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# Spatial decay of temperature nonuniformities: The weakly interacting Bose gas

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The spatial decay of temperature nonuniformities in the weakly interacting Bose gas is studied by utilizing Ma's low-temperature solutions of the Boltzmann equation. These results, which give exponential decay lengths varying as $T^{-9}$, may be relevant to He II for $T \lesssim 0.1$ K. The longest two decay lengths, $\delta_1$ and $\delta_2$, are found to be in nearly a ten-to-one ratio, with $\delta_1 \approx 3.41\times10^3$ cm for He II at 0.1 K. At higher temperatures the decay lengths are expected to decrease less rapidly than $T^{-9}$ because of phase-space considerations, but an extrapolation to 0.3 K does indicate $\delta_1$ to be on the order of 1 cm. Two types of experiments are suggested for determining these lengths. It is pointed out that information about phonon dispersion can be obtained from such measurements.

## I. INTRODUCTION

Recently,[1] we analyzed the problem of the apparent temperature discontinuity $\Delta T$ which occurs when heat flows through an interface between two materials.[2] Our point of view was that the study of heat transport requires the study of the transport equation. In the appendix of that paper it was shown that there are an infinite number of solutions to the linearized Boltzmann equation which, in the steady state, damp out exponentially in space with characteristic distances $\delta_i$, and which cannot be described in terms of local thermohydrodynamic variables (i.e., pressure $P$, temperature $T$, and velocity $\vec{v}$). It was shown that these solutions (which we called nonhydrodynamic modes) can produce an apparent temperature discontinuity $\Delta T$ at an interface.

The purpose of this paper is to provide a concrete realization of the nonhydrodynamic modes for an easily analyzed model: the weakly interacting Bose gas at low temperature (WIBGALT). This model may be relevant to liquid He II for $T\lesssim 0.1$ K,[3] for which the exponential decay lengths can be studied experimentally with a carbon resistor:

$$V(z) = V_0+\sum_i A_i e^{-z/\delta_i} . \qquad (1)$$

Here $V(z)$ is the voltage output across the carbon resistor as a function of distance from the interface $z$, at constant current input. $V_0$ is the voltage observed far from the interface. The quantities $A_i$ depend upon the nonhydrodynamic modes and the boundary conditions at the interface, and are proportional to the heat current through the interface. The $\delta_i$'s are properties of the medium alone, and are the exclusive subject of our present concern. Although there are an infinite number of them, we find that for the WIBGALT the longest ones are well separated, so that if this model is indeed relevant to He II, it should be possible to measure a number of the $\delta_i$'s.

One may appreciate the necessity for these nonhydrodynamic modes by considering the following question. Two heaters of identical geometry, but different composition (e.g., Cu or Al) are placed in a vat of fluid or against a third solid. How does the fluid (or solid) discriminate between the two heaters? At the macroscopic level, the (thermohydrodynamic) temperature profiles will be identical. Consequently any difference in behavior must be manifested on a scale shorter than that appropriate to hydrodynamics. This is the scale described by the nonhydrodynamic modes.

Section II presents a short summary of the general properties of the nonhydrodynamic modes. Section III presents our explicit results for the WIBGALT. In Sec. IV we discuss how one might observe the nonhydrodynamic modes by both direct and indirect methods. It is pointed out that experimental study of the exponential decay lengths could yield valuable information about phonon-phonon scattering and phonon dispersion.

## II. PROPERTIES OF THE NONHYDRODYNAMIC MODES

Consider the linearized Boltzmann equation for phonons:

$$\frac{\partial(\delta n_p)}{\partial t}+\frac{\partial(\delta n_p)}{\partial \vec{r}}\cdot\vec{v}_p+\frac{\partial(\delta n_p)}{\partial \vec{p}}\cdot\frac{\partial \vec{p}}{\partial t}=J_{pp'}\delta n_{p'} . \qquad (2)$$

Here $\delta n_p$ is the deviation of the phonon distribution function from its equilibrium value $n_p^{(0)}$, where $\vec{p}$ is the phonon wave vector. $J_{pp'}$ is the linearized collision operator. Note that summation on $p'$ is implied.

At this point, the $\partial\vec{p}/\partial t$ term may be dropped as an unnecessary complication. We will be interested in solutions of the form $\delta n_p \propto e^{i(\vec{q}\cdot\vec{r}-\nu t)}$. Then

$$-i(\nu - \vec{q}\cdot\vec{v}_p)\delta n_p = J_{pp'}\delta n_{p'} \ . \tag{3}$$

One may view this equation from either of two viewpoints. On the one hand, $\vec{q}$ may be fixed by geometrical considerations (with $\vec{q}$ real), and one wants the response frequencies $\nu$, which are generally complex. On the other hand, $\nu$ may be fixed by an oscillator (with $\nu$ real), and one wants the response wave vectors $\vec{q}$, which are generally complex. We take the latter point of view.

In Ref. 1 it was shown that there are an infinite number of solutions to Eq. (3). In the limit as $\nu \to 0$, only a few of these solutions correspond to modes which can be described by hydrodynamics. The rest are what we call nonhydrodynamic modes. Note that Maris has studied the finite-frequency nonhydrodynamic modes for phonons in He II, finding a number of them which are not strongly attenuated.[4] He calls these modes second second sound, third second sound, etc.

At zero frequency ($\nu = 0$), one can easily show[1] that the nonhydrodynamic modes have the following properties: (i) $q$ is imaginary; (ii) for $q \neq 0$, there is a mode with $-q$; (iii) the modes with $q \neq 0$ can transport no conserved quantities; (iv) the modes with $q \neq 0$ can contribute to quantities such as the energy density $\epsilon$.

Since $q$ is imaginary, we write it as $q = i/\delta$. It is clear that a planar heater generates these modes at its interface with the system being studied. Far from the heater, these modes should be undetectable.

## III. APPLICATION TO THE WIBGALT

Recently, Ma derived the Boltzmann transport equation for the WIBGALT.[5] This was done very nearly from first principles by starting from the quantum-mechanical model Hamiltonian, applying temperature-dependent many-body theory, and using precisely defined approximations. The form of the collision operator $J$ was the same as would have been obtained by more naive considerations. What is most significant is that Ma was able to obtain analytically the smallest eigenvalues and their eigenvectors for that collision operator. Since the eigenvalues of $J$ are inversely proportional to a corresponding decay time, the smallest eigenvalues correspond to the longest (and therefore most macroscopically accessible) decay times. We now present a more detailed summary of these results. [Note that $k_B = \hbar = m = 1$, where $m$ is the boson mass, and the volume is taken to be unity. In these units the sound velocity is unity and the interaction strength $g = (4\pi a)^{3/2} n^{1/2}$, where $a$ is the boson-boson scattering length and $n$ is the number density.]

The following integral equation describes the response $\phi_p$ of the WIBGALT to a disturbance with a source having an $\exp[i(\vec{q}\cdot\vec{x} - \nu t)]$ variation[5]:

$$(\nu' - \vec{q}'\cdot\vec{v}_p)\phi_p = -iK_{pp'}\phi_{p'} + \text{source term} \ . \tag{4}$$

Here $\nu' = \nu/g$, $\vec{q}' = \vec{q}/g$, and $\vec{v}_p$ is the group velocity of a phonon of momentum $\vec{p}$ and energy $\omega_p$. Note that $\phi_p \propto \delta n_p (\partial n_p^{(0)}/\partial\omega_p)^{-1}$, where $\delta n_p$ and $n_p^{(0)}$ are defined in Sec. II. The integral operator $K_{pp'}$ $[= -(1/g)(s)^2 J_{pp'}(s')^{-2}$ of Sec. II] is defined by

$$K_{pp'}\phi_{p'} = (2\pi)^{-6} s \int d^3p'\, d^3p''\, (s's'')^{-1} \times [\sigma(pp'p'')(\phi_p + \phi_{p'} - \phi_{p''}) + \sigma(p'p''p)\tfrac{1}{2}(\phi_p - \phi_{p'} - \phi_{p''})] \ . \tag{5}$$

Here $s = 2\sinh(\omega/2T)$ and

$$\sigma(pp'p'') = (2\pi)^4 \delta(\omega + \omega' - \omega'') \times \delta(\vec{p} + \vec{p} - \vec{p}'')|A(pp'p'')|^2 \ , \tag{6}$$

with

$$A(pp'p'') = \tfrac{1}{2}(\lambda\lambda'\lambda'')^{-1/2}(\lambda + \lambda' - \lambda'' + 3\lambda\lambda'\lambda'') \tag{7}$$

and

$$\omega = \omega_p, \quad \omega' = \omega_{p'}, \quad \lambda = p^2/2\omega_p, \quad \text{etc.} \tag{8}$$

With the inner product, or matrix element, defined as

$$\langle A|B\rangle = (2\pi)^{-3} \int d^3p\, s^{-2} A_p B_p \ , \tag{9}$$

$K$ is a real, symmetric, and positive semidefinite operator. Because of its rotational invariance, the eigenfunctions of $K$ are proportional to the spherical harmonics $Y_{lm}(\hat{p})$, and the eigenvalues are independent of $m$. For each $l$, the lowest eigenvalue is given by[3,5]

$$\lambda_l = l(l+1)[\tfrac{1}{2}l(l+1) - 1]CT^9 + O(T^{11}) \ , \tag{10}$$

where (see Appendix)

$$C = 5(3/8\pi)^5 \int_0^\infty dy\, dy'\, dy''\, (ss's'')^{-1} \times (yy'y'')^4 \delta(y - y' - y'') = 1225.8 \ . \tag{11}$$

The corresponding normalized eigenfunctions are

$$\phi_{lm} = N\chi_l(\omega/T)Y_{lm}(\hat{p}) \ , \tag{12}$$

where

$$\chi_l(y) = y - \tfrac{1}{16}l(l+1)y^3T^2 + O(T^4) \tag{13}$$

and

$$N^{-2} = \tfrac{1}{30}\pi T^3 + O(T^5) \ . \tag{14}$$

The higher eigenvalues are found to be of $O(T^5)$, and therefore very much larger than the $\lambda_l$ at low temperatures. This lack of knowledge of the higher eigenvalues means that $q'(=|\vec{q}'|)$ and $\nu'$ are restricted to be of $O(T^9)$.

To find the response of the system to the source, one must first find the solutions of the integral equation without the source. This provides a relationship between $\nu'$ and $q'$ for each of the solutions. Therefore, using the notation of Ref. 6, we wish to solve

$$\Omega'_{pp'}\phi_{p'} = \nu'\phi_p \ , \tag{15}$$

where

$$\Omega'_{pp'} = \vec{q}' \cdot \vec{v}_p \delta_{pp'} - iK_{pp'} \ . \tag{16}$$

Note that there are two differences between Ma's solution for transverse waves[6] and that which we shall obtain. First, transverse waves couple to a source with $m = \pm 1$ symmetry, whereas temperature gradients have $m = 0$ symmetry. Second, Ma considered a fixed $q'$, whereas in the problem of static temperature gradients one must fix $\nu'$ and solve for $q'$.

Restricting ourselves to $m = 0$ and using the $\phi_{l0}$'s as a basis set, so that

$$\phi_p = \sum_l a_l \phi_{l0} \ , \tag{17}$$

$\Omega'$ can be represented as[6]

$$\Omega'_{ll'} = -i\lambda_l \delta_{ll'} + q'(\eta_l \delta_{l+1,l'} + \eta_{l'} \delta_{l,l'+1}) \tag{18}$$

where, for $m = 0$,

$$\eta_l = (l+1)(2l+1)^{-1/2}(2l+3)^{-1/2} \ . \tag{19}$$

One can obtain the lowest two solutions easily in the limit as $\nu' \to 0$ by solving $\det(\Omega'_{ll'} - \nu'\delta_{ll'}) = 0$ for $l, l' = 0, 1, 2$ (i.e., a $3 \times 3$ determinant). This gives

$$q'^2 = 3\nu'^2[1 - i\tfrac{4}{5}\nu'(\lambda_2 - i\nu')^{-1}]^{-1}$$

$$\approx 3\nu'^2[1 + i(4/5\lambda_2)\nu'] \ . \tag{20}$$

This is the (damped) second-sound solution. As $\nu' \to 0$, the damping becomes negligible, so temperature gradients are not damped out by this mode.

To obtain the higher solutions we set $\nu' = 0$ and solve $\det\Omega'_{ll'} = 0$. For our purposes, only the $l, l' \geq 2$ components of $\Omega'_{ll'}$ need be considered. This may be cast into the form of an eigenvalue problem with a nonsymmetric real matrix by considering

$$\overline{\Omega}_{ll'} \equiv \Omega'_{ll'}/q'\lambda_l$$

$$= -(i/q')\delta_{ll'} + \lambda_l^{-1}(\eta_l \delta_{l+1,l'} + \eta_l \delta_{l,l'+1}) \ . \tag{21}$$

With $\delta' \equiv i/q'$, the condition $\det \overline{\Omega}_{ll'} = 0$ is equivalent to our previous condition on $\Omega'_{ll'}$, and takes the form of an equation for the eigenvalues $\delta'$. The results for the lowest nine such pairs are given in Table I. In addition, sufficient information about the eigenvectors is given to permit them to be calculated in detail. The eigenvalues are real and come in equal and opposite pairs, as described in the previous section. The corresponding damping lengths $\delta = g^{-1}\delta'$ are proportional to $(gT^9)^{-1}$, lengthening as the interaction strength and temperature decrease.

In an appendix,[3] Maris indicates that for $T \leq 0.1$ K, Eq. (10) for the low-temperature limit of the eigenvalues $\lambda_l$ is confirmed by numerical calculations on a model specific to real liquid helium. Here the phonon energy $\epsilon$ is given by

$$\epsilon = c_0 p(1 + \gamma p^2) \ , \tag{22}$$

for sufficiently small momentum $p$, where

$$c_0 = (2.383 \pm 0.001) \times 10^4 \text{ cm/sec}^{-1}$$

is the sound velocity as determined by ultrasonic measurements,[7] and $\gamma$ may be taken to be $8 \times 10^{37}$ cgs units.[3] In these units, the eigenvalues $\lambda_l$ for three-phonon processes are given by[3]

$$\lambda_l = 17\,433(l-1)l(l+1)(l+2)$$

$$\times [\gamma^2(u_0+1)^2/\rho\hbar^4](K_B T/c_0)^9 \ . \tag{23}$$

Here $u_0 = (\rho/c_0)(\partial c_0/\partial\rho) = 2.84$ is the Grüneisen constant[8] and $\rho = 0.145\,13$ g cm$^{-3}$ is the density.[9] To convert the results of the table to values relevant to He II at $T \lesssim 0.1$ K, one simply notes that

$$\delta = 1.747 \times 10^5 (\gamma')^{-2} T^{-9} \delta' \text{ cm} \ . \tag{24}$$

Here $\gamma' = \gamma/(8 \times 10^{37}$ cgs), $T$ is in 0.1 K, and the $\delta'$ are the dimensionless numbers given in the table. At $T = 0.1$ K, the first two exponential damping lengths are (with $\gamma' = 1$)

$$\delta_1 = 3.41 \times 10^3 \text{ cm}, \quad \delta_2 = 3.44 \times 10^2 \text{ cm} \ .$$

TABLE I. The largest nine (pairs) of $\delta'_i$'s. The units are $(CT^9)^{-1}$, where these symbols are defined in the text. Using $\overline{\Omega}_{ll'} a_{l'} = 0$, Eq. (21), and $a_{l=2}$ and $a_{l=3}$ from columns 3 and 4, all $a_l$'s can be generated. The solutions are not normalized. Convergence was tested by going to larger and larger basis sets, in steps of two. This explains the decreasing accuracy for increasing $i$. The largest basis set had 20 members, with the largest value of $l$ being $l = 21$.

| $i$ | $\delta'_i$ | $a_{l=2}$ | $a_{l=3}$ |
|---|---|---|---|
| 1 | $\pm 0.195\,158\,08 \times 10^{-1}$ | $0.100\,000\,00 \times 10^{1}$ | $\pm 0.461\,828\,33$ |
| 2 | $\pm 0.197\,088\,11 \times 10^{-2}$ | $0.100\,000\,00 \times 10^{1}$ | $\pm 0.466\,395\,61 \times 10^{-1}$ |
| 3 | $\pm 0.482\,759\,99 \times 10^{-3}$ | $0.994\,457\,27$ | $\pm 0.113\,608\,66 \times 10^{-1}$ |
| 4 | $\pm 0.172\,433\,34 \times 10^{-3}$ | $0.942\,239\,65$ | $\pm 0.384\,482\,56 \times 10^{-2}$ |
| 5 | $\pm 0.763\,416\,01 \times 10^{-4}$ | $0.893\,132\,83$ | $\pm 0.161\,350\,88 \times 10^{-2}$ |
| 6 | $\pm 0.388\,646\,96 \times 10^{-4}$ | $0.895\,428\,05$ | $\pm 0.823\,531\,08 \times 10^{-3}$ |
| 7 | $\pm 0.218\,317 \times 10^{-4}$ | $0.916\,008$ | $\pm 0.473\,238 \times 10^{-3}$ |
| 8 | $\pm 0.132 \times 10^{-4}$ | $0.857$ | $\pm 0.267 \times 10^{-3}$ |
| 9 | $\pm 0.79 \times 10^{-5}$ | $0.89$ | $\pm 0.17 \times 10^{-3}$ |

Note that a factor of $c_0$ must be used in converting from the dimensionless $\delta'$ to $\delta$ in cgs units.

## IV. DISCUSSION

Our purpose has been to discuss the spatial decay of temperature nonuniformities. In the previous section we presented an explicit realization of the modes by which such decay occurs. Their properties were discussed quite generally in Sec. II. We now consider some experimental questions relating to the nonhydrodynamic modes (NHM's).

First, one must determine under what circumstances they are generated. Certainly, they do not exist for a system in thermal equilibrium: they are generated only when there is a disturbance from thermal equilibrium, as is the case for steady-state heat flow. One expects that the modes are generated with amplitudes proportional to the heat current, and whose precise values are sensitive to the nature of the coupling between the heater and the medium in question (i.e., boundary conditions).

Second, one must determine under what circumstances they are observable. Clearly, one must use a probe which is small compared to the characteristic length $\delta_i$ of any mode one wishes to study. On the other hand, the apparatus must be large enough to perform the measurements. If the results of Sec. III are to be applied to $T \lesssim 0.1$ K, one can study only the smaller $\delta$'s (e.g., $\delta_7$ is about 3 cm at $T=0.1$ K) in a typical apparatus of cm dimensions. Interpretation of data in this case will be more complex than if only one or two modes were of the characteristic dimension of the apparatus. However, the values given in Table I do indicate a fairly clean separation of the modes, so that analysis in this temperature range should be tractable. Besides He II, the NHM's may also be studied in rarified gases, in which case one must employ particle number densities corresponding to mean free paths on the order of 1 cm. It should be noted that temperature-profile measurements on rarified gases have been made,[10] but because of heat-leak problems the method is questionable. The results are not inconsistent with an exponential profile.

Third, one may question what is a valid measurement of "temperature" when the system properties vary significantly over a distance on the order of, or smaller than, a mean free path. Fortunately, although it is not obvious how to establish the absolute magnitude of the NHM amplitudes (and therefore, an absolute "temperature"), one can measure a resistance or a thermoelectrically generated voltage as a function of position, as indicated in the Introduction. This enables one to obtain a spatial profile, from which the exponential damping lengths $\delta_i$ can be determined. The problem of defining a "temperature" can therefore be circumvented. It is only the shape of the profile which is important for determining the $\delta_i$'s.

It should be noted that these modes are intimately related to the problem of thermal-boundary resistance. For steady-state heat flow, the hydrodynamic mode carries the heat current (in Sec. II we pointed out that the NHM's cannot do this), and the NHM's have an associated energy density which mimics the temperature.[1] The NHM's are generated near the boundary in both materials, and produce (for coarse measuring devices) what appears to be a temperature discontinuity.[1] This is a very general phenomenon; it has even appeared in molecular dynamics calculations of heat transfer through finite systems.[11] In Fig. 3 of Ref. 11 the hint of an exponential contribution to the temperature profile appears at the interfaces of the system with the thermal reservoirs.

Since measurements of the temperature profile are expected to be rather difficult to obtain, an alternate approach would be to consider the Kapitza resistance of a bath of He II between two plates separated by a distance $2L$. (The plate width $w$ must satisfy $w \gg 2L$ to approximate the ideal of a one-dimensional situation.) It was previously found[12] that for only one exponentially damped mode

$$R_K = R \tanh(L/\delta) . \tag{25}$$

Here $R$ depends on a product of two quantities, one involving the properties of the mode and the other involving the boundary conditions at the plates (which are assumed to be essentially identical). In the presence of many such modes, this result generalizes to[13]

$$R_K = \sum_i R_i \tanh(L/\delta_i) . \tag{26}$$

It should be noted that, at fixed $T$, measurements of $R_K$ should be corrected for the fixed thermal-boundary resistance occurring at the surface of each plate. For small $\delta_i$, or $\delta_i \ll L$, one has $R_i \tanh(L/\delta_i) \approx R_i$, and the contribution appears as a pure surface effect. For large $\delta_i$, or $\delta_i \gg L$, one has $R_i \tanh(L/\delta_i) \approx R_i L/\delta_i$, and the contribution appears as a pure bulk effect. By measuring $R_K(L)$, it should be possible to obtain information about $\delta_1$, $\delta_2$, etc.

If, at sufficiently low temperatures, the $\delta_i$'s are found to have a $T^{-9}$ dependence, this would provide evidence in favor of three-phonon processes.[14] Further, given a $T^{-9}$ dependence, one would be able to simply determine the dispersion parameter $\gamma$ without having to go through an involved theoretical analysis. On the other hand, the results of Sec. III indicate that the nonhydrodynamic modes can be observed at temperatures higher than $T = 0.1$ K. Extrapolating the result for $\delta_1$ to $T = 0.25$ K, one finds $\delta_1 \approx 1$ cm. Therefore it should be possible to observe the nonhydrodynamic modes by utilizing the temperature range of 0.2–0.3 K. In this range, the detailed results of Sec. III will be inapplicable, and work similar to that of Ref. 4 would have to be done to find theoretical values for the δ's and their dependence on the phonon dispersion. One should note that an extrapolation of the $T^{-9}$ dependence involves an overestimate of the phase space available for three-phonon processes since the phonon dispersion relation starts to bend downward as the wave vector increases.[3] Therefore the damping lengths are expected to exceed the values given by Eq. (24), and it would not be surprising to find $\delta_1$ on the order of 1 cm at $T = 0.3$ K. The first clear observation of a nonhydrodynamic mode in any system, and the fact that such observation would shed light on the fundamental question of phonon dispersion, provide motivation for performing either of the suggested experiments.

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## APPENDIX

Our purpose here is to evaluate the integral

$$J = \int_0^\infty dy\, dy'\, dy''\, (ss's'')^{-1}(yy'y'')^4 \delta(y - y' - y''). \qquad \text{(A1)}$$

To do this we use the method employed by Ma (private communication) in evaluating a similar integral in Ref. 5. Note that in Ref. 3 it is pointed out that the exponent of $(yy'y'')$ should be 4, rather than 3 (the value employed in Ref. 5).

Using the identity

$$\delta(y - y' - y'') = (2\pi)^{-1} \int_{-\infty}^{\infty} dk\, e^{-ik(y - y' - y'')} , \qquad \text{(A2)}$$

one can rewrite Eq. (A1) as

$$J = (2\pi)^{-1} \int_{-\infty}^{\infty} K^2 K^* dk , \qquad \text{(A3)}$$

where

$$\begin{aligned} K &= \int_0^\infty dx\, s^{-1} x^4 e^{-ikx} \\ &= \sum_{n=0}^{\infty} \int_0^\infty dx\, x^4 e^{[-(n+1/2)-ik]x} \\ &= 4! \sum_{n=0}^{\infty} (ik + n + \tfrac{1}{2})^{-5} . \end{aligned} \qquad \text{(A4)}$$

One then has

$$J = (24)^3 \sum_{n_1 n_2 n_3} \int_{-\infty}^{\infty} (2\pi i)^{-1} dk\, (k - k_1)^{-5} (k - k_2)^{-5} \times (k - k_3)^{-5} , \qquad \text{(A5)}$$

where $k_1 = i(n_1 + \frac{1}{2})$, $k_2 = i(n_2 + \frac{1}{2})$, and $k_3 = -i(n_3 + \frac{1}{2})$. This may be evaluated by enclosing the pole at $k_3$, which is in the lower half-plane. Then

$$J = (24)^3 \sum_{n_1 n_2 n_3} (-4!)^{-1} (d^4 f / dk^4)_{k_3} , \qquad \text{(A6)}$$

where $f = (k - k_1)^{-5}(k - k_2)^{-5}$. Performing the indicated differentiation yields

$$\begin{aligned} J &= 5(24)^3 \sum_{n_1 n_2 n_3} \frac{1}{n^5 n'^5} \left( \frac{14}{n^4} + \frac{35}{n^3 n'} + \frac{45}{n^2 n'^2} + \frac{35}{n n'^3} + \frac{14}{n'^4} \right) , \\ &\equiv 5(24)^3 \mathcal{J} , \end{aligned} \qquad \text{(A7)}$$

where $n = n_1 + n_3 + 1$, $n' = n_2 + n_3 + 1$. The sum may be rearranged so that tabular values of the Riemann zeta function may be employed. Then

$$\begin{aligned} \mathcal{J} &= \sum_{n_1 n_2 n_3} \mathcal{J}(n_1, n_2, n_3) = \sum_{n_1 n_2 n_3} [\mathcal{J}(n_1, n_2, 0) + \mathcal{J}(n_1, n_2, 1) + \mathcal{J}(n_1, n_2, 2) + \cdots ] \\ &= [28\zeta(5)\zeta(9) + 70\zeta(6)\zeta(8) + 45\zeta(7)\zeta(7)] + \{28[\zeta(5) - 1][\zeta(9) - 1] + 70[\zeta(6) - 1][\zeta(8) - 1] + 45[\zeta(7) - 1]^2\} \\ &\quad + \{28[\zeta(5) - 1 - 2^{-5}][\zeta(9) - 1 - 2^{-9}] + 70[\zeta(6) - 1 - 2^{-6}][\zeta(8) - 1 - 2^{-8}] + 45[\zeta(7) - 1 - 2^{-7}]^2\} \\ &\quad + \cdots \\ &= 146.351\,240 + 0.010\,163\,570 + 0.000\,042\,330 + \cdots = 146.361\,446\ldots . \end{aligned} \qquad \text{(A8)}$$

(Extra decimal places have been kept in intermediate steps to indicate the accuracy of the method.)

From this one has

$$\begin{aligned} C &= 5(3/8\pi)^5 J \\ &= [5(3/8\pi)^5][5(24)^3]\mathcal{J} \\ &= 1225.7655 . \end{aligned}$$

Note that the integral given in (A2) of Ref. 5 can be related to $J$. One finds that

$$J=\frac{1}{4}\int_0^1 dx\,x^4(1-x)^4 \times\int_0^\infty dy\,y^{13}\coth(\tfrac{1}{2}xy)\,\mathrm{csch}^2(\tfrac{1}{2}y)\ .$$

[Reference 3 has $y^{14}$ rather than $y^{13}$; this appears to be a misprint. Further, the first line of (A2) has the factor 1012 rather than 1024; this also appears to be a misprint.]

From (A2) of Ref. 3 one has

$$J_5=(\tfrac{1}{4})(\tfrac{17\,433}{135})(1024\pi^5)=1.011\,644\times10^7\ ,$$

whereas we find

$$J_A=1.011\,650\times10^7\ .$$

The agreement is quite good.

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[14]Note that L. D. Landau and I. M. Khalatnikov obtained a collision time $\tau_{pp}\propto T^{-9}$, appropriate to equilibration of transverse disturbances in the phonon gas. The scattering mechanism they considered was the three-phonon process to second order. Since $\tau_{pp}>>\lambda_2^{-1}$, we neglect the mechanism. See Zh. Eksp. Teor. Fiz. 19, 637 (1949) [English translation in *Collected Papers of C. D. Landau* (Pergamon, Oxford, England, 1965), p. 494].