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Sound waves near T_c in a dynamic spherical model

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The solution of the isotropic n-component spin models by expanding in powers of $1/n$ has been very helpful in studying static critical phenomena. The limit $n \rightarrow \infty$ is the spherical model. There are many possible generalizations of these models for studying dynamic (time dependent) critical phenomena. In this paper we study the propagation of sound waves and heat diffusion near the critical point in a $n/2$-component complex-spin (each complex component has two real components) model in the limit of large n. This model is a particular dynamic generalization of the spherical model and is equivalent to a system of bosons obeying quantum mechanics.

I. INTRODUCTION

Recently, dynamic generalizations of the static spherical model and $1/n$ expansion have been studied by several authors.[1-4] A particularly simple generalization is the model of a $\frac{1}{2}n$-component complex field, i.e., a Bose field, with very large n. The dynamics is given at the microscopic level by quantum mechanics. The simplicity of this well-defined model suggests that a detailed analysis would lead to useful qualitative information concerning critical dynamics. Much attention has been received.[2]

The main purpose of this study is to investigate how the phenomena of sound waves and heat diffusion are realized in this model. In this study, only the leading order in $1/n$ will be kept. Thus, the status is analogous to that of the spherical model in the static $1/n$ expansion.

We want to emphasize that dynamics is vastly more complicated than statics. One can make many different generalizations from the static $1/n$ expansion for studying critical dynamics. The model studied in Ref. 2 and here is just one of them. The exploration of other models is important, but no effort will be made here in that direction. In this paper, we report only some straightforward calculations concerning sound waves in this model and the physical picture which follows. The results are quite simple and easy to understand. The calculations will reflect some complexities in $1/n$ expansions of dynamics.

There are two rather undesirable qualities of our approach. First, the calculation will be based on a perturbation expansion. The solution is obtained by a formal counting of powers in $1/n$ and then dropping all but the lowest power of $1/n$. Without a rigorous mathematical foundation, this approach is not guaranteed to give the true solution.

Second, the solution to the leading order in $1/n$ may not contain some of the information of interest to critical phenomena. Let us illustrate this point in more detail. Suppose that a physical quantity Q behaves near T_c like

$$Q = A_1 |T - T_c|^{-a_1} + A_2 |T - T_c|^{-a_2} + \cdots, \quad a_1 > a_2 > \cdots. \tag{1.1}$$

Suppose that, for $n \rightarrow \infty$, $Q = O(1)$. The most important term is the first one as far as critical behavior is concerned. The most important quantity we want to calculate is a_1. Now we calculate Q in $1/n$ expansion and keep only the leading term, i.e., $O(1)$. Then we shall get a_1 only if $A_1 = O(1)$. If $A_1 = O(1/n)$ and A_2 or $A_3 = O(1)$, then we would be out of luck. Thus one must be extremely cautious in interpreting the results.

The dynamics described by the leading order in $1/n$ turns out to be essentially that of a system of linearly coupled modes. The information input to this linear-coupling scheme will come from the statics of the spherical model. The simplicity allows an easy visualization of the basic features of first and second sound and heat diffusion. However, along with some other important features, the nonlinear mode-mode coupling of Kawasaki[5] and Kadanoff and Swift[6] can be realized only if higher-order terms in $1/n$ are kept.

Let us introduce the model and summarize briefly its important features. The model describes a complex vector field $\psi_\sigma(x)$, $\sigma = 1, 2, \ldots, \frac{1}{2}n$, in a d-dimensional cube of unit volume with periodic boundary conditions. Let $a_{\sigma k}$ be the Fourier component of $\psi_\sigma(x)$ of wave vector (or momentum) k. We assume the Hamiltonian

$$H = \sum_{k,\sigma} (k^2 a_{\sigma k}^\dagger a_{\sigma k} + \tfrac{1}{2} u_k \rho_k \rho_k^\dagger), \tag{1.2}$$

where

$$\rho_k \equiv \int d^dx\, e^{-ik\cdot x}\rho(x), \tag{1.3}$$

$$\rho(x) \equiv \sum_\sigma \psi_\sigma^\dagger(x)\psi_\sigma(x), \tag{1.4}$$

and u_k is assumed to be a smooth function of k with $u_0 > 0$. The dynamics follows quantum mechanics and the commutation rule

$$[a_{\sigma k}, a^\dagger_{\sigma' k'}] = \delta_{\sigma\sigma'}\delta_{kk'}. \tag{1.5}$$

This model is conveniently regarded as a system of bosons of mass $\frac{1}{2}$. The statistical averages will be taken over the grand-canonical ensemble

$$e^{-\beta(H-\mu N)} \tag{1.6}$$

in the usual notation. This model observes the local conservation law

$$\partial\rho/\partial t = -i[\rho, H] = -\nabla\cdot j, \tag{1.7}$$

where

$$j \equiv -i\sum_\sigma [\psi_\sigma^\dagger \nabla\psi_\sigma - (\nabla\psi_\sigma^\dagger)\psi_\sigma]. \tag{1.8}$$

This conservation law is, of course, fundamental to our discussion and results.

For large n, we take ψ_σ as $O(1)$. Then $\rho(x) = O(n)$. In order that both terms in H be of the same order, i.e., $O(n)$, we take $u_k = O(1/n)$. Thus, the coupling between bosons is small but the total interaction energy is not small.

Now we have defined a model at a microscopic level. What remains to be done is "a straightforward calculation" to bring out the collective phenomena of interest from first principles. Such an approach eliminates questions associated with artificial cutoffs, over-counting of modes, etc., in a semimacroscopic phenomenological approach. On the other hand, introducing the small parameter $1/n$ at a microscopic level is dangerous. In the process of calculation, terms which are formally small for small $1/n$ are thrown away. Sometimes, it is not easy to prevent important physics from being thrown away. The interpretation of results becomes difficult too. Such danger is less serious in a phenomenological approach, where much of the physical information is put in at a semimacroscopic level.

For static phenomena, the present model gives the same results as the spherical model. There is a Bose-Einstein condensation below a certain temperature T_c. The order parameter is the complex field ψ. The solution of the model is given by the Hartree approximation and is easily understood: While each ψ_σ can fluctuate a lot near T_c, the fractional fluctuation of $\rho(x)$ about its average is small since $\rho(x)$ is a sum of $\frac{1}{2}n\ \psi_\sigma^\dagger\psi_\sigma$'s. Approximating $\rho(x)$ by its average is precisely the Hartree approximation.

The Hartree approximation is a self-consistent approximation in which bosons move independently, i.e., are effectively noninteracting. If we took it for dynamics, we would never get the sound waves or heat diffusion. We must examine carefully the order of magnitudes involved in the dynamics of interest.

The interaction between two bosons is very weak because $u_k = O(1/n)$. The scattering cross section is proportional to $u_k^2 = O(1/n^2)$. Since the population of bosons is of $O(n)$, the collision rate and the inverse mean free path of a boson are of $O(1/n)$. Therefore, the frequency and wave number of interest are in the range of $O(1/n)$. Later it will become evident that the Hartree approximation would give the correct description of the dynamics in the $1/n \to 0$ limit if the frequency or the wave number were counted as $O(1)$.

Thus, as far as collisions are concerned, the system behaves as a weakly interacting system. However, as we noted before, the interaction energy is not small. In fact, the restoring force against a change in density is strong. The combination of these two features is what makes this model interesting and results nontrivial.

Our program is to look for long-lived collective modes with frequencies and wave numbers in the range of $O(1/n)$, for temperatures near T_c. Indeed, we find the sound wave and the heat diffusion above T_c, and below T_c the first and second sounds are found.

The temperature dependence of associated physical quantities are examined. The results are summarized as follows.

For $T > T_c$, the frequency of sound and that of the heat-diffusion mode are

$$ck - \tfrac{1}{2}i\Gamma k^2 + O(k^3), \quad -iD_Tk^2 + O(k^3), \tag{1.9}$$

respectively, where k is the wave vector and $D_T = \kappa/C_P$. C_p and κ are the specific heat at constant pressure and the thermal conductivity, respectively. We find that c, the speed of sound, Γ, the damping coefficient, and κ are nonsingular and finite for small $T - T_c$ to $O(T - T_c)$. Below T_c, the frequencies of the first and second sounds are

$$\begin{aligned} &c_1k - \tfrac{1}{2}i\Gamma_1k^2 + O(k^3),\\ &c_2k - \tfrac{1}{2}i\Gamma_2k^2 + O(k^3), \end{aligned} \tag{1.10}$$

respectively. We find that

$$\Gamma_1 = (\text{const.})\{(T_c - T)\ln[(\text{const.})/(T_c - T)]\}^{-1} + \Gamma_1' \tag{1.11}$$

and that c_1^2, c_2^2, Γ_1', and Γ_2 are nonsingular to $O(T_c - T)$. Also, we find

$$c_2 \propto (T_c - T)^{1/2}. \tag{1.12}$$

At $T = T_c$, we find that

$$\Gamma_1' = \Gamma, \quad \Gamma_2 = D_T. \tag{1.13}$$

There is little resemblance of the temperature dependences listed above to those for liquid helium near the λ point.[7,8] Various reasons will be discussed.

The outline of this paper is the following.

Section II: Thermodynamic properties of the model are summarized and the mathematical program is outlined. We introduce a dielectric function which is proportional to the inverse of the density-response function.

Section III: Much of the discussion will center around an integral equation which is a result of graph summation supplied in Appendix A. This integral equation is simply a linearized Boltzmann kinetic equation and reflects the weakly interacting nature of the system. The solution of the integral equation is then used to construct the dielectric function, whose zeros determine the frequencies of sound waves and the rate of heat diffusion. The strongly interacting nature of the system is taken care of by the dielectric function. The discussion is restricted to $T > T_c$.

Section IV: The effect of Bose-Einstein condensation below T_c on the results of Sec. III is examined. The propagation and damping of the second sound are discussed.

Section V: Further discussion of the results are given. Features missed by the leading terms are discussed. We shall also show why this model would not be very helpful in studying spin diffusion (i.e., the dynamics of ψ).

Appendixes A and B sum up some technical details in deriving the integral equations used in the main text. No background on the graphic expansion at finite temperature (Matsubara method) is needed in reading the main text.

II. PRELIMINARY

A. Review of statics

The model is completely defined by (1.2)–(1.6). It describes a strongly interacting system in the sense that the kinetic energy is of the same order of magnitude as the interacting energy, even though the coupling constant u_k is of $O(1/n)$. By virtue of the large number of components, the fractional fluctuation of $\rho(x)$ is expected to be small although the amplitudes $\psi_\sigma(x)$ may fluctuate considerably. For $n \to \infty$, $\rho(x)$ may be taken as a constant, $\langle\rho(x)\rangle = N$, and the interaction Hamiltonian is approximated by a constant $\frac{1}{2}uN^2$ ($u \equiv u_0$). This is the Hartree approximation. The physical picture is that of a noninteracting Bose gas in a self-consistently determined uniform potential. Let the effective chemical potential be

$$-r = \mu - uN. \tag{2.1}$$

Then the population distribution follows:

$$\langle a_{\sigma p}^\dagger a_{\sigma p}\rangle = f_p \equiv 1/(e^{\beta\epsilon_p} - 1), \tag{2.2}$$

$$\epsilon_p \equiv p^2 + r. \tag{2.3}$$

If $T < T_c$, there is a Bose-Einstein condensate

$$\langle a_{10}^\dagger a_{10}\rangle = N_0. \tag{2.4}$$

The condensate acts as a reservoir of bosons. Thus the effective chemical potential r vanishes:

$$r = 0, \quad \mu = uN \ \text{ for } T < T_c. \tag{2.5}$$

The population excluding the condensate is then

$$N' = \tfrac{1}{2}n(2\pi)^{-d}\int d^d p \, f_p, \tag{2.6}$$

$$N' + N_0 = N. \tag{2.7}$$

For $T \geq T_c, N_0 = 0$. All thermodynamic formulas follow from (2.1)–(2.7). We collect them in Table I. Their derivation is easy and omitted here. The quantity $\Pi_0(0)$ appearing in Table I is defined as

$$\Pi_0(0) = -\frac{\partial}{\partial r}(2\pi)^{-d}\int d^d p \, f_p. \tag{2.8}$$

Note that for $T - T_c$ very small we have

$$r \propto (T - T_c)^{2/(d-2)}, \quad \Pi_0(0) \propto r^{d/2-2}. \tag{2.9}$$

Thus $\Pi_0(0)$ blows up at T_c. The quantities C_V, C_P, and $(\partial P/\partial N)_T$ are nonanalytic functions of $T - T_c$ and show a cusp behavior.

We define a correlation length ξ as

$$\xi = r^{-1/2} \propto (T - T_c)^{-\nu}, \quad \nu = 1/(d-2). \tag{2.10}$$

It should be noted that for $T < T_c$ the specific heat apparently has no singular behavior, in contrast to that above T_c[(const.) − (const.)$(T - T_c)^{-\alpha}$, $-\alpha = (4-d)/(d-2)$]. However, if one includes the next order in $1/n$, he would find a term $(T_c - T)^{-\alpha}$ below T_c.

B. Mathematical program

As we have mentioned in Sec. I, the Hartree approximation would not be adequate for dynamic calculation. To get results from first principles we shall rely on the finite-temperature perturbation theory (Matsubara method). The large-n limit will be exactly (at least formally) accounted for by an infinite set of graphs. However, to those readers who have not been exposed to the Matsubara method, some of the technicality will not be transparent. Fortunately the result of the graph summation simply leads to an integral equation which is easily visualized as a linearized Boltzmann kinetic equation. Apart from some details this kinetic equation can be guessed on intuitive grounds. Since the technical details are not our main concern, we shall consider the details of graph summation in the appendixes. In the main text we shall appeal to intuitive arguments.

We begin with the density-response function defined by

$$\mathfrak{F}(k,\omega)=-i\int dt\,d^dx\,e^{i\omega t-ik\cdot x}\langle[\rho(x,t),\rho(0)]\rangle\,\theta(t), \tag{2.11}$$

with $\mathrm{Im}\,\omega>0$, and

$$\rho(x,t)=e^{iHt}\rho(x)e^{-iHt}.$$

The speed and damping of sound waves and the rate of heat diffusion are determined by the poles of $\mathfrak{F}(k,\omega)$, analytically continued to below the ω-real axis. The poles of $\mathfrak{F}$ are the zeros of the dielectric function ϵ;

$$\epsilon(k,\omega)=1+\tfrac{1}{2}nu\Pi(k,\omega), \tag{2.12}$$

where Π is related to $\mathfrak{F}$ via $\mathfrak{F}=-\frac{1}{2}n\Pi/\epsilon$. A more convenient form for ϵ will be obtained to replace (2.12) by taking advantage of the conservation law (1.7). This will be done shortly. Thus the nontrivial part of the program is the calculation of ϵ to the leading order in $1/n$, namely to $O(1)$. This would be trivial if the frequency ω or the wave number k can be considered as quantities of $O(1)$. However, as we have mentioned before, the range of interest is of $O(1/n)$. There is a class of graphs which are of higher orders in $1/n$, but contribute to the dielectric-function powers of $(1/n)\omega^{-1}$ or $(1/n)k^{-1}$. Such contributions must not be excluded. Thus we need to sum this infinite set of graphs to obtain $\epsilon(k,\omega)$ to $O(1)$. The summation leads to a kinetic equation in the intermediate stage, and the dielectric function can be obtained from the solution of the kinetic equation. The collision term of the kinetic equation accounts for the scattering of two bosons. For $T<T_c$, another collision term appears to account for the absorption and emission of bosons. The new processes appear as a consequence of the Bose-Einstein condensation. The standard procedure in kinetic theory will be used to obtain the solutions to the kinetic equation.

We would like to emphasize that the kinetic equation is not a complete description. Solving the kinetic equation is only an intermediate step. In particular, the sound-wave solutions of the kinetic equation do not describe the sound waves of the model. This is evident since, for example, the speed of sound given by the kinetic equation is that derived from the thermodynamics of an ideal gas, not that of the Hartree approximation. Nor will the kinetic equation give the second sound below T_c. It is crucial that we substitute the solution of the kinetic equation into the dielectric function and then locate the zeros. This last step will give the correct sound waves, including the second sound.

TABLE I. Thermodynamics at $n\to\infty$ limit. The critical temperature T_c is given by $T_c=4\pi[2N/n\,\zeta(\frac{1}{2}d)]^{2/d}$. The symbols N', N_0, K, E, S, P, and C_v represent the uncondensed density, condensate density, kinetic energy, total energy, entropy, pressure, and specific heat at constant volume, respectively. We have defined $F_m(r)\equiv\sum_{k=1}^{\infty}e^{-\beta rk}/k^m$. Then $F_m(0)=\zeta(m)$ where ζ is Riemann's ζ function. r is defined by $N=\frac{1}{2}n\,(T/4\pi)^{d/2}F_{d/2}(r)$.

Thermodynamic quantity	$T>T_c$	$T<T_c$
N'	$N=$ total density	$\frac{1}{2}n\,(T/4\pi)^{d/2}\zeta(\frac{1}{2}d)$
N_0	0	$N-N'$
K	$T\frac{1}{4}nd\,(T/4\pi)^{d/2}F_{d/2+1}(r)=N\langle p^2\rangle$	$T\frac{1}{4}nd\,(T/4\pi)^{d/2}\zeta(\frac{1}{2}d+1)=N'\langle p^2\rangle$
E	$K+\frac{1}{2}uN^2$	$K+\frac{1}{2}uN^2$
S	$(2/d+1)K/T+Nr/T$	$(2/d+1)K/T$
P	$(2/d)K+\frac{1}{2}uN^2$	$(2/d)K+\frac{1}{2}uN^2$
$\left(\frac{\partial P}{\partial N}\right)_T$	$Nu+\dfrac{2N}{n\Pi_0(0)}$	Nu
$\left(\frac{\partial P}{\partial N}\right)_S$	$\left(\frac{2}{d}+1\right)T\dfrac{F_{d/2+1}(r)}{F_{d/2}(r)}+Nu$	$\left(\frac{2}{d}+1\right)\dfrac{\zeta(\frac{1}{2}d+1)}{\zeta(\frac{1}{2}d)}+uN$
C_v	$(\frac{1}{2}d+1)K/T-d^2N^2[2Tn\Pi_0(0)]^{-1}$	$(\frac{1}{2}d+1)K/T$

C. Conservation of bosons and the dielectric function

The mathematical task is to find the zeros of the dielectric function (2.12) in the ω plane. The function $-\frac{1}{2}n\Pi$ is sometimes called the "polarization part." It is just the density-response function $\mathfrak{F}$ with only "irreducible" graphs included (see Fig. 1). Previous experience showed that, for dynamic calculation, current-response functions are better behaved and easier to calculate than density-response functions.[9] The reason is that singular behavior often comes from the large boson population with small momenta. The current is proportional to the momentum. Therefore the small-momentum bosons would affect the current response less than they would affect the density response. Let us define the longitudinal current-response function as

$$\mathfrak{F}^{jj}(k,\omega) = -i\int dt\, d^dx\, \langle[\hat{k}\cdot j(x,t), \hat{k}\cdot j(0)]\rangle \times \theta(t)e^{i\omega t - ik\cdot x}, \tag{2.13}$$

where $\hat{k}$ is the unit vector along the direction of k. It follows from the continuity equation (1.7) and the commutation rule

$$[\rho(x), j(x')] = 2i\rho(x)\nabla\delta(x-x') \tag{2.14}$$

that

$$\omega^2\mathfrak{F} = k^2(2N + \mathfrak{F}^{jj}). \tag{2.15}$$

This identity also applies to the irreducible part, i.e.,

$$-\omega^2\Pi = k^2(2N - \tfrac{1}{2}n\Pi^{jj})2/n. \tag{2.16}$$

Thus the dielectric function (2.12) is then expressed as

$$\epsilon(k,\omega) = 1 - (uk^2/\omega^2)[2N - \tfrac{1}{2}n\Pi^{jj}(k,\omega)]. \tag{2.17}$$

Thus the nontrivial mathematical task is to determine Π^{jj} to the leading order in $1/n$. Note that (2.17) evidently is not useful for static problems, i.e., for $\omega = 0$. However, since for our purpose of extracting zeros of $\epsilon(k,\omega)$ we are interested in finite ω, this formula is very useful as previous work has shown.[9]

III. SOUND WAVES AND HEAT DIFFUSION ABOVE T_c

In this section we shall determine Π^{jj} to $O(1)$, substitute it in (2.17), and then determine the zeros of $\epsilon(k,\omega)$. The speed and damping of sound waves and the heat diffusion coefficient are then obtained.

If we simply take the noninteracting gas contribution to Π^{jj}, we shall get the so called "random phase approximation" or "time-dependent Hartree approximation" to the dielectric function. This

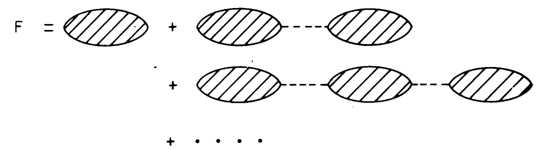

FIG. 1. Full density-response function written as a geometric sum of the "polarization parts" $-\frac{1}{2}n\Pi$. Dashed line denotes the interaction u.

would be good enough for large n if k or ω is of $O(1)$. However, as we pointed out, the range of interest is $\omega, k \sim O(1/n)$. Consequently, an infinite set of graphs must be summed to obtain Π^{jj} to $O(1)$. This summation of graphs is discussed in Appendix A. Here we shall write down the results first and then show that they are easily visualizable and almost self-evident.

A. Kinetic equation

First, let us specify some notations. Define

$$s \equiv s_p = 2\sinh\tfrac{1}{2}\beta\epsilon_p, \tag{3.1}$$

$$(\varphi,\chi) \equiv (2\pi)^{-d}\int d^dp\, s^{-2}\varphi(p)\chi(p), \tag{3.2}$$

where φ and χ are any functions of p. Equation (3.2) defines a "scalar product" in the vector space of functions of p. Note that

$$s^{-2} = (f_p+1)f_p = -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_p}f_p. \tag{3.3}$$

Then the result of summing graphs of leading order is

$$\tfrac{1}{2}n\Pi^{jj} = 2N - \tfrac{1}{2}n\beta\omega(2p\cdot\hat{k}, \varphi), \tag{3.4}$$

where φ is the solution to the integral equation

$$(\omega - k\cdot v)\varphi = 2p\cdot\hat{k} - i\mathfrak{K}\varphi, \tag{3.5}$$

with $v \equiv \nabla_p\epsilon_p = 2p$, and the integral operator $\mathfrak{K}$ is defined by

$$\mathfrak{K}\varphi \equiv \tfrac{1}{2}n(2\pi)^{-3d}\int d^dp'\, d^dp''\, d^dp'''\, s(s's''s''')^{-1} \times R(pp'p''p''')[\varphi(p)+\varphi(p')-\varphi(p'')-\varphi(p''')], \tag{3.6}$$

and R is proportional to the rate of boson-boson scattering,

$$R(pp'p''p''') = (2\pi)^{d+1}\delta(\epsilon+\epsilon'-\epsilon''-\epsilon''') \times \delta(p+p'-p''-p''')|\tilde{u}(p-p'',\epsilon-\epsilon'')|^2. \tag{3.7}$$

Some details of the effective scattering amplitude

$\tilde{u}$ are given by (A10) and (A52) and will not concern us until later. In the above formulas,

$$\epsilon \equiv \epsilon_p, \quad \epsilon' \equiv \epsilon_{p'}, \text{ etc.}, \quad s \equiv s_p, \quad s' \equiv s_{p'}, \text{ etc.} \tag{3.8}$$

Of course, $\varphi(p)$ is a function of k and ω as well as p.

These results can be visualized as those of simple kinetic theory. Let us write the boson distribution function as

$$f(x,p,t) = f_p + \delta f(x,p,t), \tag{3.9}$$

where the deviation δf from the equilibrium distribution is caused by an external field (coupled to the current) which generates a force

$$F = -\nabla V, \quad V = \lambda p \cdot \hat{k} e^{ik\cdot x - i\omega t}. \tag{3.10}$$

We expect f to satisfy a Boltzmann equation

$$\frac{\partial \delta f}{\partial t} + v \cdot \nabla \delta f + F \circ \nabla_p f_p = \left(\frac{\partial f}{\partial t}\right)_c, \tag{3.11}$$

where we keep only terms of first order in the small parameter λ. Now we relate δf to φ by

$$\delta f(x,p,t) = -\lambda \beta s^{-2}(2p\cdot\hat{k} - \omega\varphi) e^{ik\cdot x - i\omega t}. \tag{3.12}$$

Then a little algebra will reduce (3.5) to (3.11). Note that $\nabla_p f_p = -2p\beta s^{-2}$ and $\mathcal{K} p\cdot k = 0$. We have not said anything about $(\partial f/\partial t)_c$ in (3.11). Presumably one can get every detail of (3.6) by kinetic-theory argument through $(\partial f/\partial t)_c$. However, from (3.7) and (3.8), it is already clear that $\mathcal{K}$ describes the effect of collisions. Here we are only interested in the physical picture behind (3.5), not its detailed derivation, which is supplied by Appendix A.

Having linked φ to δf by (3.12), we can easily understand the result (3.4). The longitudinal current generated as a result of the external disturbance (3.10) is

$$\langle j(x)\rangle_t \cdot \hat{k} = \tfrac{1}{2} n (2\pi)^{-d} \int d^d p \, \delta f(x,p,t) 2p\cdot\hat{k}, \tag{3.13}$$

where $\langle\,\rangle_t$ means the expectation value at time t. By (3.12), we have

$$\begin{aligned}\langle j(x)\rangle_t \cdot \hat{k} &= -\lambda \tfrac{1}{2} n\beta[(2p\cdot\hat{k}, 2p\cdot\hat{k}) - \omega(2p\cdot\hat{k}, \varphi)] e^{ik\cdot x - i\omega t} \\ &= -\lambda \tfrac{1}{2} n \Pi^{jj} e^{ik\cdot x - i\omega t},\end{aligned} \tag{3.14}$$

where we have made use of the fact that

$$\tfrac{1}{2} n\beta (2p\cdot\hat{k}, 2p\cdot\hat{k}) = 2N \tag{3.15}$$

and (3.4). Thus $-\frac{1}{2}n\Pi^{jj}$ is the longitudinal current-response function predicted by the Boltzmann equation.

As we have emphasized in Sec. II, the Boltzmann equation is not the whole story. The sound waves we want are not those given directly by the solution of the Boltzmann equation (the poles of Π^{jj}), but by the zeros of the dielectric function.

B. Speed of sound

In order to compute $(2p\cdot\hat{k}, \varphi)$ in (3.4), we need to solve (3.5). The solution is obtained by inverting the operator $\omega - k\cdot v + i\mathcal{K}$, and

$$(2p\cdot\hat{k}, \varphi) = (2p\cdot\hat{k}, (\omega - k\cdot v + i\mathcal{K})^{-1} 2p\cdot\hat{k}). \tag{3.16}$$

Suppose that there is a complete set of orthonormal eigenvectors ϕ_i such that

$$(k\cdot v - i\mathcal{K})\phi_i = \omega_i \phi_i . \tag{3.17}$$

Then (3.16) can be written

$$(2p\cdot\hat{k}, \varphi) = \sum_i \frac{(2p\cdot\hat{k}, \phi_i)^2}{\omega - \omega_i} . \tag{3.18}$$

We shall determine $(2p\cdot\hat{k}, \phi_i)$ and ω_i as power series in k, following the standard steps in kinetic theory.

First, let us set $k=0$. Then (3.17) is just the eigenvector equation for $\mathcal{K}$. We shall use the symbols λ_i, χ_i for the eigenvalues and eigenvectors of $\mathcal{K}$;

$$\mathcal{K}\chi_i = \lambda_i \chi_i . \tag{3.19}$$

The properties of $\mathcal{K}$ are conveniently expressed by the quadratic form

$$\begin{aligned}(\chi, \mathcal{K}\chi) = \tfrac{1}{2} n (2\pi)^{-4d} \int & d^d p \, d^d p' \, d^d p'' \, d^d p''' (s s' s'' s''')^{-1} \\ & \times \tfrac{1}{4} R(pp'p''p''')[\chi(p) + \chi(p') - \chi(p'') - \chi(p''')]^2\end{aligned} \tag{3.20}$$

obtained from (3.6). It is evident that $\mathcal{K}$ is symmetric and positive definite and invariant under rotation and inversion in p space. It is also clear from (3.20) and the δ functions in (3.7) that $\mathcal{K}$ has a $(d+2)$-dimensional "null space" (i.e., the space of eigenvectors of zero eigenvalue) spanned by

$$\chi_0 = A_0^{-1/2}, \tag{3.21}$$

$$\chi_1 = A_1^{-1/2} p_a, \quad a = 1, \ldots, d, \tag{3.22}$$

$$\chi_\epsilon = A_\epsilon^{-1/2}(p^2 - dA_1/A_0), \tag{3.23}$$

where the A's are normalization constants,

$$A_0 = T\Pi_0(0), \tag{3.24}$$

$$A_\epsilon = T(d+2)\langle p^2\rangle N/n - d^2 A_1^2/A_0, \tag{3.25}$$

$$A_1 = TN/n. \tag{3.26}$$

One could write eigenvectors of $\mathcal{K}$ in terms of hyperspherical harmonics in d dimensions, but there is no need to do it for our purpose. As soon as we turn on the $k\cdot v$ term in (3.17), the spherical symmetry is reduced to a cylindrical symmetry around k. We shall only need to discuss eigenvectors which are cylindrically symmetric. Thus, we shall use χ_1 in (3.22) to denote $A_1^{-1/2}p\cdot\hat{k}$.

To find ω_i to first order in k, we use the first-order degenerate perturbation theory. This means diagonalize $k\cdot v$ of (3.17) in the three-dimensional space spanned by χ_0, χ_ϵ and χ_1. The matrix representation of $k\cdot v$ in this space is

$$k\cdot v=\begin{pmatrix}0 & 0 & \mu_0\\ 0 & 0 & \mu_\epsilon\\ \mu_0 & \mu_\epsilon & 0\end{pmatrix}k, \tag{3.27}$$

where

$$\mu_0=(\chi_0, k\cdot v\,\chi_1)=2(A_1/A_0)^{1/2}, \tag{3.28}$$

$$\mu_\epsilon=(\chi_\epsilon, k\cdot v\,\chi_1)=2d^{-1}(A_\epsilon/A_1)^{1/2}. \tag{3.29}$$

The eigenvalues ω_i and eigenvectors ϕ_i are then easily deduced from (3.27);

$$\omega_0=0+O(k^2), \tag{3.30}$$

$$\omega_\pm=\pm c_0 k+O(k^2), \tag{3.31}$$

$$(2p\cdot\hat{k}, \phi_0)=O(k), \tag{3.32}$$

$$(2p\cdot\hat{k}, \phi_\pm)=(2A_1)^{1/2}+O(k), \tag{3.33}$$

where [using (3.24)–(3.26)],

$$c_0^2\equiv\mu_0^2+\mu_\epsilon^2=4(A_1/A_0+A_\epsilon/d^2A_1)$$
$$=(4/d)(2/d+1)\langle p^2\rangle. \tag{3.34}$$

Taking ω as of $O(k)$ in (3.18), the above results enable us to obtain $\beta\omega(2p\cdot\hat{k}, \varphi)$ to $O(1)$. Note that because $(2p\cdot\hat{k}, \phi_i)$ vanishes unless $i=\pm$, (3.18) is simply

$$(2p\cdot\hat{k}, \varphi)=4A_1\,\omega/(\omega^2-c_0^2k^2). \tag{3.35}$$

Substituting this in (3.4) and then (2.17), we get

$$\epsilon=1-2Nuk^2/(\omega^2-c_0^2k^2), \tag{3.36}$$

whose zeros are given by

$$\omega^2=c^2k^2,\quad c^2=2Nu+c_0^2. \tag{3.37}$$

These zeros give the sound-wave speed c. By (3.34), (3.37), and $(\partial P/\partial N)_S$ in Table I, we see that

$$c^2=2(\partial P/\partial N)_S, \tag{3.38}$$

as expected. Note that 2 comes from the fact that the mass of a boson in this model is $\frac{1}{2}$.

C. Damping of sound and heat diffusion

We now examine the $O(k)$ terms in $(2p\cdot\hat{k}, \phi_i)$ and $O(k^2)$ terms in ω_i. Let Q be the projection operator excluding the null space of $\mathcal{K}$ and $P\equiv 1-Q$. We then write $\phi_i=(P+Q)\phi_i$ in (3.17) to obtain

$$(k\cdot v-\omega_i-i\mathcal{K})Q\phi_i=(\omega_i-k\cdot v)P\phi_i, \tag{3.39}$$

from which we can solve $Q\phi_i$ to $O(k)$,

$$Q\phi_i=-iQ(1/\mathcal{K})Qk\cdot vP\phi_i, \tag{3.40}$$

for $\omega_i=O(k)$. If $\omega_i=O(1)$, it would not contribute to the terms of interest. Substitute (3.40) back into (3.39). We obtain an equation for $P\phi_i$,

$$[Pk\cdot v-iPk\cdot vQ(1/\mathcal{K})Qk\cdot v]P\phi_i=\omega_i P\phi_i. \tag{3.41}$$

The operator on the left-hand side of (3.41) has the matrix representation

$$\begin{pmatrix}0 & 0 & \mu_0\\ 0 & -ik\tau_\epsilon & \mu_\epsilon\\ \mu_0 & \mu_\epsilon & -ik\tau_1\end{pmatrix}k, \tag{3.42}$$

where

$$\tau_\epsilon\equiv\sum_i{}'\frac{1}{\lambda_i}(\chi_\epsilon, \hat{k}\cdot v\chi_i)^2, \tag{3.43}$$

$$\tau_1\equiv\sum_i{}'\frac{1}{\lambda_i}(\chi_1, \hat{k}\cdot v\chi_i)^2. \tag{3.44}$$

The sum excludes χ_i in the null space of $\mathcal{K}$. Equation (3.42) is just (3.27) with an $O(k^2)$ correction. The eigenvalues and eigenvectors of (3.42) are easily determined. The dielectric function is then obtained to one more order in k. The zeros give us the heat-diffusion coefficient D_T and the sound-wave damping coefficient Γ as well as the speed of sound. The algebra is straightforward. Let us record a few intermediate steps. Equations (3.30)–(3.33) now become

$$\omega_0=-iD_0k^2+O(k^3), \tag{3.45}$$

$$\omega_\pm=\pm c_0k-i\tau_+k^2+O(k^3), \tag{3.46}$$

$$(2p\cdot\hat{k}, \phi_0)=2A_1^{1/2}(\mu_\epsilon/\mu_0c_0)iD_0k+O(k^2), \tag{3.47}$$

$$(2p\cdot\hat{k}, \phi_\pm)=\pm(2A_1)^{1/2}(1\mp ik\tau_-/2c_0)+O(k^2). \tag{3.48}$$

The new symbols are defined by

$$D_0=\tau_\epsilon\mu_0^2/c_0^2, \tag{3.49}$$

$$\tau_\pm=\tfrac{1}{2}(\tau_1\pm\tau_\epsilon\mu_\epsilon^2/c_0^2). \tag{3.50}$$

The dielectric function is

$$\epsilon(k,\omega)=1-2Nuk^2\left(1+\frac{ik^2(D_0\mu_\epsilon^2/\mu_0^2)}{\omega}\right)\times\frac{1}{\omega^2-c_0^2k^2+2i\tau_+\omega k^2}+2Nu\frac{k^4}{\omega}\frac{D_0^2\mu_\epsilon^2}{\mu_0^2c_0^2}\frac{1}{\omega+iD_0k^2}. \tag{3.51}$$

The zeros are

$$\omega = -iD_T k^2, \quad (3.52)$$

$$\omega_\pm = \pm ck - \tfrac{1}{2} i\Gamma k^2, \quad (3.53)$$

with

$$D_T = \tau_\epsilon (2Nu + \mu_0^2)/c^2 \equiv \kappa/C_P, \quad (3.54)$$

$$\Gamma = \tau_1 + \tau_\epsilon \mu_\epsilon^2/c^2, \quad (3.55)$$

and c^2 is supplied by (3.37). The thermal conductivity κ defined by $D_T = \kappa/C_P$ can be obtained from (3.54) and Table I. We find

$$\kappa = \tau_\epsilon C_v. \quad (3.56)$$

Note that $C_v/C_P = 2(\partial P/\partial N)_T/c^2$ and $\mu_0^2 = 4N/[n\Pi_0(0)]$.

We have thus completed our description of the phenomena of sound propagation and heat diffusion in the large-n limit. We now turn our attention to the temperature dependence near T_c of the various quantities associated with these phenomena. We are mainly interested in the nonanalytic dependence on $T - T_c$. The quantities c^2, μ_0^2, μ_ϵ^2, and C_P are thermodynamic derivatives and their leading temperature dependences are shown in Table II.

There are still two quantities in (3.54) and (3.55) which are not yet calculated, namely τ_1 and τ_ϵ, defined by (3.43) and (3.44). The calculation is very difficult. What we shall do is to make a qualitative estimate of their temperature dependence. It is quite nontrivial even to make such an estimate. As suggested by (3.43) and (3.44), we need to know the spectrum of $\mathcal{K}$ better.

D. Spectrum of $\mathcal{K}$

So far we know that the lowest eigenvalue of $\mathcal{K}$ is zero and we know the null space quite well through the conservation laws. The other eigenvectors χ_r and eigenvalues are "decaying modes" with decay rate λ_r. The important modes are those with small decay rates, as (3.43) and (3.44) indicate.

Let us write $\mathcal{K}$ in the standard form of kinetic theory[10]

$$\mathcal{K}\chi = \nu\chi + K\chi, \quad (3.57)$$

where ν is simply the coefficient of $\varphi(p)$ in (3.6);

$$\nu(p^2) \equiv s\tfrac{1}{2}n(2\pi)^{-3d} \int d^dp'\, d^dp''\, d^dp'''\,(s's''s''')^{-1} \times R(pp'p''p'''), \quad (3.58)$$

which is the mean collision rate of a boson with momentum p. $K\chi$ is the rest and is an integral operator. Note that $\nu(p^2)$ is not an integral operator; it is just a multiplicative factor. The integral defining K is well defined owing to the $(s's''s''')^{-1}$ factor which vanishes exponentially at large momenta. A counting of powers shows that K is also well defined for small momenta. In general, the spectrum of the sum of an integral operator and a multiplicative factor will consist of a continuous spectrum and a discrete spectrum. The continuous spectrum extends over possible values of $\nu(p^2)$.

The minimum of the continuous spectrum is $\nu(0)$, which is proportional to r. This is because $\nu(p^2)$ is proportional to the factor s in front of (3.58);

$$s \approx \beta(p^2 + r) \text{ for small } p, r, \quad (3.59)$$

and the integral in (3.58) approaches a constant for small p and r. Note that $p^2 + r$ is the energy of a boson of momentum p. It is a characteristic feature of a Bose system that the collision rate (the imaginary part of the self energy) is proportional to the energy. This is very different from the case for a classical dilute gas.

For $T \to T_c$, we have $r \to 0$, and therefore the continuous spectrum would start immediately

TABLE II. Critical behavior of quantities derivable from thermodynamics for $n \to \infty$. Const. = constant $O(1)$. It may contain positive integral powers of $(T - T_c)$. c is the speed of sound above T_c, c_1 the speed of first sound below T_c, and c_2 the speed of second sound. $\gamma = 2/(d-2)$, $\alpha = (d-4)/(d-2)$.

Quantity	$T > T_c$	$T < T_c$
r	(const.) $(T - T_c)^\gamma$	0
A_0, μ_0^2	(const.) $(T - T_c)^{-\alpha}$	0
A_ϵ, μ_ϵ^2	(const.) − (const.) $(T - T_c)^{-\alpha}$	const.
$C_{v/n}$, $C_{p/n}$, $\left(\frac{\partial P}{\partial N}\right)_T$	(const.) − (const.) $(T - T_c)^{-\alpha}$	const.
c	const.	
c_1		const.
c_2		(const.) $(T_c - T)^{1/2}$

above zero.

The eigenvector χ_q of an eigenvalue $\nu(q^2)$ in the continuous spectrum is not normalizable. In general χ_q has the form

$$\chi_q(p) \propto \delta(p-q) + \eta_q(p). \tag{3.60}$$

The physical meaning of such a mode is very simple. The δ function describes a stream of bosons of momentum q, and $\eta_q(p)$ describes the distribution of scattered bosons. Of course, the damping rate of the stream is just $\nu(q^2)$, the collision rate. The special feature is that slow streams are long lived near T_c.

Note that, mathematically, (3.57) resembles the Hamiltonian operator in a Schrödinger's equation in momentum representation. The "kinetic energy" corresponds to $\nu(p^2)$ and the "potential" corresponds to K. The energy eigenvectors of the Schrödinger equation consist of scattering states (with eigenvalue = kinetic energy) and bound states (with discrete eigenvalues). The above qualitative conclusions concerning $\mathcal{K}$ are thus quite obvious.

The discrete spectrum can appear only between zero and the minimum of the continuous spectrum, since the continuous spectrum has no upper limit. There may be none, a finite number, or an infinite number of them. We cannot say anything definite without a more detailed study. However, since K is well defined at $r=0$, we expect that there will be no discrete spectrum if r is small enough.

E. Temperature dependence of κ and Γ

We shall estimate the leading temperature dependence of κ and Γ using the qualitative conclusions reached above concerning the spectrum of $\mathcal{K}$. We would like to emphasize that, for computational and perhaps other purposes, the set of eigenvectors χ_i of $\mathcal{K}$ are often not the most convenient basis, especially because the eigenvectors for the continuous spectrum are not normalizable, and very difficult to calculate. It is easier to use the matrix representation of $\mathcal{K}$ over a known convenient discrete basis and then compute the inverse to obtain τ_1 and τ_ϵ.

The essence of the argument below is just power counting. The spectrum of $\mathcal{K}$ will serve only as a vehicle.

Before we begin, let us factor out from (3.43) the nonanalytic temperature dependence of χ_ϵ. Since $(1, v\cdot\hat{k}\chi_i) = (2p\cdot\hat{k}, \chi_i) = 0$, we have

$$(\chi_\epsilon, \hat{k}\cdot v\chi_i)^2 = A_\epsilon^{-1}(p^2, \hat{k}\cdot v\chi_i)^2,$$

where A_ϵ is given by (3.26) and has a nonanalytic term in $T-T_c$. It follows from (3.43) and (3.56) that

$$\tau_\epsilon \mu_\epsilon^2 = (4/d^2A_1)\tau, \tag{3.61}$$

$$\kappa = (n/2T^2)\tau, \tag{3.62}$$

$$\tau \equiv \sum_i{}' \frac{(p^2, \hat{k}\cdot v\chi_i)^2}{\lambda_i}. \tag{3.63}$$

Note that $\mu_\epsilon^2 = 4A_\epsilon/d^2A_1$, $A_\epsilon = (2/n)T^2C_v$.

We proceed to estimate τ. We assume that there is no discrete eigenvalue between zero and $\nu(0)$. Let us also drop $\eta_q(p)$ in (3.60). Then χ_q is simply a δ function. The proportionality constant is determined by the condition

$$\chi = \sum_q \chi_q(\chi_q, \chi). \tag{3.64}$$

We obtain

$$\chi_q(p) = s_q(2\pi)^d\delta(p-q). \tag{3.65}$$

The sum (3.63) becomes

$$\tau = (2\pi)^{-d}\int \frac{d^dq\, s_q^{-2}q^4(2q\cdot\hat{k})^2}{\nu(q)}. \tag{3.66}$$

Since $\nu(q)\sim s_q \sim r+q^2$, we see that the integral (3.66) is well behaved when $r\to 0$. The total power of q is d. Thus the first nonanalytic power of r is $r^{d/2}$, which is smaller than r. We conclude that

$$\tau = (\text{const.}) + O(T-T_c) + O(r). \tag{3.67}$$

Note that $r\sim(T-T_c)^{2/(d-2)} < T-T_c$, since $2<d<4$.

The same argument can be applied to τ_1, given by (3.44). We obtain

$$\tau_1 = A_1^{-1}(2\pi)^{-d}\int \frac{d^dq\, s_q^{-2}(2q\cdot\hat{k})^4}{\nu(q)}. \tag{3.68}$$

Now there are two powers fewer of q in (3.68) than in (3.66). Thus the leading nonanalytic power of r is $r^{d/2-1}$. However, we know that $r^{d/2-1}\propto T-T_c$. Therefore, we have

$$\tau_1 = (\text{const.}) + O(T-T_c) + O(r). \tag{3.69}$$

In view of (3.55), (3.61), and (3.62) we conclude from (3.67) and (3.69) that the thermal conductivity κ and the sound damping constant Γ are nonsingular to $O(T-T_c)$.

As we mentioned, the above argument is essentially power counting. We do not have to use the eigenvectors of $\mathcal{K}$. We can use an orthonormal discrete set. Power counting will show that matrix elements of interest are nonsingular to $O(T-T_c)$.

To sum up, we find that the speed and the damping of the sound wave are nonsingular to $O(T-T_c)$. The thermal diffusion coefficient D_T is finite at T_c but has an infinite upward slope for $4>d>3$;

$$D_T \sim (\text{const.}) + (T-T_c)^{-\alpha}, \quad -\alpha = (4-d)/(d-2) > 0. \tag{3.70}$$

This nonanalytic behavior comes from the C_P in the denominator of (3.54). The thermal conductivity κ is found to be nonsingular to $O(T-T_c)$.

IV. SOUND WAVES BELOW T_c

A. New features in the kinetic equation

The origin of many new features below T_c is the Bose-Einstein condensation. Second sound is one of the most interesting.

We describe the condensate by a nonzero average of $\langle a_{10}\rangle$,

$$\langle a_{10}\rangle = \langle a_{10}^\dagger\rangle = (N_0)^{1/2}. \tag{4.1}$$

In the language of spin vector, this means a nonzero average of the spin in the Reψ_1 direction in spin space. The direction can be unambiguously defined by introducing a small "external field" h, i.e., adding the term

$$\tfrac{1}{2}h(a_{10}+a_{10}^\dagger) \tag{4.2}$$

to the Hamiltonian H. The limit $h\to 0$ can be taken whenever no ambiguity arises.

The density operator ρ_k now has the form

$$\rho_k = (N_0)^{1/2}(a_{1k}+a_{1-k}^\dagger) + \sum_{\sigma,p} a_{\sigma p}^\dagger a_{\sigma p+k}. \tag{4.3}$$

With $(N_0)^{1/2}$ explicitly written, the momentum subscripts of a_σ and $a_\sigma^\dagger$ will be restricted to nonzero values. When (4.3) is substituted in the Hamiltonian (1.2), interaction terms proportional to $N_0 a_1 a_1$, $(N_0)^{1/2} a_1 a_\sigma^\dagger a_\sigma$, etc., appear. The number of bosons of component 1 is no longer conserved. Processes of creation and annihilation of bosons are the major source of new features in dynamics although they play no part in the thermodynamics in the large-n limit. The static results show that $N_0/N \propto (T_c-T)/T_c$ for T close to T_c. We shall take N_0/N as a small parameter.

Again, we shall study the sound waves through the dielectric function (2.17), and the mathematical task is still the calculation of Π^{ij} by summing graphs. Now we have many new graphs owing to the new processes mentioned above. Here we shall appeal to the physical arguments and leave the analysis of graphs to Appendix B.

In Sec. III, we determined Π^{ij} by solving kinetic equation (3.5). This kinetic equation will be modified qualitatively when $T<T_c$. There is a special direction, namely the direction 1 in spin space, and there are non-boson-conserving processes. Therefore, we generalize φ to a pair

$$\varphi = \begin{pmatrix}\varphi_1\\ \varphi_\perp\end{pmatrix}, \tag{4.4}$$

namely, we introduce the component label $\sigma = 1, 2, \ldots, \frac{1}{2}n$. Since $2, 3, \ldots, \frac{1}{2}n$ are equivalent, we use $\perp$. It is just another label in addition to the momentum label p. The scalar product (3.2) is generalized to

$$(\chi,\varphi) = \frac{2}{n}(2\pi)^{-d}\int d^dp\, s^{-2}[\chi_1(p)\varphi_1(p) + (\tfrac{1}{2}n-1)\chi_\perp(p)\varphi_\perp(p)]. \tag{4.5}$$

We write the new kinetic equation as

$$(\omega - k\cdot v)\varphi = 2\hat{k}\cdot p - i\mathcal{K}\varphi, \tag{4.6}$$

where $\mathcal{K}$ can be expressed as a 2×2 matrix. As we learned in the previous section, a convenient way to extract relevant information about $\mathcal{K}$ is to examine the quadratic form $(\chi,\mathcal{K}\chi)$. Let us write

$$(\chi,\mathcal{K}\chi) = (\chi,\mathcal{K}_4\chi) + (\chi,\mathcal{K}_3\chi), \tag{4.7}$$

where $(\chi,\mathcal{K}_4\chi)$ is just the contribution of the process of boson-boson scattering and is adequately given by (3.20) with $\chi_\perp$ replacing χ,

$$(\chi,\mathcal{K}_4\chi) = \tfrac{1}{2}n(2\pi)^{-4d}\int d^dp\, d^dp'\, d^dp''\, d^dp'''\,(ss's''s''')^{-1} \times \tfrac{1}{4}R(pp'p''p''')[\chi_\perp(p)+\chi_\perp(p')-\chi_\perp(p'') - \chi_\perp(p''')]^2. \tag{4.8}$$

χ_1 can be ignored in (4.8) because it will be $O(1/n)$ smaller. R will be given by (3.7) evaluated at $r=0$ plus correction terms of $O(N_0/N)$. The last term $(\chi,\mathcal{K}_3\chi)$ in (4.7) comes from the new processes due to the presence of the condensate. To the first order in the small parameter N_0/N, these are the processes of emission and absorption of component-1 bosons:

$$1p + \perp p' \to \perp p'' + \text{(condensate boson)}, \tag{4.9}$$

$$1p + 1p \to 1p'' + \text{(condensate boson)}. \tag{4.10}$$

Owing to the overwhelming population of the bosons of $\perp$ components, only (4.9) needs to be taken into account.

The following argument allows us to go quite far in determining $(\chi,\mathcal{K}_3\chi)$. The equilibrium population of the component-1 bosons is

$$f_p + N_0(2\pi)^d\delta(p), \tag{4.11}$$

where $f_p = (e^{\beta p^2}-1)^{-1}$ and the δ-function term accounts for the condensate bosons. We can thus regard the emission and absorption processes (4.9) as a scattering process with one of the initial or final bosons belonging to the δ-function portion of (4.11). The s^{-1} factors in $(\chi,\mathcal{K}_4\chi)$ all came from the Bose distribution function f_p. Thus, to obtain $(\chi,\mathcal{K}_3\chi)$, all we have to do is to replace one of the s^{-1} factors by a δ function in $(\chi,\mathcal{K}_4\chi)$ and take into account that the emitted or absorbed boson must be of component 1. One easily obtains

$$(\chi, \mathcal{K}_3\chi) = N_0(2\pi)^{-3d}\int d^dp\, d^dp'\, d^dp''\, (ss's'')^{-1} \times \gamma(pp'p'')[\chi_1(p) + \chi_\perp(p') - \chi_\perp(p'')]^2, \tag{4.12}$$

where $\gamma(pp'p'')$ is proportional to the rate of the process (4.9),

$$\gamma(pp'p'') = (2\pi)^{d+1}\delta(p + p' - p'')\delta(p^2 + p'^2 - p''^2) \times |\tilde{u}(p,p^2)|^2. \tag{4.13}$$

Substituting (4.8) and (4.12) in (4.7), we get $(\chi, \mathcal{K}\chi)$ to the zeroth and first order in N_0/N. We shall not attempt to study the higher orders.

In view of (4.8) and (4.12), $(\chi, \mathcal{K}\chi)$ is non-negative. All eigenvalues of $\mathcal{K}$ are therefore positive except a $(2+d)$-fold degenerate eigenvalue zero, which is a consequence of the conservation of the number of $\perp$-component bosons, energy, and momentum. In other words, $\mathcal{K}$ has a $2+d$-dimensional null space. The three-dimensional subspace in the null space with cylindrical symmetry around k is spanned by the vectors

$$\chi_0 = A_0^{-1/2}\begin{pmatrix}0\\1\end{pmatrix}, \quad \chi_\epsilon = A_\epsilon^{-1/2}\begin{pmatrix}1\\1\end{pmatrix}p^2,$$

$$\chi_v = A_v^{-1/2}\begin{pmatrix}1\\1\end{pmatrix}p\cdot\hat{k}, \tag{4.14}$$

where the normalization factors are given by (3.24)–(3.26) with $r \to 0$;

$$A_\epsilon = T(d+2)\langle p^2\rangle N'/n, \tag{4.15}$$

$$A_v = N'/\beta n, \tag{4.16}$$

$$A_0 \to \infty. \tag{4.17}$$

Note that we use the subscript v instead of 1 to avoid confusion. As a consequence of $A_0 \to \infty$, matrix elements of interest involving χ_0 drop out. We can thus ignore χ_0 from now on. One could keep A_0 finite by keeping a finite h [defined by (4.2)], and let h go to zero later to obtain the same results. The null space of interest is effectively two dimensional.

All other eigenvalues are positive and will be examined further, later.

B. Speed of propagation

From the solution φ of the kinetic equation (4.6), we can obtain

$$\tfrac{1}{2}n\Pi^{jj} = 2N' - \tfrac{1}{2}n\beta\omega(2p\cdot\hat{k}, \varphi), \tag{4.18}$$

from which the dielectric function can be determined. Note that (4.18) is the same as (3.4) except that the first term is not N but N', the total number of bosons excluding those in the condensate. As was done in the previous section, let ϕ_i and ω_i be the eigenvectors and eigenvalues of $k\cdot v - i\mathcal{K}$. Then (4.6) implies that

$$(2p\cdot\hat{k}, \varphi) = (2p\cdot\hat{k}, (\omega - k\cdot v + i\mathcal{K})^{-1}2p\cdot\hat{k}) = \sum_i \frac{(2p\cdot\hat{k}, \phi_i)^2}{\omega - \omega_i}. \tag{4.19}$$

Again our task is to determine ω_i and ϕ_i. Since $2p\cdot\hat{k}$ is a vector in the null space, it is sufficient to know the projection of ϕ_i in the null space. We shall expand ω_i and ϕ_i in powers of k and only those ω_i which vanish as $k \to 0$ are of interest, just as in the case studied in Sec. III.

First, let us keep only the lowest order terms in k, i.e., ϕ_i to $O(1)$ and ω_i to $O(k)$. This means diagonalizing $k\cdot v$ in the two-dimensional space spanned by χ_v and χ_ϵ [see (4.14)–(4.16)]. The matrix representation of $k\cdot v$ is just

$$\begin{pmatrix}0 & c_0\\ c_0 & 0\end{pmatrix}k, \tag{4.20}$$

$$c_0 = (\chi_v, \hat{k}\cdot v\chi_\epsilon) = 2A_\epsilon^{1/2}A_v^{-1/2}/d = 2[(d+2)\langle p^2\rangle]^{1/2}/d. \tag{4.21}$$

The eigenvalues ω_i are $\pm c_0 k$ and the eigenvectors ϕ_i are

$$\phi_\pm = (1/\sqrt{2})(\chi_\epsilon \pm \chi_v). \tag{4.22}$$

Putting these results in (4.19) and in turn in (4.18), we obtain

$$\tfrac{1}{2}n\Pi^{jj} = 2N' - [4\beta\omega^2/(\omega^2 - c_0^2k^2)]A_v\tfrac{1}{2}n. \tag{4.23}$$

The dielectric function is then

$$\epsilon(k,\omega) = 1 - (uk^2/\omega^2)(2N - \tfrac{1}{2}n\Pi^{jj}) = 1 - u[2N_0k^2/\omega^2 + 2N'k^2/(\omega^2 - c_0^2k^2)], \tag{4.24}$$

where we have identified $N - N'$ as N_0, and $n\beta A_v = N'$. The zeros of $\epsilon(k,\omega)$ are thus easily found. They are given by

$$\omega^2/k^2 = c_1^2 = 2Nu + c_0^2 - [2Nu/(2Nu + c_0^2)](N_0/N)c_0^2, \tag{4.25a}$$

$$\omega^2/k^2 = c_2^2 = [2Nu/(2Nu + c_0^2)](N_0/N)c_0^2. \tag{4.25b}$$

As the notation suggests, c_1 and c_2 are to be interpreted as the speed of the first and second sounds, respectively. Terms proportional to higher powers of N_0/N are dropped.

Note that the result (4.25) has been obtained without any reference to the two-fluid model. Mathematically, (4.25) is the result of graph summation. It is interesting to see if it is consistent with the prediction of the two-fluid model,

$$c_1^2 = 2\left(\frac{\partial P}{\partial N}\right)_S, \quad c_2^2 = \frac{2\rho_s TS^2}{\rho_n NC_v}. \tag{4.26}$$

Note that the mass of a boson in our units is $\frac{1}{2}$. From Table I, we find that for our model

$$2(\partial P/\partial N)_S = 2Nu + c_0^2 N'/N, \tag{4.27}$$

$$2ST^2/NC_v = c_0^2. \tag{4.28}$$

The normal fluid density ρ_n can be calculated as the static current-response function to a transverse vector potential. It is easily calculated in the large-n limit to give just N'. Thus $\rho_s = N - N' = N_0$, $\rho_s/\rho_n \approx N_0/N$.

It is clear then that (4.26) is consistent with (4.25) if

$$Nu \gg c_0^2, \tag{4.28a}$$

which essentially says that the potential energy of a boson is much greater than the kinetic energy. This also means that

$$\frac{C_P}{C_V} = \frac{(\partial P/\partial N)_S}{(\partial P/\partial N)_T} = \frac{2Nu + c_0^2 N'/N}{2Nu} \tag{4.29}$$

must be nearly unity, which is an assumption behind (4.26). Thus we conclude that the prediction of the two-fluid model on c_1 and c_2 is consistent with (4.25).

C. Damping of sound waves

The $O(k^2)$ terms in ω_i and $O(k)$ terms in ϕ_i are obtained in the same way as in Sec. III. We define

$$\tau_v = \sum_i{}' \frac{(\chi_v, \hat{k}\cdot v\chi_i)^2}{\lambda_i}, \tag{4.30}$$

$$\tau_\epsilon = \sum_i{}' \frac{(\chi_\epsilon, \hat{k}\cdot v\chi_i)^2}{\lambda_i}, \tag{4.31}$$

$$\tau_\pm \equiv \tfrac{1}{2}(\tau_v \pm \tau_\epsilon), \tag{4.32}$$

where λ_i and χ_i are eigenvalues and eigenvectors of $\mathcal{K}$, and, as before, the sum over i includes only terms with $\lambda_i > 0$. With $O(k^2)$ terms included, (4.20) becomes

$$\begin{pmatrix} -i\tau_\epsilon k^2 & c_0 k \\ c_0 k & -i\tau_v k^2 \end{pmatrix}, \tag{4.33}$$

from which one obtains

$$\omega_\pm = \pm c_0 k - i\tau_+ k^2, \tag{4.34}$$

$$(\phi_\pm, 2p\cdot\hat{k})^2 = 2A_v(1 \mp ik\tau_-/c_0). \tag{4.35}$$

These results are substituted in (4.19) and the dielectric function follows:

$$\epsilon(k,\omega) = 1 - \frac{2N_0 uk^2}{\omega^2} - \frac{2N'uk^2(1 + i\tau_\epsilon k^2/\omega)}{\omega^2 - c_0^2k^2 + 2i\tau_+\omega k^2}. \tag{4.36}$$

The zeros of ϵ are

$$\omega = \pm c_1 k - \tfrac{1}{2} i\Gamma_1 k^2, \quad \omega = \pm c_2 k - \tfrac{1}{2} i\Gamma_2 k^2, \tag{4.37}$$

with c_1, c_2 supplied by (4.25) and

$$\Gamma_{1,2} = (\tau_v + \tau_\epsilon c_0^2/c_{1,2}^2)[1 - (N_0/N')(1 - c_0^2/c_{1,2}^2)]^{-1}. \tag{4.38}$$

D. Temperature dependence of Γ_1 and Γ_2

The temperature dependence of c_1^2 and c_2^2 can be easily deduced from thermodynamic formulas. We shall concentrate on Γ_1 and Γ_2.

We now face the sums (4.30) and (4.31) for τ_v and τ_ϵ. They are as difficult as (3.43) and (3.44). We can estimate the temperature dependence following the arguments in Sec. III A, B. For $T < T_c$, the continuous spectrum of $\mathcal{K}$ starts from immediately above zero. There is some special feature due to $\mathcal{K}_3$ [see (4.12)] which will be discussed first.

Because of $\mathcal{K}_3$, the number of bosons of component 1 is not conserved and the vector

$$\chi_a = A_a^{-1/2}\begin{pmatrix}1\\0\end{pmatrix} \tag{4.39}$$

is no longer in the null space of $\mathcal{K}$. (The normalization factor A_a can be shown to be proportional to $|T - T_c|^{(d-4)/(d-2)}$.) The vectors of (4.14) together with χ_a span the null space of $\mathcal{K}_4$. Since $\mathcal{K}_3 \propto N_0$ and χ_a is an eigenvector of $\mathcal{K}$ when $N_0 = 0$, χ_a must retain much of its discrete character when N_0 is very small although nonzero. We can imagine that when N_0 is turned on, a splitting of the eigenvalue 0 of $\mathcal{K}$ occurs and χ_a is lifted into the continuum. In terms of the Schrödinger-equation analogy given in Sec. III D, χ_a is expected to be very different from the "stream modes" in the continuum. It should be a "sharp resonance."

In view of this qualitative picture of χ_a, we shall separate the contribution of χ_a from the rest of the continuum and write

$$\tau_v = (\chi_v, \hat{k}\cdot v\chi_a)^2/(\chi_a, \mathcal{K}_3\chi_a) + \tau_v' \tag{4.40}$$

and a similar expression for τ_ϵ, and we expect τ_v' and τ_ϵ' to follow the arguments in Sec. III D and III E.

We can also do a calculation of the dielectric function starting from the null space of $\mathcal{K}_4$ [i.e., using (4.14) and χ_a on equal footing]. The end result turns out to be just that given by (4.40).

Note that the normalization constant A_a cancels out in (4.40). We shall ignore A_a.

Setting $\chi_1 = 1$ and $\chi_\perp = 0$ in (4.12), we get

$$(\chi_a, \mathcal{K}_3 \chi_a) = N_0 (2\pi)^{-3d} \times \int d^d p \, d^d p' \, d^d p'' \, (s s' s'')^{-1} \gamma(p p' p''). \tag{4.41}$$

In view of (4.13) and the fact that

$$\tilde{u}(p, p^2) \sim (1/n) p^{4-d}, \quad s^{-1} \sim p^{-2}, \tag{4.42}$$

for small p, it is evident that the integral in (4.41) is logarithmic divergent in small p. A closer analysis shows that there is an effective lower cutoff for p of the order, as one might have expected,

$$\xi'^{-1} = (N_0/N)^{1/(d-2)} \propto (T_c - T)^\nu. \tag{4.43}$$

For our purpose, it is sufficient to know that (4.41), (4.42), and (4.43) give

$$(\chi_a, \mathcal{K}_3 \chi_a) \sim (N_0/n^2)[\ln(N/N_0) + (\text{const.})]. \tag{4.44}$$

It is trivial to verify that

$$(\chi_v, \hat{k} \cdot v \chi_a) = (4/n) A_v^{1/2}, \quad (\chi_\epsilon, \hat{k} \cdot v \chi_a) = 0. \tag{4.45}$$

Combining (4.44) and (4.45), we get from (4.39),

$$\tau_v = \frac{a}{n} \frac{N/N_0}{\ln(bN/N_0)} + \tau'_v, \tag{4.46}$$

$$\tau_\epsilon = \tau'_\epsilon, \tag{4.47}$$

where a and b are constants of $O(1)$.

Having singled out the contribution of χ_a, τ'_v and τ'_ϵ can be examined by power counting. Note that because of the overwhelming population of bosons in the $\perp$ components, τ'_v and τ'_ϵ are dominated by these bosons. The modes of the $\perp$-component bosons behave in a similar way to those described in Sec. III D. The eigenvalues form a continuum starting immediately above zero. The collision rate $\nu(p^2)$ for a $\perp$ boson is

$$\nu(p^2) \propto \beta p^2 (1 + N_0/N) p^{2-d} \times (\text{const.}), \tag{4.48}$$

for small p. It is straightforward to obtain this formula from (4.8) and (4.12) or from (B24). The p^2 term comes from s as in the case $T > T_c$ [see (3.59) with $r = 0$]. The correction term can be qualitatively understood as follows. The effect of N_0 comes in the manner of (4.11). Namely, a power p^{-2} (the f_p for small p) gets a correction $N_0 p^{-d}$ (the δ function scales as p^{-d}). This change of power from -2 to $-d$ is also observed when N_0 enters in other ways involving momentum variables. Equation (4.48) defines the scale ξ' given by (4.43). Following the same power-counting arguments as in Sec. III E, we arrive at the conclusion

$$\tau'_\epsilon, \; \tau'_v \sim (\text{const.}) + O(T_c - T) + O(\xi'^{-2}). \tag{4.49}$$

They are essentially nonsingular.

We now substitute (4.46) and (4.47) in (4.38) to obtain

$$\Gamma_1 = \Gamma'_1 + (1/n)(\text{const.}) \{(T_c - T) \ln[(\text{const.})/(T_c - T)]\}^{-1}, \tag{4.50}$$

with

$$\Gamma'_1 = (\tau_v + \tau_\epsilon c_0^2/c_1^2)[1 + O(N_0/N)]. \tag{4.51}$$

At T_c, Γ'_1 agrees with the sound damping Γ given by (3.55). We also have, from (4.38),

$$\Gamma_2 = \tau_\epsilon [2Nu/(2Nu + c_0^2)][1 + O(N_0/N)]. \tag{4.52}$$

At T_c, $\Gamma_2 = D_T = \kappa/C_P$ [see (3.54)]. It must be noted that the divergence in Γ_1 does not mean that there is any real divergence in the frequency of any mode at T_c. In fact there is not even any discontinuity. The frequencies of various modes change continuously in this model when T changes from above to below T_c. Immediately below T_c, some degeneracy is lifted by the condensate and some splitting of frequencies occurs. However, this splitting, although continuous, vanishes at T_c and cannot be expanded in powers of k immediately below T_c. This fact is the cause of the divergence in Γ_1 because we have expanded in powers of k and kept only up to k^2 terms.

V. DISCUSSION

We shall make further comments on the qualitative aspects of the results, in connection with the mode-mode coupling scheme and the dynamic scaling hypothesis. We shall also mention that it is not straightforward to study the dynamics of the spin amplitude using this model.

A. Mode-mode coupling

We can visualize the structure of the dynamics described above as a linearized mode-mode coupling scheme. Since we have been speaking in terms of bosons, it might be instructive to make a change and use the physical picture of a spin-vector system.

We began with a set of spin-fluctuation modes. They are not damped if they are not coupled. The coupling gives rise to damping. It also creates new modes characterizing the fluctuations of the square of the spin vectors. These are the sound waves, heat diffusion, and higher decaying modes, which we got out of the kinetic equation and dielectric function. All modes interact. The interaction strengths in various cases are determined by the modes themselves. This nonlinear picture is effectively linearized by dropping all but the leading terms in $1/n$. Nontrivial coupling occurs only in a small range ω, $k=O(1/n)$. The modes in this range do not affect those outside this range and couple to themselves linearly. The nonlinear coupling outside the range is done by the Hartree approximation. The collision integral and the dielectric function are built by combining spin modes in a rather complicated nonlinear manner. The result is a machinery for the linear coupling of the modes of final interest here. Evidently, if we carry out the calculation to the next order in $1/n$, we shall need to include nonlinear coupling between modes within the small range.

The mode-mode coupling scheme put forth by Kawasaki, Kadanoff, Swift and others is a closed self-consistent nonlinear scheme. In it the frequencies of the lowest modes determine themselves under the constraints of symmetry, conservation laws, and static limits. On the other hand, the $1/n$ expansion would be an open-ended successive perturbation scheme. In practice both have weaknesses, but should be complementary. The present model, however, seems a bit inconvenient in its present form for an effective $1/n$ expansion.

B. Smoothness of results and the dynamic scaling hypothesis

In this model, the effect of the large spin-amplitude fluctuation near T_c is largely suppressed by the large n. As a result, singular behavior often does not appear in the leading terms. This is true even in statics. For example, in the specific heat below T_c (see Sec. II), the term $(T_c-T)^{-\alpha}$ does not appear in the leading term but in the next order in $1/n$. Since we have kept only the leading terms in our analysis, we expect to miss some singular terms in our results.

In fact most of our results are smooth. Frequencies of modes are continuous at T_c. The thermal diffusion coefficient D_T shows a dip at T_c instead of blowing up. The thermal conductivity, speed of sound, and damping of sound above T_c are all smooth. Below T_c, similar smooth behavior persists except for the divergent term in τ_v which leads to a divergent Γ_1. All our results are consistent with the two-fluid hydrodynamics. In particular, the relationship

$$\Gamma_1 \propto \rho_s^{-1}\Gamma_2 + (\text{const.}) \tag{5.1}$$

is observed in our results.

The interesting question now is to what extent our results follow the pattern set by the extended dynamic-scaling hypothesis.[11] This hypothesis says that, below T_c, the frequency of the second sound should be a function of $\xi' k$ apart from an over-all multiple of a power of k. This means that, for small $\xi' k$, we should have

$$c_2 k[1+\xi' k\times(\text{const.})] = c_2 k + \tfrac{1}{2} i \Gamma_2 k^2, \tag{5.2}$$

namely,

$$\Gamma_2 \propto c_2 \xi'. \tag{5.3}$$

Since $c_2 \propto (T_c-T)^{1/2}$, $\xi' \propto (T_c-T)^{-\nu}$ and $\nu = 1/(d-2)$, then (5.1) and (5.3) imply that

$$\Gamma_1 \propto (T_c-T)^{-d/2(d-2)}, \tag{5.4}$$

$$\Gamma_2 \propto (T_c-T)^{(d-4)/2(d-2)}. \tag{5.5}$$

The hypothesis also says that $\Gamma_2 \propto D_T$. Therefore it says that Γ_2 and D_T should diverge at T_c and Γ_1 diverges more than $(T_c-T)^{-1}$ since $\frac{1}{2}d(d-2)^{-1}>1$. This pattern is not consistent with the smooth behavior given by the leading terms in $1/n$ found above. If the behavior predicted by (5.4) and (5.5) exists, then we are in a situation discussed in Sec. I around (1.1). It is also possible that the hypothesis does not apply to this system in the above manner.

C. Semimacroscopic description and the renormalization group approach

The nonanalytic behavior of physical quantities near T_c are expected to be a result of interactions among long-wavelength modes. In a renormalization-group approach, short-wavelength modes are successively eliminated. That is why a semimacroscopic description is important. Such a description, and hence a convenient formulation of the renormalization group, is not easy for dynamics for the following reason. In such a description, one uses a model of interacting modes of wave vectors $k<\Lambda$, with Λ much less than an inverse microscopic length, and $\Lambda \gg \xi^{-1}$. For statics, the effect of modes of $k>\Lambda$ shows up in coupling parameters in the model Hamiltonian. For dynamics, the effect is much more complicated. In short, the modes of $k>\Lambda$ act as a reservoir of energy, spin, etc., for modes of $k<\Lambda$. They not only cause dissipation, but also combine to form new long-wavelength modes. Sound waves are examples, as our calculations above have shown. Therefore, to study critical behavior of sound waves via a renormalization-group approach, one has to account for the complicated effect of modes with $k>\Lambda$. This is by no means easy. So far the renormalization-group analysis of dynamics has been restricted to models without propagating modes.[12,13] Generalizations to include propagating modes should complement our results here and remove some of the uncertainties.

D. Concluding remarks

We have made calculations for the sound waves and heat diffusion in the many-component Bose system. The solution illustrates some important physical features qualitatively, although it does not allow us to make useful extrapolations to real systems.

To those who are familiar with the static $1/n$ expansion, what turned out in the above analysis might be surprising. One probably expected that a generalization from the static calculation should be straightforward—just include finite frequencies in the same graphs. We have shown that this expectation is false for the particular model and phenomena analyzed above.

Several authors have studied this model recently to derive the dynamic exponent for the order parameter.[2] In view of what we have learned here, more attention should have been paid to the frequency range of $O(1/n)$. It seems that further study would be important.

APPENDIX A

In this appendix, we carry out the summation of the graphs shown in Fig. 2 for $\Pi^{jj}(k,\omega)$ for the case $T > T_c$. We are interested only in the limit of small $1/n$, ω, and k. The ratios $(1/n)\omega^{-1}$ and k/ω are not necessarily small. Thus k and ω are considered $O(1/n)$.

We introduce the vertex function $\Lambda(p,\epsilon;\,k,\omega)$ (see Fig. 3) from which $\Pi^{jj}(k,\omega)$ can be obtained by closing the legs;

$$\Pi^{jj}(k,\omega) = T\sum_{p,\epsilon} 2p\cdot\hat{k}\,\Lambda(p,\epsilon;\,k,\omega). \tag{A1}$$

In (A1), ϵ and ω are integral multiples of $2\pi i T$. The analytic continuation of Λ from these discrete points to the space of two complex variables ω and ϵ has cuts along $\mathrm{Im}\epsilon = 0$, $\mathrm{Im}\omega = 0$, and $\mathrm{Im}(\epsilon+\omega) = 0$. These cuts divide up the space into six regions as shown in Fig. 4. We shall be interested in $\mathrm{Im}\omega > 0$ only, i.e., the regions I, II, and III. The sum over ϵ in (A1) can be converted into a contour integral,

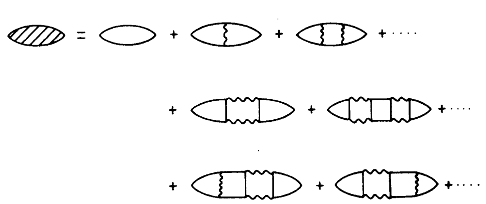

FIG. 2. Longitudinal current-response function Π^{jj}. Each vertex gives a factor $2p\cdot\hat{k}$.

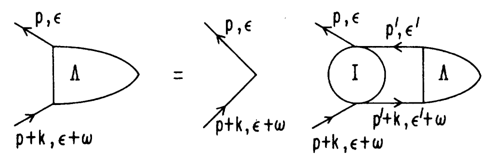

FIG. 3. Bethe-Salpeter equation satisfied by the vertex function Λ.

$$\Pi^{jj}(k,\omega) = (2\pi)^{-d}\int d^dp\,(2p\cdot\hat{k}) \times \int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi i}\{\Lambda_2(p,\epsilon)[f(\epsilon+\omega)-f(\epsilon)] +\Lambda_1(p,\epsilon)f(\epsilon)-\Lambda_3(p,\epsilon)f(\epsilon+\omega)\}, \tag{A2}$$

where the subscripts 1, 2, and 3 stand for the regions I, II, and III, respectively, and $f(\epsilon) = (e^{\beta\epsilon}-1)^{-1}$. The arguments k and ω in Λ are understood. All frequency variables in (A2) are now real.

Figure 3 shows that Λ satisfies an integral equation

$$\Lambda(p,\epsilon;k,\omega) = G(p,\epsilon)G(p+k,\epsilon+\omega) \times\left(2\vec{p}\cdot\hat{k} + T\sum_{p',\epsilon'} I(p,\epsilon,p',\epsilon';k,\omega) \times\Lambda(p',\epsilon';k,\omega)\right), \tag{A3}$$

where I contains no isolated pair of boson lines. The G's in (A3) denote the full Green's function

$$G(p,\epsilon) = \int_0^{\beta} d\tau\, e^{\epsilon\tau}\langle a_{\sigma p}(\tau)a^{\dagger}_{\sigma p}\rangle, \tag{A4}$$

which is independent of σ since $T > T_c$ is assumed. In terms of the self energy $\Sigma(p,\epsilon)$, G^{-1} can be written

$$G^{-1}(p,\epsilon) = G_0^{-1}(p,\epsilon) + \Sigma(p,\epsilon) - \Sigma(0,0), \tag{A5}$$

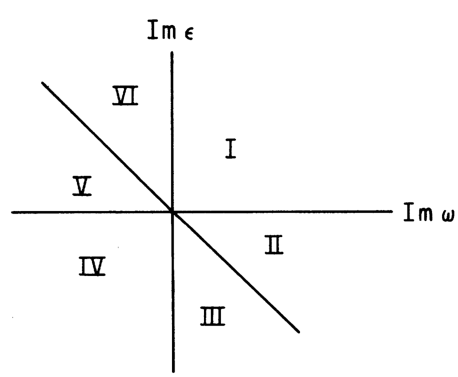

FIG. 4. Regions of analyticity of the vertex function.

$$G_0^{-1}(p,\epsilon)=\epsilon_p-\epsilon,\quad \epsilon_p\equiv p^2+r, \tag{A6}$$

$$r\equiv G^{-1}(0,0). \tag{A7}$$

G_0 is the free-boson propagator. For simplicity, we shall write G for $G(p,\epsilon)$ and

$$G'\equiv G(p+k,\epsilon+\omega),$$

$$\Sigma\equiv\Sigma(p,\epsilon),\quad \Sigma'=\Sigma(p+k,\epsilon+\omega). \tag{A8}$$

Then (A3) is, symbolically,

$$\Lambda=GG'(2p\cdot\hat{k}+I\Lambda). \tag{A9}$$

The lowest order self-energy graph contributing to $\Sigma(p,\epsilon)-\Sigma(0,0)$ is given in Fig. 5(a), which exhausts graphs of $O(1/n)$,

$$\Sigma_c(p,\epsilon)=(2\pi)^{-d}\int d^dq\, T\sum_\nu G_0(p+q,\epsilon+\nu)\tilde{u}(q,\nu). \tag{A10a}$$

The wavy line represents the geometric sum shown in Fig. 5(b),

$$\tilde{u}(q,\nu)=u/[1+\tfrac{1}{2}nu\Pi_0(q,\nu)] \tag{A10b}$$

with each dashed line representing u. The "bubble" gives the function $-\frac{1}{2}n\Pi_0(q,\nu)$ which is the lowest-order term of the polarization part given by

$$\Pi_0(q,\nu)=(2\pi)^{-d}\int d^dp\, T \times\sum_\epsilon G_0(p,\epsilon)G_0(p+q,\epsilon+\nu). \tag{A10c}$$

Since $u=O(1/n)$ and each closed loop implies a factor of $\frac{1}{2}n$, the summation of the diagrams in Fig. 5(b) is sufficient to $O(1/n)$. Thus

$$G^{-1}-G'^{-1}=\omega-2p\cdot k-\Sigma_c'+\Sigma_c+O(n^{-2}), \tag{A11}$$

$$GG'=(G-G')(G'^{-1}-G^{-1})^{-1}. \tag{A12}$$

Regarding k and ω as of $O(1/n)$, then in regions I and III of Fig. 3, $G-G'=O(1/n)$, and therefore

$$GG'=O(1)\ \text{ in I, III}. \tag{A13}$$

However, in region II, $\mathrm{Im}\epsilon$ and $\mathrm{Im}(\omega+\epsilon)$ have opposite signs so that

$$G'-G=2\pi i\delta(\epsilon-\epsilon_p)+O(1/n),$$

$$GG'=\frac{2\pi i\delta(\epsilon-\epsilon_p)}{\omega-2p\cdot k-2i\,\mathrm{Im}\Sigma_c}+O(1). \tag{A14}$$

Thus, in II, $GG'=O(n)$. Since Λ is proportional to GG' according to (A9), Λ_2 is much larger than Λ_1 and Λ_3. It turns out that $I=O(1/n)$. Thus only Λ_2 is needed in evaluating $I\Lambda$ in (A9) for large n. Furthermore, Λ_2 is proportional to $2\pi i\delta(\epsilon-\epsilon_p)$. So we define $\varphi(p)$ by

$$\Lambda_2=2\pi i\delta(\epsilon-\epsilon_p)\varphi(p). \tag{A15}$$

It is understood that $\varphi(p)$ also depends on ω and k. Thus in the large n limit, (A2) gives

$$\Pi^{jj}(k,\omega)=(2\pi)^{-d}\int d^dp\,(2p\cdot\hat{k})\times\left(\varphi(p)\omega\frac{\partial}{\partial\epsilon_p}f(\epsilon_p)+2p\cdot\hat{k}\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi i}f(\epsilon)\times[(\epsilon-\epsilon_p+io)^{-2}-(\epsilon-\epsilon_p-io)^{-2}]\right), \tag{A16}$$

where we have set Λ_1, Λ_3 in (A2) to simply $(2p\cdot\hat{k})G_0^2$. Let us introduce the following useful notation of a scalar product:

$$(\chi,\varphi)\equiv(2\pi)^{-d}\int d^dp\, s^{-2}\chi(p)\varphi(p), \tag{A17}$$

where

$$s^{-2}\equiv-\frac{1}{\beta}\frac{\partial}{\partial\epsilon_p}f(\epsilon_p)=(2\sinh\tfrac{1}{2}\beta\epsilon_p)^{-2}. \tag{A18}$$

Then (A16) takes the simple form

$$\Pi^{jj}(k,\omega)=\beta[(2p\cdot\hat{k},2p\cdot\hat{k})-\omega(2p\cdot\hat{k},\varphi)]. \tag{A19}$$

To get the equation for φ, we examine (A3) for Λ_2. We shall first convert the sum over ϵ' into an integral over real ϵ' and meanwhile continue $I\Lambda$ to real ϵ and ω. There will be contributions from Λ_1, Λ_2, and Λ_3. The analytic continuation of $I(\epsilon,\epsilon';\omega)$ is rather complicated in general, but since we shall keep only Λ_2, we need only consider the case

$$\mathrm{Im}\epsilon,\ \mathrm{Im}\epsilon'<0$$
$$\mathrm{Im}(\epsilon+\omega),\ \mathrm{Im}(\epsilon'+\omega)>0. \tag{A20}$$

Let us carry out the ϵ' sum. Figures 6(a)–6(c) show the contributions to I to $O(1/n)$. Now

$$I=I_a+\tfrac{1}{2}n(I_b+I_c),$$

I_b and I_c are of $O(n^{-2})$, but since an additional closed

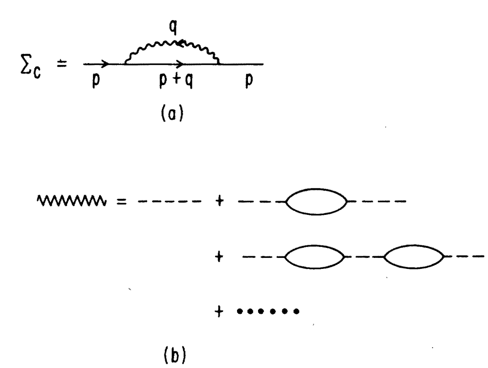

FIG. 5. (a) Self-energy diagram. (b) "Effective" interaction $\tilde{u}(k)$.

loop of solid lines is generated everytime either one appears, a factor $\frac{1}{2}n$ must be included. In the region $\mathrm{Im}\epsilon' < 0, \mathrm{Im}(\epsilon' + \omega) > 0$, of the ϵ' plane, I_a and I_b have a cut along $\mathrm{Im}(\epsilon - \epsilon') = 0$ and I_c has a cut along $\mathrm{Im}(\epsilon + \epsilon' + \omega) = 0$. Let us write

$$I_{a,b} = I_{a,b}(\epsilon - \epsilon'), \quad I_c = I_c(\epsilon + \epsilon' + \omega),$$

and leave all other variables understood. Then we have

$$\begin{aligned} T\sum_{\epsilon'} I_a(\epsilon - \epsilon')\Lambda(\epsilon') &= \int \frac{d\epsilon'}{2\pi i}[f(\epsilon' - \epsilon) - f(\epsilon')]2i\,\mathrm{Im}I_a(\epsilon - \epsilon')\Lambda(\epsilon'), \\ T\sum_{\epsilon'} I_b(\epsilon - \epsilon')\Lambda(\epsilon') &= \int \frac{d\epsilon'}{2\pi i}[f(\epsilon' - \epsilon) - f(\epsilon')]2i\,\mathrm{Im}I_b(\epsilon - \epsilon')\Lambda(\epsilon'), \\ T\sum_{\epsilon'} I_c(\epsilon + \epsilon' + \omega)\Lambda(\epsilon') &= \int \frac{d\epsilon'}{2\pi i}[f(\epsilon' + \epsilon) - f(\epsilon')]2i\,\mathrm{Im}I_c(\epsilon + \epsilon')\Lambda(\epsilon'). \end{aligned} \tag{A21}$$

The contours of integration for performing the above summations over ϵ' in the region $\mathrm{Im}\epsilon' < 0$, $\mathrm{Im}(\epsilon' + \omega) > 0$, of the ϵ' plane are shown in Fig. 7.

We substitute (A15) for Λ in (A21). The energy integrals can be done easily. Utilizing the fact that $I = I_a + \frac{1}{2}n(I_b + I_c)$, we have from (A3) and (A14)

$$\begin{aligned} [\omega + i\nu(p) - k\cdot v]\varphi(p) = 2p\cdot\hat{k} + 2i\int\frac{d^dp'}{(2\pi)^d} &\times\{[f(\epsilon' - \epsilon) - f(\epsilon')] \\ &\times[\mathrm{Im}I_a(\epsilon - \epsilon') + \tfrac{1}{2}n\,\mathrm{Im}I_b(\epsilon - \epsilon')] \\ &+ \tfrac{1}{2}n[f(\epsilon' + \epsilon) - f(\epsilon')] \\ &\times\mathrm{Im}I_c(\epsilon' + \epsilon)\}\varphi(p'), \end{aligned} \tag{A22}$$

where

$$\epsilon' \equiv p'^2 + r, \quad \epsilon \equiv p^2 + r, \quad v = 2p = \partial\epsilon/\partial p, \tag{A23}$$

and we have defined

$$\nu(p) = -2\,\mathrm{Im}\Sigma_c. \tag{A24}$$

Figure 6(a) shows that $I_a(\epsilon - \epsilon') = \tilde{u}(\epsilon - \epsilon')$, where the momentum dependence is still understood. Then from (A10b) we have

$$\mathrm{Im}I_a(\epsilon - \epsilon') = -|\tilde{u}(\epsilon - \epsilon')|^2\tfrac{1}{2}n\,\mathrm{Im}\Pi_0(\epsilon - \epsilon'). \tag{A25}$$

Substituting for G_0 in the expression (A10c) for Π_0 we have

$$\Pi_0(q,\nu) = (2\pi)^{-d}\int d^dp''\,T \times\sum_{\omega''}\frac{1}{(\epsilon_{p''} - \omega'')(\epsilon_{p''+q} - \omega'' - \nu)}, \tag{A26}$$

where $\epsilon_{p''} \equiv p''^2 + r$ and $\epsilon_{p''+q} = (p'' + q)^2 + r$. The frequency sum in (A26) is easily calculated. We obtain

$$\Pi_0(q,\nu) = (2\pi)^{-d}\int d^dp''\,\frac{f(\epsilon_{p''}) - f(\epsilon_{p''+q})}{\epsilon_{p''+q} - \epsilon_{p''} - \nu - io}. \tag{A27}$$

We make the change of variables $q = p - p'$ and $\nu = \epsilon - \epsilon'$ in (A27) and take the imaginary part,

$$\mathrm{Im}\Pi_0(q,\nu) = (2\pi)^{-d}\pi\int d^dp''[f(\epsilon_{p''}) - f(\epsilon_{p''+p-p'})] \times\delta(\epsilon_{p''+p-p'} - \epsilon_{p''} - \epsilon + \epsilon'). \tag{A28}$$

Equation (A28) can be written

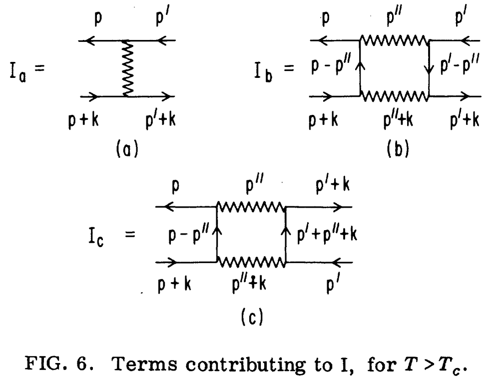

FIG. 6. Terms contributing to I, for $T > T_c$.

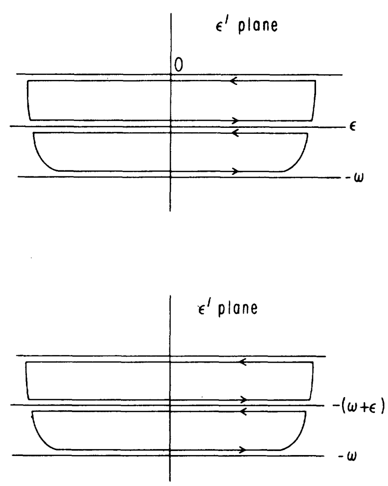

FIG. 7. Contours of integration for performing the summations over ϵ' in the region $\mathrm{Im}\epsilon' < 0$ and $\mathrm{Im}(\epsilon' + \omega) > 0$ of the ϵ' plane [see (A21)].

$$\mathrm{Im}\Pi_0(q,\nu)=(2\pi)^{-d}\pi\int d^dp''\,d^dp'''[f(\epsilon'')-f(\epsilon''')]\times\delta(\epsilon-\epsilon'+\epsilon''-\epsilon''')\delta(p+p''-p'-p''').\tag{A29}$$

The variables ϵ, ϵ'', ϵ', and ϵ''' are defined by (A23). We can now write (A25) in the form

$$\mathrm{Im}I_a(\epsilon-\epsilon')=\tfrac{1}{2}n(2\pi)^{-2d}\int d^dp''\,d^dp'''[f(\epsilon''')-f(\epsilon'')]\times\tfrac{1}{2}R(pp''p'p'''),\tag{A30}$$

where

$$R(pp''p'p''')\equiv(2\pi)^{d+1}\delta(\epsilon+\epsilon''-\epsilon'-\epsilon''')\times\delta(p+p''-p'-p''')|\tilde u(p-p',\epsilon-\epsilon')|^2\tag{A31}$$

and the order in which the p's appear is important.

From the graph of Fig. 6(b) we have

$$I_b(\epsilon-\epsilon')=\int\frac{d^dp''}{(2\pi)^d}T\sum_{\omega''}G(\epsilon-\omega'')G(\epsilon'-\omega'')\times\tilde u(\omega'')\tilde u(\omega''+\omega).\tag{A32}$$

We transform the sum over ω'' to an integral along the real ω'' axis, and analytically continue the G's and $\tilde u$'s. The singularities in the complex ω'' plane are along the cuts $\mathrm{Im}\omega''=0$, $\mathrm{Im}(\epsilon-\omega'')=0$, $\mathrm{Im}(\epsilon'-\omega'')=0$, and $\mathrm{Im}(\omega+\omega'')=0$. Let us assume $\mathrm{Im}\epsilon>\mathrm{Im}\epsilon'$. Then the contour of integration for performing the summation over ω'' is shown in Fig. 8.

We have

$$\begin{aligned}I_b(\epsilon-\epsilon')=(2\pi)^{-d}\int d^dp''\int_{-\infty}^{\infty}\frac{d\omega''}{2\pi i}f(\omega'')[&G(\epsilon-\omega''-io)G(\epsilon'-\omega''-io)2i\,\mathrm{Im}\tilde u(\omega'')\tilde u(\omega+\omega''+io)\\&+2i\,\mathrm{Im}G(-\omega''+io)G(\epsilon'-\epsilon-\omega''+io)\tilde u(\omega''+\epsilon-io)\tilde u(\epsilon+\omega+\omega''+io)\\&+G(\epsilon-\epsilon'-\omega''-io)2i\,\mathrm{Im}G(-\omega''+io)\tilde u(\omega''+\epsilon'-io)\tilde u(\epsilon'+\omega+\omega''+io)\\&+G(\epsilon-\omega''+\omega+io)G(\epsilon'-\omega''+\omega+io)\tilde u(\omega''-\omega-io)2i\,\mathrm{Im}\tilde u(\omega'')].\end{aligned}\tag{A33}$$

By assumption ω,k are of $O(1/n)$. This means that to leading order

$$\tilde u(p+k,\epsilon+\omega)\approx\tilde u(p,\epsilon).\tag{A34}$$

Similar expansions are carried out for all functions of ω and k in (A33). Then the sum of the first and last term in (A33) is real. So it does not contribute to $\mathrm{Im}I_b(\epsilon-\epsilon')$. After making some change of variables, we have

$$\mathrm{Im}I_b(\epsilon-\epsilon')=2\int\frac{d^dp''}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{d\omega''}{2\pi}[f(\omega''-\epsilon')-f(\omega''-\epsilon)]\times\mathrm{Im}G(\epsilon-\omega'')\,\mathrm{Im}G(\epsilon'-\omega'')|\tilde u(p'',\omega'')|^2.\tag{A35}$$

To the order we are considering, we can use the zeroth-order approximation for the Green's function G. We have

$$\mathrm{Im}G(p-p'',\epsilon-\omega''-io)=-\pi\int d^dp'''\times\delta(\epsilon-\omega''-\epsilon''')\delta(p-p''-p''').\tag{A36}$$

Substituting the identity (A36) in (A35) we can perform the ω'' integral easily. After some simple algebra we have

$$\mathrm{Im}I_b(\epsilon-\epsilon')=\tfrac{1}{2}\int\frac{d^dp''\,d^dp'''}{(2\pi)^{2d}}\times[f(\epsilon'')-f(\epsilon''')]R(pp'''p''p'),\tag{A37}$$

where R is defined by (A31). [Note the comment after (A31).]

Next we will consider I_c. Figure 6(c) shows that

$$I_c(\epsilon+\epsilon'+\omega)=\int\frac{d^dp''}{(2\pi)^d}T\sum_{\omega''}G(\epsilon-\omega'')G(\epsilon'+\omega''+\omega)\times\tilde u(\omega'')\tilde u(\omega''+\omega).\tag{A38}$$

We follow the same mathematical steps as were reported for $I_b(\epsilon-\epsilon')$. In this case the cuts in the complex ω'' plane are at $\mathrm{Im}\omega''=0$, $\mathrm{Im}(\epsilon-\omega'')=0$, $\mathrm{Im}(\epsilon'+\omega+\omega'')=0$, and $\mathrm{Im}(\omega+\omega'')=0$. We assume $\mathrm{Im}(\epsilon+\epsilon'+\omega)>0$. The contour of integration is similar to that of Fig. 8, except now the cut at $\mathrm{Im}(\epsilon'-\omega'')=0$ is replaced by the cut at $\mathrm{Im}(\epsilon'+\omega+\omega'')=0$. We have

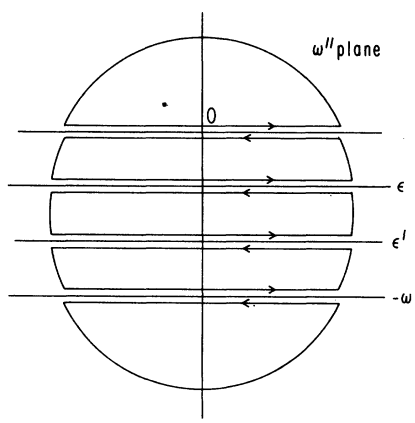

FIG. 8. Contours of integration for evaluating $I_b(\epsilon-\epsilon')$.

$$I_c(\epsilon+\epsilon'+\omega)=(2\pi)^{-d}\int d^dp''\int_{-\infty}^{\infty}\frac{d\omega''}{2\pi i}f(\omega'')[G(\epsilon-\omega''-io)G(\epsilon'+\omega+\omega''+io)2i\,\mathrm{Im}\tilde{u}(\omega'')\tilde{u}(\omega+\omega''+io)$$
$$+2i\,\mathrm{Im}G(-\omega''+io)G(\epsilon'+\omega+\epsilon+\omega''+io)\tilde{u}(\epsilon+\omega''-io)\tilde{u}(\omega+\omega''+\epsilon+io)$$
$$+G(\epsilon+\epsilon'+\omega-\omega''-io)2i\,\mathrm{Im}G(\omega''+io)\tilde{u}(\omega''-\epsilon-\omega-io)\tilde{u}(\omega''-\epsilon'+io)$$
$$+G(\epsilon-\omega''+\omega+io)G(\epsilon'+\omega''-io)\tilde{u}(\omega''-\omega-io)2i\,\mathrm{Im}\tilde{u}(\omega''-io)]\,. \quad \text{(A39)}$$

Taking into account the fact that ω, k is $O(1/n)$, the sum of the first and last term in (A39) is real and therefore does not contribute to $\mathrm{Im}I_c(\epsilon+\epsilon'+\omega)$. We have from the remaining terms

$$\mathrm{Im}I_c(\epsilon+\epsilon')=(2\pi)^{-d}\int d^dp''\int_{-\infty}^{\infty}\frac{d\omega''}{\pi}[1+f(\epsilon-\omega'')+f(\epsilon'+\omega'')]\,\mathrm{Im}G(\epsilon-\omega'')\,\mathrm{Im}G(\epsilon'+\omega'')|\tilde{u}(\omega'')|^2. \quad \text{(A40)}$$

Using (A36) for the Green's functions we finally get

$$\mathrm{Im}I_c(\epsilon+\epsilon')=\tfrac{1}{2}\int\frac{d^dp''\,d^dp'''}{(2\pi)^{2d}}[1+f(\epsilon'')+f(\epsilon''')]R(pp'p''p'''). \quad \text{(A41)}$$

Substituting (A30), (A37), and (A41) into (A22) we have

$$[\omega+i\nu(p)-k\cdot v]\varphi(p)=2p\cdot\hat{k}+\tfrac{1}{2}ni(2\pi)^{-3d}\int d^dp'\,d^dp''\,d^dp'''[f(\epsilon'-\epsilon)-f(\epsilon')][f(\epsilon''')-f(\epsilon'')]R(pp''p'p''')\varphi(p')+\tfrac{1}{2}ni(2\pi)^{-3d}$$
$$\times\int d^dp'\,d^dp''\,d^dp'''[f(\epsilon'-\epsilon)-f(\epsilon')][f(\epsilon'')-f(\epsilon''')]R(pp'''p''p')\varphi(p')+\tfrac{1}{2}ni(2\pi)^{-3d}$$
$$\times\int d^dp'\,d^dp''\,d^dp'''[f(\epsilon'+\epsilon)-f(\epsilon')][1+f(\epsilon'')+f(\epsilon''')]R(pp'p''p''')\varphi(p'). \quad \text{(A42)}$$

We now make a change of variables $p'\to p''$ in the first term, $p'\to p'''$ in the second term, of (A42). Using the symmetry of the integrand, the δ functions in R, and the identity

$$e^{\beta\epsilon}=[1+f(\epsilon)]/f(\epsilon),$$

We get, after some algebra, the kinetic equation

$$[\omega+i\nu(p)-k\cdot v]\varphi(p)=2p\cdot\hat{k}-i(2\pi)^{-3d}\tfrac{1}{2}n\int d^dp'\,d^dp''\,d^dp'''$$
$$\times s(s's''s''')^{-1}R(pp'p''p''')$$
$$\times[\varphi(p')-\varphi(p'')-\varphi(p''')]. \quad \text{(A43)}$$

Of course s, s', s'' and s''' are defined by (A18) with ϵ_p replaced by ϵ, ϵ', ϵ'' and ϵ''', respectively. We have also used the identity

$$f(\epsilon')+f(\epsilon')f(\epsilon'')+f(\epsilon'')f(\epsilon''')-f(\epsilon'')f(\epsilon''')=s(s's''s''')^{-1}$$
$$\text{if } \epsilon+\epsilon'=\epsilon''+\epsilon'''. \quad \text{(A44)}$$

To find an expression for $\nu(p,\epsilon)$ we consider (A10a),

$$\Sigma_c(p,\epsilon)=(2\pi)^{-d}\int d^dq\,T\sum_{\nu}G_0(p+q,\epsilon+\nu)\tilde{u}(q,\nu).$$

We transform the sum over ν to an integral along the real ν axis and analytically continue G_0 and $\tilde{u}$. The singularities in the complex ν plane are along the cuts $\mathrm{Im}\nu=0$ and $\mathrm{Im}(\epsilon+\nu)=0$. Noting that $\mathrm{Im}\epsilon<0$ in region II, and performing the contour integration we get

$$\Sigma_c(p,\epsilon)=(2\pi)^{-d}\int d^dq\int_{-\infty}^{\infty}\frac{d\nu}{2\pi i}f(\nu)$$
$$\times[G_0(p+q,\epsilon+\nu-io)2i\,\mathrm{Im}\tilde{u}(q,\nu+io)$$
$$+2i\,\mathrm{Im}G_0(p+q,\nu+io)\tilde{u}(q,\nu-\epsilon+io)]. \quad \text{(A45)}$$

Taking the imaginary part of Σ_c, the frequency integral can easily be performed using

$$\mathrm{Im}G(p+q,\epsilon+\nu+io)=\pi\delta(\epsilon_{p+q}-\epsilon-\nu). \quad \text{(A46)}$$

After some change of variables and use of the property $\tilde{u}(p)=\tilde{u}(-p)$ we have

$$\mathrm{Im}\Sigma_c(p,\epsilon)=-(2\pi)^{-d}\int d^dp'[f(\epsilon'-\epsilon)-f(\epsilon')]$$
$$\times\mathrm{Im}\tilde{u}(p-p',\epsilon-\epsilon'), \quad \text{(A47)}$$

where ϵ and ϵ' are defined in (A23). We substitute (A30) for $\mathrm{Im}\tilde{u}(p-p',\epsilon-\epsilon')$ in (A47) and use definition (A24) to get

$$\nu(p,\epsilon)=(2\pi)^{-3d}\tfrac{1}{2}n\int d^dp'\,d^dp''\,d^dp'''[f(\epsilon'-\epsilon)-f(\epsilon')]$$
$$\times[f(\epsilon''')-f(\epsilon'')]R(pp''p'p'''). \quad \text{(A48)}$$

Making the transformation $p'\to p''$ in (A48) and using the identity (A44) we have

$$\nu(p,\epsilon)=(2\pi)^{-3d}\tfrac{1}{2}n\int d^dp'\,d^dp''\,d^dp'''\,s(s's''s''')^{-1}$$
$$\times R(pp'p''p'''). \quad \text{(A49)}$$

We finally substitute (A49) in the kinetic equation (A43) and get

$$(\omega - k\cdot v)\,\varphi(p) = 2p\cdot\hat{k} - i\,\mathcal{K}(\varphi), \tag{A50}$$

where $\mathcal{K}$ is a real symmetric integral operator,

$$\mathcal{K}(\varphi) = (2\pi)^{-3d}\tfrac{1}{2}n\int d^dp'\,d^dp''\,d^dp'''\,s(s's''s''')^{-1} \times R(pp'p''p''')[\varphi(p)+\varphi(p')-\varphi(p'')-\varphi(p''')]. \tag{A51}$$

This is the collision operator with the collision rate R given by (A31). It can be shown from (A10) that, for $r=0$ and small q and ν,

$$\Pi_0(q,\nu)\sim q^{d-4}g(\nu/q^2),\qquad \tilde{u}(q,\nu)\sim(2/n)q^{4-d}g^{-1}(\nu/q^2), \tag{A52}$$

where $g(y)$ is a nonzero function except for $y\to\infty$. Thus the effective scattering amplitude $\tilde{u}$ vanishes for forward scattering ($q,\nu\to 0$). If q and ν are not small or $\nu/q^2\to\infty$, $\tilde{u}(q,\nu)\sim u$. In short, $\tilde{u}$ is always finite and suppressed when $q\to 0$. $\tilde{u}$ is effectively a short-range interaction at T_c and causes no divergence.

APPENDIX B

In this appendix we examine the effect of the condensate, when $T<T_c$, on the results obtained in Appendix A. Some of the details omitted in Sec. IV will be supplied here.

The presence of the condensate breaks the rotation symmetry in the spin space. We apply an "external field" h in the $\mathrm{Re}\,a_1$ direction as shown by (4.2). Thus, the component 1 is special. The Green's function G_1 is different from $G_2=G_3=\cdots=G_{n/2}\equiv G_\perp$. We define

$$\lim_{p\to 0}G_1^{-1}(p,0)=r_1, \tag{B1}$$

$$\lim_{p\to 0}G_\perp^{-1}(p,0)=r_\perp. \tag{B2}$$

They are the inverse of magnetic susceptibilities parallel and perpendicular to the direction of h, respectively,

$$r_1=\partial h/\partial\sqrt{N_0}\,, \tag{B3}$$

$$r_\perp=h/\sqrt{N_0}\,. \tag{B4}$$

Equation (B4) follows from the fact that $r_\perp=\Delta h/\Delta\langle\psi\rangle$, with $\Delta h\perp h$, and that Δh and $\Delta\langle\psi\rangle$ can be accounted for by an infinitesimal rotation $\Delta\theta$ of h and $\langle\psi\rangle$; $\Delta h=h\Delta\theta,\ \Delta\langle\psi\rangle=\langle\psi\rangle\Delta\theta$. For the unperturbed Green's functions, we write

$$G_{01}^{-1}=r_1+p^2-\epsilon,\quad G_{0\perp}^{-1}=r_\perp+p^2-\epsilon. \tag{B5}$$

In the limit $h\to 0$, both r_1 and $r_\perp$ vanish, thus G_{01} and $G_{0\perp}$ become the same. We can simply set $r_1=r_\perp=0$ unless there is ambiguity.

The presence of the condensate modifies the density operator ρ_k [see (4.3)] and generates more terms in the Hamiltonian H. These additional terms are shown in Fig. 9. The current operator j_k is also modified,

$$j_k=\sqrt{N_0}\,k(a_{1k}-a_{1-k}^\dagger)+\sum_{\sigma,p}a_{\sigma p}^\dagger a_{\sigma p+k}(2p+k). \tag{B6}$$

The quantity N_0 is $O(n)$ and is supplied by the static analysis. For T close to T_c, $N_0/N\propto T_c-T$ is a small parameter. The effect of the condensate to the lowest order in N_0/N on the results of Appendix A can be summarized as follows:

(a) A correction

$$-\tilde{u}^2\,2N_0k^2/(\omega^2-k^4) \tag{B7}$$

to $\tilde{u}(k)$ (see Fig. 10). This will modify all $|\tilde{u}|^2$ in the kinetic equation. In the derivation of the kinetic equation, $|\tilde{u}|^2$ appeared as a result of $\mathrm{Im}\tilde{u}=-|\tilde{u}|^2\tfrac{1}{2}n\,\mathrm{Im}\Pi_0$ [see (A25)] as well as directly as $|\tilde{u}|^2$ [see (A29)–(A37)]. Thus the correction (B7) implies a term

$$\mathrm{Im}[2N_0k^2/(\omega^2-k^4)]=-N_0\pi[\delta(\omega-k^2)-\delta(\omega+k^2)] \tag{B8}$$

added to $\tfrac{1}{2}n\,\mathrm{Im}\Pi_0$ as well as a direct correction to the $|\tilde{u}|^2$ that finally appears in the kinetic equation.

(b) An additional self-energy term to G_1 (see Fig. 11). This term Σ_1 is of $O(1)$, not $O(1/n)$, although it is small because it is proportional to N_0/N,

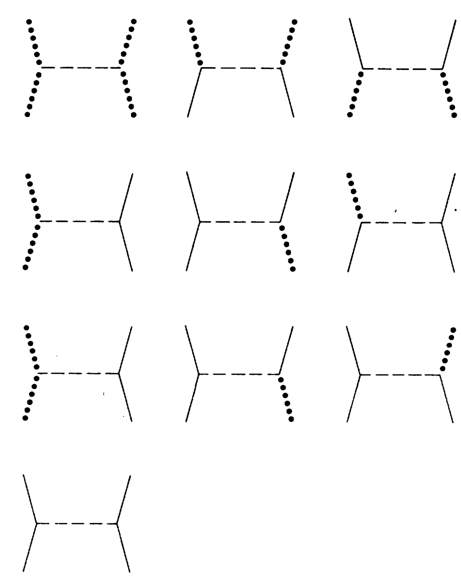

FIG. 9. Additional interaction terms for $T<T_c$. Dashed line, bare interaction u; dotted lines, factor $\sqrt{N}_0$ for the condensate particles.

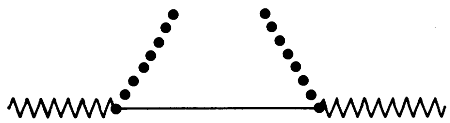

FIG. 10. Correction to $\tilde{u}(k)$. Solid line, G_{01}; dotted line, factor $\sqrt{N_0}$; wavy line, as defined in Fig. 5(b).

$$\Sigma_1(p,\epsilon) = uN_0/[1+u\tfrac{1}{2}n\Pi_0(p,\epsilon)]. \tag{B9}$$

The imaginary part of Σ_1 will account for the rate of absorption and decay. The real part gives a correction to the energy of a boson of component 1. For very small p, the p^2 in G_1^{-1} will be overshadowed by $\mathrm{Re}\Sigma_1$ since

$$\mathrm{Re}\Sigma_1 \propto p^{4-d} N_0/N. \tag{B10}$$

However, for our discussion of sound waves to the first order in N_0/N, $\mathrm{Re}\Sigma_1$ will play no role, only the imaginary part of Σ_1 does.

(c) Additional graphs for I. These graphs are shown in Fig. 12. Together with (B8), these graphs account for the absorption and decay processes.

(d) We need to use the component label σ for Λ and φ just as we do for G. Instead of one kinetic equation, we shall have two coupled equations.

(e) Finally, there are graphs which can be broken into two pieces by cutting one solid line (see Fig. 13). However, these graphs for Π^{ij} turn out to be negligible. They are $O(1/n)$ smaller.

Now it is a matter of repeating the procedures of Appendix A with the above five modifications taken into account. The results are conveniently summarized as follows. We use the additional label σ so that we have $\varphi_\sigma(p)$, $\epsilon_{\sigma p}$, $v_\sigma = \partial\epsilon_{\sigma p}/\partial p$, and $s_\sigma(p) = 2\sinh\frac{1}{2}\beta\epsilon_{\sigma p}$. For $h\to 0$, the label σ is not needed on ϵ_p, v, and s for the results we are interested in here, but we keep the label so that the reader can make generalizations more easily. Furthermore, if $\sigma=1$, we add a term to the inverse of s_1,

$$s_\sigma^{-1}(p) = (2\sinh\tfrac{1}{2}\beta\epsilon_{\sigma p})^{-1} + N_0(2\pi)^d\delta(p)\delta_{\sigma 1}. \tag{B11}$$

We also set

$$[\varphi_1(p)]_{p=0} = 0. \tag{B12}$$

Then we can write a set of kinetic equations as

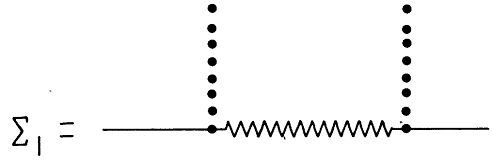

FIG. 11. Additional self-energy term $\Sigma_1(p,\epsilon)$ which is nonzero below T_c.

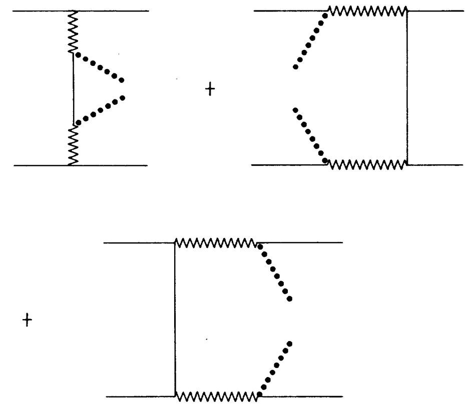

FIG. 12. Additional graphs for I as a result of the condensate below T_c.

the result of graph summation,

$$(\omega - k\cdot v)\varphi = 2p\cdot\hat{k} - i\mathcal{K}\varphi, \tag{B13}$$

where φ is a column matrix of $\frac{1}{2}n$ entries, $k\cdot v$ is a diagonal matrix with $k\cdot v_\sigma$ on the diagonal, and

$$\mathcal{K}\varphi = s\sum_{\sigma'\sigma''\sigma'''=1}^{n/2}(2\pi)^{-3d}\int d^dp'\,d^dp''\,d^dp'''\,(s's''s''')^{-1} \times R(pp'p''p''')(\varphi+\varphi'-\varphi''-\varphi''')\delta_{\sigma\sigma''}\delta_{\sigma'\sigma'''}, \tag{B14}$$

where R is the same as (A31), and

$$\varphi\equiv\varphi_\sigma(p),\quad \varphi'\equiv\varphi_{\sigma'}(p'),\quad \text{etc.},$$
$$s\equiv s_\sigma(p),\quad s'\equiv s_{\sigma'}(p'),\quad \text{etc.} \tag{B15}$$

We can generalize the scalar product (4.5) to

$$(\chi,\varphi) = \frac{2}{n}\sum_{\sigma=1}^{n/2}(2\pi)^{-d}\int d^dp\, s_\sigma^{-2}\chi_\sigma(p)\varphi_\sigma(p). \tag{B16}$$

Then

$$\Pi^{ij} = \beta[(2p\cdot\hat{k}, 2p\cdot\hat{k}) - \omega(2p\cdot\hat{k},\varphi)]. \tag{B16}$$

One easily shows that

$$\beta(2p\cdot\hat{k}, 2p\cdot\hat{k}) = \frac{2}{n}\sum_{\sigma=1}^{n/2}(2\pi)^{-d}\int d^dp\, f_{\sigma p} = (2/n)N'. \tag{B17}$$

The quadratic form $(\chi,\mathcal{K}\chi)$ now becomes

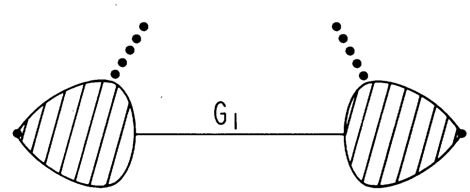

FIG. 13. Nonproper graphs (i.e., graphs which can be broken into two pieces by cutting one solid line). Heavy dots are density (or current) vertices.

$$(\chi, \mathcal{K}\chi) = \frac{2}{n} \sum_{\sigma\sigma'\sigma''\sigma'''} (2\pi)^{-4d} \int (ss's''s''')^{-1} d^d p\, d^d p'\, d^d p''\, d^d p'''\; \tfrac{1}{4} R(pp'p''p''') \delta_{\sigma\sigma''} \delta_{\sigma'\sigma'''} (\chi + \chi' - \chi'' - \chi''')^2 . \tag{B18}$$

This is a form slightly more general than the ones given in Sec. IV. It will still hold if more than one direction becomes preferred, or if $h \neq 0$. It is a form more easily generalizable. Equations (4.7), (4.8), and (4.12) are obtained from (B18) by summing over equivalent components $\sigma = 2, \ldots, \frac{1}{2}n$. Remember that (B11) must be taken into account.

Note added in proof. Recently Halperin has made a general study of multicomponent Bose systems.[14] The special role of modes which are not symmetric in spin space is pointed out. The reader is referred to Halperin's paper for a broader view of the dynamics of this model.

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[1]B. I. Halperin, P. C. Hohenberg, and S. Ma, Phys. Rev. Lett. 29, 1548 (1972).

[2]P. Szepfalusy and I. Kondor, to be published; R. Abe and S. Hikami, to be published.

[3]For discussions on the static spherical model, see H. E. Stanley, *Introduction to Phase Transition and Critical Phenomena* (Oxford U. P., Oxford, 1971), Chap. 8.

[4]There is a very large literature on the static l/n expansion. See, for example, R. Abe, Prog. Theor. Phys. 49, 113 (1973); S. Ma, Phys. Rev. A 7, 2172 (1973); E. Brezin and D. J. Wallace, Phys. Rev. B 7, 1969 (1973).

[5]K. Kawasaki, Phys. Rev. 148, 375 (1966).

[6]L. P. Kadanoff and J. Swift, Phys. Rev. 166, 89 (1968).

[7]B. I. Halperin and P. C. Hohenberg, Phys. Rev. 177, 952 (1969).

[8]R. A. Ferrell, N. Menyhand, H. Schmit, F. Schwabl, and P. Szepfalusy, Ann. Phys. (N.Y.) 47, 565 (1968).

[9]S. Ma and C. W. Woo, Phys. Rev. 159, 165 (1967); S. Ma, H. Gould, and V. K. Wong, Phys. Rev. A 3, 1453 (1971).

[10]Discussions on mathematical problems involved in kinetic theory can be found in *The Boltzmann Equation*, edited by E. G. D. Cohen and W. Thirring (Springer, New York, 1973).

[11]See, for example, Refs. 1, 5, 8, and Chaps. 13–15 of Ref. 3.

[12]B. I. Halperin, P. C. Hohenberg, and S. Ma, Phys. Rev. B (to be published).

[13]B. I. Halperin, P. C. Hohenberg, and E. D. Siggia, Phys. Rev. Lett. 32, 1289 (1974).

[14]B. I. Halperin, Phys. Rev. to be published.