\beta\to\infty$), condensation into the lowest self-consistent state is complete in this approximation, and $n^{\mathrm{nc}}(\vec{p}_1)\to0$ while $\bar{\mu}=E_{0S\nu}^{\min}$ and $\gamma=1$. The full ground-state problem is then solved by finding the (possibly degenerate) ground-state wave function $\phi_{0S\nu}(\vec{p}_1)$ with eigenvalue $\bar{\mu}$ according to Eq. (3.19). The full $n(\vec{p}_1)$ is then given by Eq. (3.32) with $\gamma=1$.

At nonzero temperature there is some depletion of the lowest composite state, and some exitation to the higher states. In this case, $n(\vec{p}_1)=n^c(\vec{p}_1)+n^{\mathrm{nc}}(\vec{p}_1)$ with condensate and noncondensate parts given by Eqs. (3.32) and (3.34), respectively. The quantity γ is determined by normalization

$$n=\int\frac{d^3p_1}{(2\pi)^3}n(\vec{p}_1)=\gamma n+\int\frac{d^3p_1}{(2\pi)^3}n^{\mathrm{nc}}(\vec{p}_1).$$

Thus, γ is determined by

$$(1-\gamma)=\frac{1}{n}\int\frac{d^3p_1}{(2\pi)^3}n^{\mathrm{nc}}(\vec{p}_1).\tag{3.35}$$

Below a certain critical temperature, obtained by setting $\gamma=0$ and solving for the temperature at which (3.35) is satisfied when $\bar{\mu}=E_{0S\nu}^{\min}$, there exists some degree of condensation into the lowest self-consistent composite states. For lower temperatures, $\bar{\mu}$ remains equal to $E_{0S\nu}^{\min}$, and the parameter γ is determined from Eq. (3.35). Above the critical temperature, $\gamma=0$, there is no condensation, and Eq. (3.35) then determines $\bar{\mu}(\beta,n)<E_{0S\nu}^{\min}$ as a function of density and temperature. These observations parallel similar results for the free Bose gas, of course. However, in the free-Bose-gas problem, one need not solve a set of coupled wave (or Green's) function and momentum distribution equations; only the normalization of a known form for $n(\vec{p}_1)$ is involved.

In many problems of physical interest, particularly in cases of low density, one expects that the composite internal wave function ϕ (Eq. 3.19) for the many-body medium resembles closely the free composite function ψ (eq. 2.26). Although ϕ clearly approaches ψ as density goes to zero, the deviations are of interest, and the free composite states form a convenient basis for examining the structure of these deviations. In particular, we may use the ψ basis for examining $\tilde{\mathcal{G}}_S$.

We may express $\langle\vec{k}|\mathcal{G}_S(\vec{P},t)|\vec{k}'\rangle$ or $\langle\vec{k}|\hat{\mathcal{G}}_S(\vec{P},z)|\vec{k}'\rangle$

in the free composite basis by

$$\langle\vec{\mathrm{k}}\,|\tilde{\mathcal{G}}_S(\vec{\mathrm{P}},t)\,|\vec{\mathrm{k}}'\rangle=\frac{1}{\Omega}\sum_{\lambda\lambda'}\psi_{\lambda S}(\vec{\mathrm{k}})\langle\lambda\,|\tilde{\mathcal{G}}_S(\vec{\mathrm{P}},t)\,|\lambda'\rangle[1+(-1)^S\pi]\,\psi^*_{\lambda' S}(\vec{\mathrm{k}}')\,[1-n(\tfrac{1}{2}\vec{\mathrm{P}}+k')-n(\tfrac{1}{2}\vec{\mathrm{P}}-\vec{\mathrm{k}}')]\,, \tag{3.36}$$

with the same relation between $\langle\vec{\mathrm{k}}\,|\hat{\mathcal{G}}_S(\vec{\mathrm{P}},z)|\vec{\mathrm{k}}'\rangle$ and $\langle\lambda\,|\hat{\mathcal{G}}_S(\vec{\mathrm{P}},z)\,|\lambda'\rangle$. Equation (3.14), together with (2.26). and (2.27) then implies immediately

$$\langle\lambda\,|\hat{\mathcal{G}}_S(\vec{\mathrm{P}},z)\,|\lambda'\rangle=\langle\lambda\,|\mathcal{G}_S(\vec{\mathrm{P}},z)\,|\lambda'\rangle_0+\sum_{\lambda''\,\lambda'''}\langle\lambda\,|\hat{\mathcal{G}}_S(\vec{\mathrm{P}},z)\,|\lambda''\rangle_0\langle\lambda''\,|K_S(\vec{\mathrm{P}})\,|\lambda'''\rangle\,\langle\lambda'''\,|\hat{\mathcal{G}}_S(\vec{\mathrm{P}},z)\,|\lambda'\rangle\,, \tag{3.37}$$

where

$$\langle\lambda\,|\mathcal{G}_S(P,z)\,|\lambda'\rangle_0\equiv\frac{i\,\delta_{\lambda\lambda'}}{z+\bar{\mu}-P^2/4m-\omega_{\lambda S}} \tag{3.38}$$

refers to the free composite solution, and the integral equation kernal $K_S(\vec{\mathrm{P}})$ is given by

$$\langle\lambda\,|K_S(\vec{\mathrm{P}})\,|\lambda'\rangle\equiv\frac{-i}{\Omega}\sum_{k}\psi^*_{\lambda S}(\vec{\mathrm{k}})\bigg[\left(\frac{k^2}{m}-\omega_{\lambda' S}\right)[n(\tfrac{1}{2}\vec{\mathrm{P}}+\vec{\mathrm{k}})+n(\tfrac{1}{2}\vec{\mathrm{P}}-\vec{\mathrm{k}})]$$

$$+\frac{1}{\Omega}\sum_{\vec{\mathrm{q}}}\langle\tfrac{1}{2}(\vec{\mathrm{k}}-\vec{\mathrm{q}})\,|\,U\,|\,\tfrac{1}{2}(\vec{\mathrm{k}}-\vec{\mathrm{q}})\rangle[n(\tfrac{1}{2}\vec{\mathrm{P}}+\vec{\mathrm{q}})+n(\tfrac{1}{2}\vec{\mathrm{P}}-\vec{\mathrm{q}})]\bigg]\,\psi_{\lambda' S}(\vec{\mathrm{k}})\,, \tag{3.39}$$

with the potential $\langle\vec{\mathrm{k}}\,|U|\vec{\mathrm{k}}'\rangle$ defined previously in Eq. (3.7). The function $K_S(\vec{\mathrm{P}})$ is small for low density, and approaches zero as density goes to zero. An iterative solution of Eq. (3.37) then provides a density expansion, the first term of which represents the free composite solution.

Thus, the Green's function $\tilde{\mathcal{G}}^<_S(\vec{\mathrm{P}},0)$ is given by

$$\langle\lambda\,|\tilde{\mathcal{G}}_S(\vec{\mathrm{P}},0)\,|\lambda'\rangle=\delta_{\lambda\lambda'}f_B\left(\frac{P^2}{4m}+\omega_{\lambda S}-\bar{\mu}\right)+\frac{i}{\beta}\sum_{n_{\mathrm{even}}}\,[\langle\lambda\,|\hat{\mathcal{G}}_S(\vec{\mathrm{P}},\,i\pi n/\beta)\,|\lambda'\rangle-\langle\lambda\,|\hat{\mathcal{G}}_S(\vec{\mathrm{P}},i\pi n/\beta)\,|\lambda'\rangle_0]\,e^{-i\pi n o^+}\,. \tag{3.40}$$

The first term of this expression, together with Eq. (3.36), now produces a result for $\langle\vec{\mathrm{k}}\,|\tilde{\mathcal{G}}^<_S(\vec{\mathrm{P}},0)\,|\vec{\mathrm{k}}'\rangle$ identical in form to the general result (3.23), but with ϕ replaced by ψ and $E_{\vec{\mathrm{P}}S\nu}\to P^2/4m+\omega_{\lambda S}$.

Finally, Eq. (3.34) can be written in the free composite representation

$$n^{\mathrm{nc}}(\vec{\mathrm{p}}_1)=\frac{1}{2\langle N\rangle}\Bigg(\frac{1}{\Omega}\sum_{\substack{\vec{\mathrm{p}}_2 S\\ \lambda\lambda'}}\psi_{\lambda S}(\vec{\mathrm{k}})\langle\lambda\,|\tilde{\mathcal{G}}^<_S\,(\vec{\mathrm{P}},0)\,|\lambda'\rangle$$

$$\times D_{S\lambda'}\,\psi^*_{\lambda' S}(\vec{\mathrm{k}})[1-n(\vec{\mathrm{p}}_1)-n(\vec{\mathrm{p}}_2)]$$

$$+\,n^{\mathrm{nc}}(\vec{\mathrm{p}}_1)-\frac{2}{\beta}\,\frac{\partial}{\partial\bar{\mu}}\,n^{\mathrm{nc}}(\vec{\mathrm{p}}_1)\Bigg)\,, \tag{3.41}$$

where $D_{S\lambda'}\equiv(\delta_{S,0}+3\delta_{S,1})\frac{1}{2}[1+(-1)^S\pi_{\lambda'}$ and we again recall that $\vec{\mathrm{P}}$ and $\vec{\mathrm{k}}$ refer to $\vec{\mathrm{p}}_1+\vec{\mathrm{p}}_2$ and $\frac{1}{2}(\vec{\mathrm{p}}_1-\vec{\mathrm{p}}_2)$, respectively. Again, use of the first term of (3.40) in (3.41) produces an expression like (3.34), with the zero density replacements $\phi\to\psi$ and $E\to(P^2/4m)+\omega$.

The specific composites discussed here have been fermion pairs and the composite formulation adopted is designed to retain the correct over-all Fermi symmetry. Thus, the "composite Hartree-Fock" approximation generates a pair wave function ϕ whose use in the many-body theory maintains this Fermi symmetry. Since "pairing with proper symmetry" is also a feature of the familiar Bardeen-Cooper-Schrieffer (BCS) theory,[4] it is perhaps useful to make a comparison. If we consider the ground-state solution to Eq. (3.19), assuming a singlet state and taking $\bar{\mu}=E_{000}$, we have

$$2\epsilon(\vec{\mathrm{k}})\phi_{000}(\vec{\mathrm{k}})+[1-2n(\vec{\mathrm{k}})]\int\frac{d^3q}{(2\pi)^3}\langle\vec{\mathrm{k}}\,|\,v_0\,|\,\vec{\mathrm{q}}\,\rangle\,\phi_{000}(\vec{\mathrm{q}})=0\,, \tag{3.42}$$

with $\epsilon(\vec{\mathrm{k}})=\epsilon^{\mathrm{HF}}(\vec{\mathrm{k}})-\frac{1}{2}\bar{\mu}$. Defining the quantity

$$\Delta(\vec{\mathrm{k}})\equiv\alpha\int\frac{d^3q}{(2\pi)^3}\,\langle\vec{\mathrm{k}}\,|\,v_0\,|\,\vec{\mathrm{q}}\rangle\,\phi_{000}(\vec{\mathrm{q}})\,, \tag{3.43}$$

with α an arbitrary constant, we see from (3.42) that $\Delta(\vec{\mathrm{k}})$ satisfies

$$\Delta(\vec{\mathrm{k}})=-\int\frac{d^3q}{(2\pi)^3}\,\langle\vec{\mathrm{k}}\,|\,v_0\,|\,\vec{\mathrm{q}}\rangle\,\frac{[1-2n(\vec{\mathrm{q}})]}{2\epsilon(\vec{\mathrm{q}})}\,\Delta(\vec{\mathrm{q}})\,. \tag{3.44}$$

This is indeed the BCS gap equation, provided that the fermion distribution $n(\vec{\mathrm{q}})$ has the form appropriate to BCS theory, i.e.,

$$n(\vec{\mathrm{q}})=\frac{1}{2}\left(1+\frac{\epsilon(\vec{\mathrm{q}})}{\omega(\vec{\mathrm{q}})}\right)f[\omega(\vec{\mathrm{q}})]+\frac{1}{2}\left(1-\frac{\epsilon(\vec{\mathrm{q}})}{\omega(\vec{\mathrm{q}})}\right)f[-\omega(\vec{\mathrm{q}})]\;,$$

with $\omega(\vec{q}) \equiv [\epsilon^2(\vec{q}) + \Delta^2(q)]^{1/2}$ and $\frac{1}{2}\bar{\mu}$ in $\epsilon(\vec{q})$ the usual fermion chemical potential μ. In this case, the factor $[1 - 2n(\vec{q})]/2\epsilon(\vec{q})$ becomes simply $\tanh\frac{1}{2}\beta\omega(\vec{q})/2\omega(\vec{q})$, and Eq. (3.44) takes on its familiar BCS form.

The point, of course, is that "pairing with proper symmetry" is achieved by utilizing the composite Hartree-Fock function ϕ in Eq. (3.42). The same information is contained in Eq. (3.44) for Δ, and "properly symmetric pairing" is independent of the specific form for $n(\vec{q})$. [The $n(\vec{q})$ noted above together with the interpretation of Δ as a gap in the single-particle energy spectrum is, of course, peculiar to BCS theory.] In other words, "pairing" with self-consistent "Hartree-Fock composites" whose wave function enforces over-all Fermi symmetry is a concept quite independent of the specific form for $n(\vec{q})$. In the weak-coupling superconductor, $n(\vec{q})$ is close to the "Fermi-sea" form, and $\epsilon(\vec{k}=0)$ can be both positive and negative. On the other hand, a system with a strongly bound two-body state can exhibit complete condensation in the composite Hartree-Fock approximation, and $n(\vec{q}) = n|\phi_{000}(\vec{q})|^2$ at $T=0$ for density n sufficiently low so that $\epsilon(\vec{k}=0)$ is always positive. This latter situation is described and illustrated with a simple two-body potential in Ref. 9.

IV. CONCLUSION

We have given a conceptually complete formalism for the composite language description of many-particle systems. Although the specific formalism discussed is applicable to composites of two-fermion constituents, its extension to more general composites is obvious. The composite fields are neither Bose nor Fermi; rather they satisfy more complicated commutation relations which reflect the internal structure. It is meaningless to ask whether the composite operators exhibit elementary Bose or Fermi behavior (i.e., elementary commutation rules) in any prescribed limit. To recover the Ehrenfest-Oppenheimer limit,[1] one need show only that the expectation values of certain operator products, corresponding to physical observables, approach at low over-all density the value one would have obtained had the composites been truly "elementary." Hence, the operators themselves need not embody any simple "low-density limit." Having defined the composite-field operators, we have then constructed a Green's-function prescription for calculation of physical observables.

The formulation given here is, of course, completely equivalent to the usual quantization in terms of fermions, and if one starts with the composites and all their symmetries, one would be led inevitably to the fermion language. In fact, one of the quantities appearing in the composite dynamics is $\langle\rho_{11}\rangle$, which is given in terms of the composites but which has physical significance only in the fermion language, i.e., the fermion momentum distribution. Although the energy, e.g., Eq. (3.10), does not formally depend on $\langle\rho_{11}\rangle$, the determination of G, which does appear, requires an evaluation of $\langle\rho_{11}\rangle$. In the special case where the composite is very tightly bound, and where $\langle\rho_{11}\rangle \ll 1$, explicit evaluation of $\langle\rho_{11}\rangle$ is unnecessary, the energy and other microscopic properties are then independent of the internal composite structure, as one would expect. The composite language is appropriate when a dominant feature of the system is the tight binding of two or more of the elementary constituents, and this language does facilitate approximation methods which include correctly all the symmetries, and which further include this tight binding aspect.

Such an approximation is that given in Eq. (3.1), a Hartree-Fock-like factorization which neglects dynamical correlations between composites. This approximation leads to a simple physical picture of how the composite internal states are distorted by the presence of the surrounding composites, with their enforcement of the necessary symmetries.

Numerical results for a model problem using this formalism and this approximation appear in Ref. 9. Specifically, it is shown there how a system with an available two-body bound state condenses into this bound state at low density and zero temperature, and how as the density is increased the binding becomes weaker as a result of the intercomposite symmetries. This is one of the types of problems for which the methods given here would prove useful. The analysis of such a system is being extended to nonzero temperature and to larger densities such that the bound state has, in fact, merged with the continuum. Results for this problem, of some interest in itself, will be reported in a subsequent article.

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