Nonperturbative approach to the theory of transition probabilities by the stroboscopic method

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The exact differential equations with periodic coefficients characterizing the transition in a two-level quantum system are solved by the stroboscopic method which is a nonperturbative approach. This method is based on an averaging over the fluctuations in the system. The solution is unitary and is valid for large times also, as opposed to the short-time validity of the perturbative solution. It should also be noted that this method gives Rabi's well-known solution to the two-level system at resonance by the so-called transformation to the rotating coordinate frame.

I. INTRODUCTION

Several authors[1-6] have questioned the validity of classical time-dependent perturbation theory in calculating the transition probabilities in quantum systems. The main defect of the perturbative methods lies in the contradiction between the validity of the solution in the neighborhood of a given state and, on the other hand, large departures implied by transitions to a different state.[7] Thus the classical time-dependent perturbation solutions are essentially valid for short times. A serious implication of the conventional time-dependent perturbation calculations is the nonunitarity of the solution and the appearance of singularities in physical attributes such as, e.g., dynamic polarizability[6-9] under a resonant harmonic external field.[10]

The aim of this paper is to present a nonperturbative analytic method for the study of transitions in quantum systems due to resonant harmonic external fields that will be valid throughout the whole time domain. The method utilized is the "stroboscopic method" initiated by Minorsky[11] for the study of the periodic solutions of nonlinear differential equations as well as their stability. The method has been applied to nonlinear lattice waves[12] in lieu of perturbative approaches achieving the same goal.[13] The stroboscopic method essentially consists of smearing out, i.e., averaging over, the high-frequency oscillations in the system, and thus substitutes a nonautonomous system (i.e., time-dependent) by an equivalent autonomous (i.e., time-independent) system.

The equations describing the system here have periodic coefficients; the system executes many oscillations during the time necessary for a transition and thus falls naturally in the realm of the stroboscopic method.

Exact numerical solutions have been obtained for similar systems[1]; these will be reproduced (though details are not given) in order to compare with the approximate solution here.

It is also interesting to note the close relationship between this solution and that of Rabi's well-known formula for transition probabilities[14,15] obtained by the so-called transformation to the rotating coordinate frame. It should also be noted that Rabi's result has been put in a rigorous framework by Salzman by extracting the rapidly oscillating terms[4,5] by a reasoning analogous to the one in the present paper.

II. GENERAL EQUATIONS FOR THE PROBABILITY OF TRANSITIONS

Consider the time-dependent Schrödinger equation

$$H\psi = i\hbar\partial\psi/\partial t\,. \tag{2.1}$$

Let the Hamiltonian be made of two parts:

$$H(\vec{x}, t) = H_0(\vec{x}) + \lambda V(\vec{x}, t)\,, \tag{2.2}$$

where H_0 is a time-independent operator and λV is a small time-dependent operator. λ is a coefficient denoting the various orders of magnitudes and will be taken as 1 in the final results. Note, however, that although $V(\vec{x}, t)$ is thought to be small, its effect in the solution is large, since it is regarded as the agent causing the transition between eigenstates of H_0 that would be stationary in its absence.

Let u_n and E_n denote, respectively, the eigenstates and energies of the unperturbed Hamiltonian and consider an expansion of the wave function in terms of the eigenfunctions $u_n e^{-iE_n t/\hbar}$, where the expansion coefficients evidently depend on the time, i.e.,

$$\psi(\vec{x}, t) = \sum_n a_n(t) u_n e^{-iE_n t/\hbar}\,, \tag{2.3}$$

The substitution of (2.3) into (2.1) gives

$$\sum_n \left(i\hbar \frac{da_n}{dt} + E_n a_n\right) u_n e^{-iE_n t/\hbar} = \sum_n a_n (H_0 + \lambda V) u_n e^{-iE_n t/\hbar} . \tag{2.4}$$

Replacing $H_0 u_n$ by $E_n u_n$, multiplying through on the left by u_k, and integrating over all space, making use of the orthonormality of the u_n's, we obtain

$$i\hbar \frac{da_k}{dt} e^{-iE_k t/\hbar} = \lambda \sum_n \langle k | V | n \rangle e^{-iE_n t/\hbar} . \tag{2.5}$$

Defining the Bohr angular frequency by

$$\omega_{kn} = (E_k - E_n)/\hbar \tag{2.6}$$

and the matrix elements by

$$V_{kn}(t) = \langle k | V | n \rangle , \tag{2.7}$$

Eq. (2.5) yields[4]

$$i\hbar \frac{da_k}{dt} = \lambda \sum_n V_{kn}(t) e^{-i\omega_{kn} t} a_n . \tag{2.8}$$

The set of equations in (2.8) is exactly equivalent to the time-dependent Schrödinger equation, i.e., no approximation has been made so far.

The transition probability is defined as the probability of finding the system in a given state $|k\rangle$ if the initial state as in our case is $|m\rangle$, i.e.,

$$W_{m\to k} = |\langle \psi | u_k \rangle|^2 . \tag{2.9}$$

Utilizing the expansion of ψ as in (2.3), in view of the orthogonality conditions we find

$$W_{m\to k} = |a_k(t)|^2 . \tag{2.10}$$

III. PERTURBATIVE SOLUTION FOR TRANSITION PROBABILITIES IN A TWO-LEVEL SYSTEM

The perturbation calculations can be pursued formally in full generality by employing an infinite-dimensional basis as is given in standard text books.[7] However, we specialize our equations at this stage to a two-level system to gain simplicity in the presentation and for comparison with the stroboscopic solution treated here for the two-level case.

Let u_1 and u_2 denote respectively the eigenfunctions for the initial and final states so that the wave function in (2.3) is

$$\psi = a_1(t) u_1 e^{-iE_1 t/\hbar} + a_2(t) u_2 e^{-iE_2 t/\hbar} . \tag{3.1}$$

In this case the equations corresponding to (2.8) read in matrix form

$$i\hbar \begin{bmatrix} da_1/dt \\ da_2/dt \end{bmatrix} = \lambda \begin{bmatrix} 0 & V_{12}(t) e^{i\omega_{21} t} \\ V_{21}(t) e^{-i\omega_{21} t} & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} , \tag{3.2}$$

where the definition of $\omega_{kn} = (E_k - E_n)/\hbar$ as in (2.6) is employed along with $\omega_{21} = -\omega_{12}$. The diagonal elements are also presumed to be zero since the perturbation is assumed not to couple a state with itself.

An important case in applications is that of harmonic perturbations:

$$V(\vec{x}, t) = v(\vec{x}) \cos(\omega t) . \tag{3.3}$$

Then the matrix elements become

$$V_{12}(t) = V_{21}(t) = \langle u_- | v | u_2 \rangle \cos(\omega t) . \tag{3.4}$$

Defining

$$\mu = \langle u_1 | v | u_2 \rangle \tag{3.5}$$

the governing equations in (3.2) became

$$i\hbar \begin{bmatrix} da_1/dt \\ da_2/dt \end{bmatrix} = \tfrac{1}{2}\lambda\mu \begin{bmatrix} 0 & e^{i(\omega_{21}+\omega)t} + e^{i(\omega_{21}-\omega)t} \\ e^{-i(\omega_{21}+\omega)t} + e^{-i(\omega_{21}-\omega)t} & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} . \tag{3.6}$$

Let us now express the a's as a power series of λ in order to solve the set of Eqs. (2.8) by a perturbative scheme. Thus

$$a_n(t;\lambda) = a_n^{(0)}(t) + \lambda a_n^{(1)}(t) + \lambda^2 a_n^{(2)}(t) + \cdots . \tag{3.7}$$

Substituting (3.7) into (2.8) and equating coefficients of corresponding powers of λ, one obtains

$$\frac{da_1^{(0)}}{dt} = 0 , \quad i\hbar \frac{da_1^{(s+1)}}{dt} = \tfrac{1}{2}\mu (e^{i(\omega_{21}+\omega)t} + e^{i(\omega_{21}-\omega)t}) a_2^{(s)} , \tag{3.8}$$

$$\frac{da_2^{(0)}}{dt} = 0 , \quad i\hbar \frac{da_2^{(s+1)}}{dt} = \tfrac{1}{2}\mu (e^{-i(\omega_{21}+\omega)t} + e^{-i(\omega_{21}-\omega)t}) a_1^{(s)} .$$

These can, in principle, be integrated successively to obtain approximate solutions to any desired order in the perturbation.

The first of Eqs. (3.8) shows that the zero-order coefficients $a_n^{(0)}$ are constant in time. Their values are the initial conditions of the problem. We assume that the system is initially at the definite unperturbed energy state u_1. Thus, we have

$$a_1^{(0)} = 1 , \quad a_2^{(0)} = 0 . \tag{3.9}$$

Integration of the first order equations in (3.8) in view of (3.9) gives

$$a_1^{(1)} = 0 ,$$
$$a_2^{(1)} = \frac{\mu}{2\hbar} \left(\frac{e^{-i(\omega_{21}-\omega)t} - 1}{\omega_{21} - \omega} + \frac{e^{i(\omega_{21}+\omega)t} - 1}{\omega_{21} + \omega} \right) . \tag{3.10}$$

Let us consider the expression for $a_2^{(1)}$ for the frequency of the external perturbation ω approaching ω_{21}. Then the term with $(\omega_{21}-\omega)$ in the denominator becomes the most important term. In this range of ω,

$$a_2^{(1)} \cong \frac{\mu}{2\hbar}\left(\frac{e^{-i(\omega_{21}-\omega)t}-1}{\omega_{21}-\omega}\right). \tag{3.11}$$

With this expression of $a_2^{(1)}$ by (2.10) we find

$$W_{1\to 2} = \frac{\mu^2}{\hbar^2}\frac{\sin^2[\frac{1}{2}(\omega_{21}-\omega)t]}{(\omega_{21}-\omega)^2}. \tag{3.12}$$

At resonance, i.e., for $\omega=\omega_{21}$, $W_{1\to 2}$ becomes

$$W_{1\to 2} = \tfrac{1}{4}(\mu^2/\hbar^2)t^2. \tag{3.13}$$

Here appears the main defect of the time-dependent perturbation theory: For large times, i.e., for times of $O(\hbar/\mu)$, the perturbation scheme cannot be valid since it was assumed that $|a_2^{(1)}| \ll 1$ (mathematical defect). Furthermore probabilities greater than 1 have no physical meaning (physical defect). Also, the operator H being Hermitian, the norm should be conserved; that is, the sum of the probabilities should be unity. It is of course not the only instance where unitarity is destroyed. But in other instances, e.g., time-independent perturbation theory, the error is of $O(\lambda^2)$, which is not as serious as here where it goes to ∞ for large times. Perturbation solutions which are not valid in the whole domain of the independent variable (in this case, time) are denoted as "singular perturbations."[6,16] Another shortcoming of the singular perturbation method is that it cannot provide any measure of the time necessary for the transition.

It should also be remarked that each further order approximation proves to be worse for large t. In fact, a simple integration of (3.8) shows that $a_2^{(2)} = -\mu^2 t^2/4\hbar^2$ and, in general, $a_2^{(k)} = (-i\mu t/2\hbar)^k$; i.e., each step gives a stronger singularity. This behavior is actually typical of singular perturbations.

The same result can also be obtained by first considering the differential equation at resonance. For $\omega=\omega_{21}$, (3.6) becomes

$$i\hbar\begin{bmatrix} da_1/dt \\ da_2/dt \end{bmatrix} = \tfrac{1}{2}\lambda\mu\begin{bmatrix} 0 & 1+e^{2i\omega t} \\ 1+e^{-2i\omega t} & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \tag{3.14}$$

The perturbation solution as in (3.7) in view of the zero-order behavior as in (3.9) gives

$$i\hbar\frac{da_2^{(1)}}{dt} = \tfrac{1}{2}\mu(1+e^{-2i\omega t}). \tag{3.15}$$

By integration we get

$$a_2^{(1)} = -\frac{i}{2}\frac{\mu}{\hbar}\left(t+\frac{1-e^{-2i\omega t}}{2i\omega}\right). \tag{3.16}$$

To obtain the same result as in (3.13) one has to neglect the oscillatory term. Note, however, this is also an unjustifiable attitude since this would be true for $t \gg 1$, where the validity of the perturbative solution is dubious. In any case, if the oscillatory term in (3.16) is dropped, the transition probability becomes the same as in (3.12).

IV. EXACT SOLUTION FOR THE PROBABILITY OF TRANSITION IN A TWO-LEVEL SYSTEM

The system of equations in (3.14) has periodic coefficients. The theory that deals with such differential equations is called the "Floquet theory."[17,18] According to Floquet theory, for a two-unknown system, a solution exists of the type

$$\begin{aligned} a_1 &= c_1 e^{\rho_1 t}P_1(t) + c_2 e^{\rho_2 t}Q_1(t), \\ a_2 &= \alpha_1 c_1 e^{\rho_1 t}P_2(t) + \alpha_2 c_2 e^{\rho_2 t}Q_2(t), \end{aligned} \tag{4.1}$$

where ρ_1, ρ_2 are called the characteristic exponents and P_i, Q_i are periodic functions of time with period $T=\pi/\omega$. Furthermore Poincaré has shown[13,16-18] that the real part of ρ_1 can be chosen as zero and the real part of ρ_2 is given as

$$\mathrm{Re}(\rho_2) = \frac{1}{T}\int_0^T [V_{11}(t)+V_{22}(t)]\,dt. \tag{4.2}$$

In our case $V_{11}(t)=V_{22}(t)=0$, so that we find

$$\mathrm{Re}(\rho_2) = 0. \tag{4.3}$$

Consequently the general behavior of the solution in this case is

$$\begin{aligned} a_1 &= c_1 e^{i\beta_1 t}P_1(t) + c_2 e^{i\beta_2 t}Q_1(t), \\ a_2 &= \alpha_1 c_1 e^{i\beta_1 t}P_2(t) + \alpha_2 c_2 e^{i\beta_2 t}Q_2(t). \end{aligned} \tag{4.4}$$

However, the Floquet theory gives no method to determine the characteristic exponents.

An exact solution to the system of equations in (3.14) can be obtained by numerical integration based on a discretization of the derivative.[19]

$$\frac{da_k(t)}{dt} \simeq \frac{a_k(t+\Delta t)-a_k(t)}{\Delta t}. \tag{4.5}$$

By this scheme, from (3.14) one gets

$$\begin{aligned} a_1(t+\Delta t) &= a_1(t) - \frac{i}{2}\frac{\mu}{\hbar}(1+e^{2i\omega t})a_2(t)\Delta t, \\ a_2(t+\Delta t) &= a_2(t) - \frac{i}{2}\frac{\mu}{\hbar}(1+e^{-2i\omega t})a_1(t)\Delta t. \end{aligned} \tag{4.6}$$

The results of the numerical calculations for $\mu=0.1$, 0.2, 0.5, and 1.0 are given in Fig. 1 along with the perturbative solution. In the numerical calculations the norm of the wave function, $|a_1|^2$

$+|a_2|^2$, has been computed at each step to check the numerical accuracy and satisfactory results (errors within 1%) have been found throughout the computation. In these figures, it is seen that the solution is made of two periodic functions: one with a large amplitude of period $\pi/\mu\omega$ and another with small amplitude of period π/ω. This observation is in accord with the general character of the solution in Eq. (4.4) as predicted by the Floquet theory. The high-frequency oscillations with period π/ω correspond to P_i, Q_i and the small-frequency oscillations with period $\pi/\mu\omega$ correspond to $e^{i\beta t}$.

V. STROBOSCOPIC METHOD

This method has been originated by Minorsky[11] for finding periodic solutions of nonlinear differential equations and to study the stability of solutions of nonlinear differential equations. The main idea in the system is the assumption that there is little change in the system during one period and therefore one can ignore what happens between times 0 and T, T and $2T$, etc. In other words, as in viewing apparatus under stroboscopic light in a physical experiment we do not observe the system continuously but rather at discrete intervals of time T. Then, since the changes occur slowly, one can assume T to vary continuously at the end of the calculations so that we can go back to a continuous description. Of course this solution ignores the fast oscillations that can exist in the intervals of time T. An expose of the method is outlined here and then applied to our system in Sec. VI.

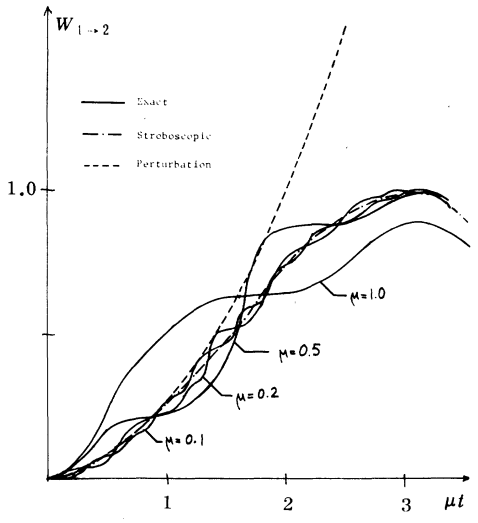

FIG. 1. Comparison of the exact (numerical), singular perturbation, and stroboscopic solutions for $\mu=0.1$, 0.2, 0.5, and 1.0.

Consider the nonautonomous system

$$\frac{dx}{dt}=\mu F(x(t), y(t), t), \quad \frac{dy}{dt}=\mu G(x(t), y(t), t), \tag{5.1}$$

where F and G are periodic in t with period 2π and μ is a small parameter. Integrate the equations in (5.1) from 0 to 2π:

$$x(2\pi)-x(0)=\mu\int_0^{2\pi}F(x(t), y(t), t)\,dt,$$
$$y(2\pi)-y(0)=\mu\int_0^{2\pi}G(x(t), y(t), t)\,dt. \tag{5.2}$$

Since small changes are assumed to occur,

$$\frac{1}{2\pi}\int_0^{2\pi}F(x(t), y(t), t)\,dt\simeq\frac{1}{2\pi}\int_0^{2\pi}F(x(0), y(0), t)\,dt$$
$$=f(x(0), y(0)),$$
$$\frac{1}{2\pi}\int_0^{2\pi}G(x(t), y(t), t)\,dt\simeq\frac{1}{2\pi}\int_0^{2\pi}G(x(0), y(0), t)\,dt$$
$$=g(x(0), y(0)). \tag{5.3}$$

Consider $x(2\pi)-x(0)$ and $y(2\pi)-y(0)$ as increments, i.e.,

$$\Delta x=x(2\pi)-x(0), \quad \Delta y=y(2\pi)-y(0). \tag{5.4}$$

Then (5.2) reads

$$\Delta x=2\pi\mu f(x(0), y(0)),$$
$$\Delta y=2\pi\mu g(x(0), y(0)). \tag{5.5}$$

It is convenient to introduce here the temporal element which we have lost in integration in (5.3) between 0 and 2π by defining as an element of the new "stroboscopic" time τ

$$\Delta\tau=2\pi\mu. \tag{5.6}$$

The transformation equations (5.5) acquire now a more familiar form:

$$\frac{\Delta x}{\Delta\tau}=f(x(0), y(0)), \quad \frac{\Delta y}{\Delta\tau}=g(x(0), y(0)). \tag{5.7}$$

These may be regarded as difference equations by which starting from $x(0)$, $y(0)$ we determine the increments $\Delta x/\Delta\tau$ and $\Delta y/\Delta\tau$ which we add to $x(0)$, $y(0)$ so as to obtain the initial conditions $x(2\pi)$, $y(2\pi)$ for the next interval $(2\pi, 4\pi)$, and so on.

This avoids the cumulative error that might otherwise occur owing to the presence of higher-order terms omitted in the approximation in (5.3) as we let $t\to\infty$. In the physical analogy this process amounts to determining the successive stroboscopic points starting from $x(0)$, $y(0)$.

We may still be guided in our analogy by intro-

ducing a passage to the limit; in fact, in this analogy, the persistence of vision produces the impression of a slow (quasicontinuous) motion instead of a set of discrete points appearing successively after short intervals of 2π. This merely amounts to replacing the set of discrete points in the x, y plane by a continuous curve and by considering (approximately) Δx, Δy, and $\Delta\tau$ in (5.7) as dx, dy, and $d\tau$. It is clear that we can do that if the total duration of the process is long enough compared to one period 2π and, moreover, $\Delta\tau$ is sufficiently small, as is seen in (5.6); the approximation is the better, the smaller is μ. We thus obtain the stroboscopic equations

$$\frac{dx}{d\tau}=f(x,y)\,,\quad \frac{dy}{d\tau}=g(x,y)\,. \tag{5.8}$$

It should be noted by comparing (5.1) with (5.8) that the stroboscopic transformation reduces the nonautonomous system (i.e., with time-dependent coefficients) in (5.1) to the autonomous system (i.e., with constant coefficients) in (5.8). It should also be remarked that in the case where $F(x,y,t)$, $G(x,y,t)$ in (5.1) are linear in x and y, the $f(x,y)$, $g(x,y)$ appearing in the stroboscopic equations in (5.8) become also linear in x and y so that the stroboscopic differential equations can be integrated in closed form.

VI. STROBOSCOPIC SOLUTION FOR THE TRANSITION PROBABILITY IN A TWO-LEVEL SYSTEM

The stroboscopic method described in the previous paragraph is readily applied to our system in (3.14). By the reasoning of Sec. V, the stroboscopic equations corresponding to (3.14) are

$$i\hbar\frac{da_1}{d\tau}=\tfrac{1}{2}a_2\,,\quad i\hbar\frac{da_2}{d\tau}=\tfrac{1}{2}a_1\,, \tag{6.1}$$

where

$$d\tau=2\pi\mu\,. \tag{6.2}$$

Eliminating a_2 between the two equations above, we find

$$\frac{d^2a_1}{d\tau^2}+\frac{1}{4\hbar^2}a_1=0\,, \tag{6.3}$$

which for the initial conditions

$$a_1(0)=1\,,\quad \frac{da_1(0)}{d\tau}=0 \tag{6.4}$$

gives

$$a_1(\tau)=\cos(\tau/2\hbar)\,; \tag{6.5}$$

in a similar manner, we find

$$a_2(\tau)=-i\sin(\tau/2\hbar)\,. \tag{6.6}$$

Substituting now dt for $\Delta t=2\pi$ in the expression (6.2), i.e., replacing τ by μt and letting t vary continuously, we have

$$a_1(t)=\cos(\mu t/2\hbar)\,,\quad a_2(t)=-i\sin(\mu t/2\hbar)\,. \tag{6.7}$$

Then according to (2.10) the transition probability becomes

$$W_{1\to 2}=|a_2(t)|^2=\sin^2(\mu t/2\hbar)\,, \tag{6.8}$$

which is a bounded function. Observe also that the stroboscopic solution is unitary, i.e., conserves the norm of the wave function:

$$|a_1(t)|^2+|a_2(t)|^2=1\,. \tag{6.9}$$

The stroboscopic solution is compared in Fig. 1 with the exact and perturbative solutions. It is seen that the stroboscopic solution ignores the small oscillations, i.e., is a solution averaged over the small fluctuations. Notice also that for small t the stroboscopic method gives the perturbation solution since from (6.8)

$$\lim_{t\to 0}|a_2(t)|^2\to\frac{1}{4}\frac{\mu^2}{\hbar^2}t^2\,. \tag{6.10}$$

At this stage it is worthwhile pointing out that the relationship between the stroboscopic solution and the multiple-time-scale perturbation solution.[2,6,16] The latter is based on a stretching of the time measure as in Ref. 2 and then eliminating the terms causing the singularity. According to this well-known method,[6,16] let us introduce two time measures t and $\tau=\mu t$ and consider

$$a_1=a_1(t,\tau)\,,\quad a_2=a_2(t,\tau)\,. \tag{6.11}$$

then, for

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\frac{\mu\partial}{\partial\tau}\,,\qquad \frac{da_k}{dt}=\frac{\partial a_k}{\partial t}+\mu\frac{\partial a_k}{\partial\tau}\,. \tag{6.12}$$

Considering now a perturbation expansion of a_1 and a_2 as

$$a_k=a_k^{(0)}+\mu a_k^{(1)}+\mu^2a_k^{(2)}+\cdots \tag{6.13}$$

and introducing (6.13) into (3.6), in view of (6.12) we have for the zeroth order

$$\frac{\partial a_1^{(0)}}{\partial t}=0\,,\quad \frac{\partial a_2^{(0)}}{\partial t}=0\,, \tag{6.14}$$

and for the first order

$$\frac{\partial a_1^{(1)}}{\partial t}=-\frac{\partial a_1^{(0)}}{\partial\tau}-\frac{i}{2\hbar}(1+e^{-2i\omega t})a_2^{(0)}\,,\qquad \frac{\partial a_2^{(1)}}{\partial t}=-\frac{\partial a_2^{(0)}}{\partial\tau}-\frac{i}{2\hbar}(1+e^{2i\omega t})a_1^{(0)}\,. \tag{6.15}$$

From the zeroth-order equations, we find

$$a_1^{(0)} = a_1^{(0)}(\tau), \quad a_2^{(0)} = a_2^{(0)}(\tau). \tag{6.16}$$

Thus, substituting (6.16) into (6.15), we get

$$\frac{\partial a_1^{(1)}}{\partial t} = -\left(\frac{da_1^{(0)}(\tau)}{d\tau} + \frac{i}{2\hbar} a_2^{(0)}(\tau)\right) - \frac{i}{2\hbar} e^{-2i\omega t} a_2^{(0)}(\tau),$$
$$\frac{\partial a_2^{(1)}}{\partial t} = -\left(\frac{da_2^{(0)}(\tau)}{d\tau} + \frac{i}{2\hbar} a_1^{(0)}(\tau)\right) - \frac{i}{2\hbar} e^{2i\omega t} a_1^{(0)}(\tau). \tag{6.17}$$

In integrating (6.17) with respect to t, the terms which do not contain t would contribute linear terms in t similar to those in (3.16) according to the conventional singular perturbation theory. Therefore, for obtaining bounded solutions, the secular terms causing the singularity have to be extracted. Thus, we set

$$\frac{da_1^{(0)}(\tau)}{d\tau} + \frac{i}{2\hbar} a_2^{(0)}(\tau) = 0,$$
$$\frac{da_2^{(0)}(\tau)}{d\tau} + \frac{i}{2\hbar} a_1^{(0)}(\tau) = 0. \tag{6.18}$$

These terms are nothing but the stroboscopic equations in (6.1). We therefore recognize the stroboscopic equations as the zeroth-order solution of a multiple-time-scale perturbation method.

VII. CONCLUSIONS

The main conclusions based on Fig. 1 are as follows:

(i) The stroboscopic approximation is seen to be well justified for $\mu = 0.1$ and $\mu = 0.2$ while it is poor for the larger values of μ, 0.5 and 1. This is of course in accord with the stroboscopic assumption.

(ii) The stroboscopic approximation is uniformly valid, i.e., for all times, unlike the perturbative solution.

(iii) The stroboscopic approximation is unitary, i.e., the norm of the wave function is conserved at all times.

(iv) The method can be used to study transient behavior in dynamical systems in quantum mechanics.

(v) It is also observed in (6.8) that the stroboscopic method gives Rabi's well-known solution to the two-level problem at resonance. It should also be interesting to consider the near-resonant case and investigate whether one obtains Rabi's line shape, starting from Eq. (3.6) rather than (3.14).

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