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Creation of circular polarization by a twisted birefringent polarizing medium*

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We consider the passage of initially unpolarized light through a medium possessing both birefringence and differential attenuation with respect to the same pair of perpendicular axes. Circular polarization is produced if the axes undergo rotation as the light path progresses. The twisting rate that maximizes N_2 (the circular Stokes parameter) is found to be an elliptic function with modulus depending on the path length. Various limits are discussed, and the maximized N_2 is tabulated and graphed against path length for several ratios of birefringence to differential attenuation. These results are pertinent to the polarization of hard x rays by a slightly tilted crystal.

I. INTRODUCTION

The usual way of imparting circular polarization to initially unpolarized light makes use of two media through which the light passes successively. The first medium possesses differential absorption with respect to two perpendicular directions of linear polarization. The second medium possesses differential refraction with respect to axes oriented at 45° to those of the first medium. The first medium (polarizer) transforms unpolarized into linearly polarized light. The second medium ($\lambda/4$ plate) transforms linearly into circularly polarized light.

In this paper we shall assume that only one medium is available, possessing differential absorption *and* birefringence with respect to the same pair of perpendicular directions. Mathematically, this means that the complex index of refraction is a 2×2 matrix whose eigenvectors are real and perpendicular, but whose eigenvalues are complex. Our interest in this situation arises from recent work[1-3] on the polarization of high-energy photons by crystals.

With such a medium we can create circular polarization by placing two samples of the medium one after the other, with the second rotated 45° relative to the first. The first acts as a linear polarizer; its birefringence is no embarrassment because the transmitted polarization is also on a principal axis for refraction. The second acts as a quarter-wave plate, but at the same time its absorptive properties reduce the final intensity.

As a result, the intensity of circularly polarized light (to be precise, the Stokes parameter N_2) obtainable in this way is not as great as the intensity of linear polarized light (N_3) obtained from a single untwisted sample of the same *total* length. We shall call the ratio of the former quantity to the latter the "circular conversion efficiency" or just "efficiency."

We may now vary our strategy by using any number of samples of whatever length and orientation we choose. To be utterly general, we may let the orientation of the principal axes change continuously as a function of distance measured along the path of the light. Our purpose is to choose this function so as to maximize the circular Stokes parameter N_2. For a given total path length, this is the same as maximizing the circular conversion efficiency.

In Secs. II–IV we shall set up a formalism to deal with this problem. It has two noteworthy features. First, the density matrix of the light is regarded as a four-dimensional vector $|m\rangle$. Second, a dual vector $\langle p|$ is introduced, which changes in such a way that its inner product with $|m\rangle$ is constant along the path of the light. The development of $|m\rangle$ along the path is then conveniently described in terms of various bilinear forms on $\langle p|$ and $|m\rangle$.

In Sec. V the variational problem is solved. The ordinary Euler-Lagrange method seems inapplicable because the quantity N_2 is not given as an integral and therefore the variational derivatives are not easily found. However, the formalism of the preceding sections makes it possible to show that the final N_2 is stationary with respect to changes in the twisting function, if and only if a certain bilinear form on $\langle p|$ and $|m\rangle$ vanishes everywhere along the path.

In Secs. VI and VII we use the vanishing of this form to derive a differential equation which enables us to express the optimal twisting function in terms of an elliptic function whose modulus is related indirectly to the length of the crystal. [See Eq. (73).] We do not obtain a closed expression for the density matrix at any point. However, in Secs. VIII and IX we study the maximized quantity N_2 by an indirect method and show that it depends on a complete elliptic integral of the third kind. [See Eq. (101).]

The remaining sections are devoted to subsidiary results. In Sec. X we obtain inequalities which show that the solution defined by (73) is really optimal, although an infinite set of other solutions exists which renders N_2 stationary. The limit of short path is discussed in Sec. XI, and that of long path in Sec. XII. Section XIII deals with the choice of "best" path length for a given over-all attenuation rate. In Sec. XIV we develop some numerical tables, and in Sec. XV we summarize our findings.

II. DEFINITIONS AND NOTATION

Let x be the distance from the entrance face to an arbitrary point on the light path.

Let L be the distance from the entrance face to the exit face.

Let the column vectors $\binom{1}{0}$, $\binom{0}{1}$ represent light of unit amplitude polarized along either of the two principal axes. Any combination of polarized and unpolarized light is represented, then, by a density matrix

$$M = M_0\sigma_0 + M_1\sigma_1 + M_2\sigma_2 + M_3\sigma_3, \tag{1}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

We shall have occasion to regard M_0, M_1, M_2, M_3 also as components of a column four-vector,

$$|m\rangle = \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{pmatrix}, \tag{3}$$

and we write the relation between M and $|m\rangle$ contained in (1) and (3) as

$$M \Rightarrow |m\rangle. \tag{4}$$

It will be useful also to define 4×4 matrices Σ_i^L, Σ_i^R whose effect on $|m\rangle$ corresponds to that of multiplying M either on the left or on the right by any of the σ_i. Thus if

$$A \Rightarrow |a\rangle, \tag{5}$$

then

$$\begin{aligned} \sigma_i A &\Rightarrow \Sigma_i^L |a\rangle \quad (i=1,2,3), \\ A\sigma_i &\Rightarrow \Sigma_i^R |a\rangle \quad (i=1,2,3). \end{aligned} \tag{6}$$

From (6) and the σ commutation laws

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \tag{7}$$

we obtain the Σ commutation laws

$$\begin{aligned} [\Sigma_i^L, \Sigma_j^L] &= 2i\epsilon_{ijk}\Sigma_k^L, \\ [\Sigma_i^R, \Sigma_j^R] &= -2i\epsilon_{ijk}\Sigma_k^R, \\ [\Sigma_i^L, \Sigma_j^R] &= 0. \end{aligned} \tag{8}$$

We write the product of the complex index of refraction and the wave number as

$$n + n'\sigma_3 = \begin{pmatrix} n+n' & 0 \\ 0 & n-n' \end{pmatrix} \tag{9}$$

and the angle by which the axes at x (and hence our coordinate system) are rotated from their direction at the entrance face as $\phi(x)$. The rate of twisting is

$$\psi(x) = \frac{d}{dx}\phi(x). \tag{10}$$

We may note here that in the application to high-energy photons, the polarizing power of the crystal is achieved[3] by tilting a principal axis at a very small angle θ from the direction of propagation (the x direction in our treatment). The plane of this tilt is fixed in relation to the crystal axes, and it in turn determines the orientation of the principal (transverse) axes for refraction. Thus, our "twisting angle" ϕ is just the azimuth of this whole system about the direction of propagation.

III. EQUATION OF MOTION

As a result of (9) and (10), the propagation of light in a pure polarization state can be described by $\binom{a(x)}{b(x)}$, where

$$\begin{aligned} \frac{da}{dx} &= i(n+n')a - b\psi, \\ \frac{db}{dx} &= i(n-n')b + a\psi \end{aligned} \tag{11}$$

or

$$\frac{d}{dx}\begin{pmatrix} a \\ b \end{pmatrix} = iH\begin{pmatrix} a \\ b \end{pmatrix}, \tag{12}$$

where

$$H = n\sigma_0 + n'\sigma_3 - \psi\sigma_2. \tag{13}$$

Therefore, the density matrix M develops according to

$$\begin{aligned} \frac{dM}{dx} &= iHM - iMH^\dagger \\ &= -2(\mathrm{Im}\,n)M + in'\sigma_3 M - in'^* M\sigma_3 - i\psi[\sigma_2, M]. \end{aligned} \tag{14}$$

Equation (14) is a direct consequence of (11) when M represents pure polarized light, $M = \begin{pmatrix} aa^* & ab^* \\ ba^* & bb^* \end{pmatrix}$. If the light is not completely polarized, M cannot be written in this form. But we can always write

$M = M_1 + M_2$, where M_1 and M_2 represent two pure polarized components of random relative phase. Then the dependence of M on x is found by letting M_1 and M_2 each satisfy (14) and equating M to their sum at each x. Since (14) is linear in M, this means that M also satisfies (14) whether the light is completely polarized or not.

Using (6), we can write (14) as

$$\frac{d}{dx}|m\rangle = (-\lambda + \alpha - \psi\beta)|m\rangle, \tag{15}$$

where

$$\lambda = 2\,\mathrm{Im}\,n, \tag{16}$$

$$\alpha = in'\Sigma_3^L - in'^*\Sigma_3^R, \tag{17}$$

$$\beta = i\Sigma_2^L - i\Sigma_2^R. \tag{18}$$

As the initial condition for unpolarized light with unit intensity, we set

$$M(0) = \tfrac{1}{2}\sigma_0 \tag{19}$$

so that M and hence $|m\rangle$ are determined for all x by (14) or (15).

It will be convenient to introduce a vector $\langle p|$ in the dual four-dimensional space which develops with x according to

$$\frac{d}{dx}\langle p| = \langle p|(\lambda - \alpha + \psi\beta), \tag{20}$$

with the *final* condition (at $x = L$)

$$\langle p(L)|a\rangle = \mathrm{Tr}\,\sigma_2 A \tag{21}$$

for arbitrary $|a\rangle$ and A satisfying (5).

The circular Stokes parameter of the emergent light is given by

$$N_2 = \mathrm{Tr}\,\sigma_2 M(L) = \langle p(L)|m(L)\rangle \tag{22}$$

on account of (21). But from (15) and (20) we see that $\langle p(x)|m(x)\rangle$, or $\langle p|m\rangle$, is independent of x. Therefore we have

$$N_2 = \langle p|m\rangle = \langle I\rangle. \tag{23}$$

In (23) and henceforth, we use the notation

$$\langle\omega\rangle = \langle p|\omega|m\rangle \tag{24}$$

for any 4×4 matrix ω. In general, $\langle\omega\rangle$ may depend on x through the changing vectors $|m\rangle$ and $\langle p|$. In fact, from (15) and (20) we obtain

$$\frac{d}{dx}\langle\omega\rangle = \langle[\omega, \alpha - \psi\beta]\rangle \tag{25}$$

for any $\langle\omega\rangle$. If ω is the unit matrix I, the commutator vanishes, showing that $\langle I\rangle$ is constant as asserted above.

IV. COMMUTATORS

To make the use of (25) easy, we define

$$\gamma = in'\Sigma_1^L - in'^*\Sigma_1^R, \tag{26}$$

$$\alpha' = n'\Sigma_3^L + n'^*\Sigma_3^R, \tag{27}$$

$$\beta' = \Sigma_2^L + \Sigma_2^R, \tag{28}$$

$$\gamma' = n'\Sigma_1^L + n'^*\Sigma_1^R. \tag{29}$$

Using (8) with (17), (18), and (26)–(29), we obtain

$$[\alpha, \beta] = -[\beta, \alpha] = 2\gamma, \tag{30}$$

$$[\gamma, \alpha] = 2(in'^2\Sigma_2^L - in'^{*2}\Sigma_2^R) = 2[\beta\,\mathrm{Re}(n'^2) - \beta'\,\mathrm{Im}(n'^2)], \tag{31}$$

$$[\gamma, \beta] = -2\alpha, \tag{32}$$

$$[\beta', \alpha] = -2\gamma', \tag{33}$$

$$[\beta', \beta] = 0, \tag{34}$$

$$[\gamma', \alpha] = 2(n'^2\Sigma_2^L + n'^{*2}\Sigma_2^R) = 2[\beta\,\mathrm{Im}(n'^2) + \beta'\,\mathrm{Re}(n'^2)], \tag{35}$$

$$[\gamma', \beta] = -2\alpha', \tag{36}$$

$$[\alpha', \alpha] = 0, \tag{37}$$

$$[\alpha', \beta] = 2\gamma'. \tag{38}$$

From these equations we can determine $[\omega, \alpha - \psi\beta]$, where ω is any combination of $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$. We note that although there are 16 independent 4×4 matrices, these six span a closed Lie subalgebra, which is the direct sum of two three-dimensional Lie algebras as shown by (8).

V. STATIONARY CONDITION

We wish to choose the function $\psi(x)$ so that the functional N_2 will be stationary against small changes in ψ. Suppose that ψ were to be altered only in the interval $x_0 < x < x_0 + \epsilon$, where ϵ is small. Using (15), we have

$$|m(x_0+\epsilon)\rangle = |m(x_0)\rangle + \epsilon(-\lambda + \alpha - \psi\beta)|m(x_0)\rangle + O(\epsilon^2), \tag{39}$$

and since $\langle p|m\rangle$ is x independent, we may substitute $x = x_0 + \epsilon$ in the right side of (23), obtaining

$$N_2 = \langle p(x_0+\epsilon)|m(x_0+\epsilon)\rangle = \langle p(x_0+\epsilon)|1 + \epsilon(-\lambda + \alpha - \psi\beta)|m(x_0)\rangle + O(\epsilon^2). \tag{40}$$

Now, if ψ is replaced by $\psi + \delta\psi$, where $\delta\psi(x)$ vanishes for $x \leqslant x_0$ and for $x \geqslant x_0 + \epsilon$, the vectors $|m(x_0)\rangle$ and $\langle p(x_0+\epsilon)|$ will be unaltered since the boundary conditions (19) and (21) fix $|m(0)\rangle$ and

$\langle p(L)|$. Therefore N_2 will be replaced by $N_2 - \delta N_2$, where

$$\begin{aligned}\delta N_2 &= \langle p(x_0+\epsilon)|\beta|m(x_0)\rangle \int_{x_0}^{x_0+\epsilon} \delta\psi(x)\,dx + O(\epsilon^2)\\ &= \langle p(x_0)|\beta|m(x_0)\rangle \int_{x_0}^{x_0+\epsilon} \delta\psi(x)\,dx + O(\epsilon^2).\end{aligned} \tag{41}$$

Since ϵ can be arbitrarily small and x_0 can be anything between 0 and L, we find that N_2 is stationary with respect to ψ if and only if

$$\langle\beta\rangle = 0 \tag{42}$$

for all x, in the notation of (24).

VI. DIFFERENTIAL EQUATION

From (42) and 5) we shall derive a differential equation for $\psi(x)$. We begin by observing that $\alpha - \psi\beta$ commutes with itself, and therefore

$$\frac{d}{dx}\langle\alpha\rangle - \psi\frac{d}{dx}\langle\beta\rangle = 0. \tag{43}$$

The second term on the left vanishes on account of (42). Therefore the first term is also zero and $\langle\alpha\rangle$ is a quantity independent of x.

Differentiating (42) with respect to x, and using (25) with (30), we have

$$\langle\gamma\rangle = 0 \tag{44}$$

for all x. Differentiating again and using (25) with (31) and (32), we find

$$\langle\beta\rangle\mathrm{Re}(n'^2) - \langle\beta'\rangle\mathrm{Im}(n'^2) + \psi\langle\alpha\rangle = 0 \tag{45}$$

or, in view of (42),

$$\langle\beta'\rangle = C\psi, \tag{46}$$

where C is given by

$$C = +\langle\alpha\rangle/\mathrm{Im}(n'^2) \tag{47}$$

and is x independent because of (43).

Differentiating a third time, with the help of (33) and (34), we have

$$2\langle\gamma'\rangle = -C\frac{d\psi}{dx}. \tag{48}$$

A fourth differentiation, with (35) and (36), yields

$$C\frac{d^2\psi}{dx^2} = 4[-\langle\beta\rangle\mathrm{Im}(n'^2) - \langle\beta'\rangle\mathrm{Re}(n'^2) - \langle\alpha'\rangle\psi], \tag{49}$$

into which we substitute from (42) and (46) to obtain

$$C\frac{d^2\psi}{dx^2} = 4\eta\psi, \tag{50}$$

where

$$\eta = -\langle\alpha'\rangle - C\,\mathrm{Re}(n'^2). \tag{51}$$

We now differentiate (51), using (25) with (37) and (38). The result is

$$\frac{d\eta}{dx} = 2\langle\gamma'\rangle\psi = -C\psi\frac{d\psi}{dx} \tag{52}$$

on account of (48). This equation can be integrated at once, yielding

$$\eta = -C(\tfrac{1}{2}\psi^2 + g), \tag{53}$$

where g is an undetermined constant.

Putting (53) into (50), we obtain

$$\frac{d^2\psi}{dx^2} + 2\psi^3 + 4g\psi = 0, \tag{54}$$

which is our equation for $\psi(x)$.

VII. ELLIPTIC FUNCTIONS

Multiplying (54) by $2d\psi/dx$ and integrating, we find

$$\left(\frac{d\psi}{dx}\right)^2 = -\psi^4 - 4g\psi^2 + h, \tag{55}$$

where h is another undetermined constant. Real solutions to this equation are of the form

$$\psi = \pm k\rho\,\mathrm{cn}\rho(x+l) \tag{56}$$

if $h > 0$, or

$$\psi = \pm\rho\,\mathrm{dn}\rho(x+l) \tag{57}$$

if $h < 0$. Here cn$u$ and dn$u$ are elliptic functions of modulus k, defined by

$$\begin{aligned}&\mathrm{cn}u = (1-\mathrm{sn}^2u)^{1/2},\\ &\mathrm{dn}u = (1-k^2\,\mathrm{sn}^2u)^{1/2},\\ &\frac{d}{du}\mathrm{sn}u = \mathrm{cn}u\,\mathrm{dn}u,\\ &\mathrm{cn}0 = \mathrm{dn}0 = 1.\end{aligned} \tag{58}$$

The quantities k and ρ depend on g and h, and l is a new undetermined constant.

We shall determine k, ρ, l and choose between (56) and (57) by examining the boundary conditions on ψ. From (6) and (19) we infer

$$\Sigma_i^L|m(0)\rangle = \Sigma_i^R|m(0)\rangle \tag{59}$$

for $i = 1, 2, 3$. Therefore, comparing (17) with (27) and (26) with (29), we have, for $x = 0$,

$$\langle\alpha'\rangle = -(\mathrm{Re}n'/\mathrm{Im}n')\langle\alpha\rangle, \tag{60}$$

$$\langle\gamma'\rangle = -(\mathrm{Re}n'/\mathrm{Im}n')\langle\gamma\rangle. \tag{61}$$

On the other hand, (6) and (21) give

$$\langle p(L)|\Sigma_i^L = -\langle p(L)|\Sigma_i^R \tag{62}$$

for $i=1$ or 3. A similar comparison yields, for $x=L$,

$$\langle\alpha'\rangle=(\mathrm{Im}n'/\mathrm{Re}n')\langle\alpha\rangle, \tag{63}$$

$$\langle\gamma'\rangle=(\mathrm{Im}n'/\mathrm{Re}n')\langle\gamma\rangle. \tag{64}$$

Substituting (44) and (48) into (61) and (64), we have

$$\frac{d\psi}{dx}=0 \tag{65}$$

for $x=0$ or L.

Substituting (47) and (60) into (51), we find

$$\begin{aligned}\eta(0)&=+(\mathrm{Re}n'/\mathrm{Im}n')\langle\alpha\rangle-C\,\mathrm{Re}(n'^2)\\&=C[(\mathrm{Re}n'/\mathrm{Im}n')\,\mathrm{Im}(n'^2)-\mathrm{Re}(n'^2)]\\&=C[2(\mathrm{Re}n')^2-(\mathrm{Re}n')^2+(\mathrm{Im}n')^2]\\&=C|n'|^2.\end{aligned} \tag{66}$$

Replacing (60) by (63), we have

$$\begin{aligned}\eta(L)&=-(\mathrm{Im}n'/\mathrm{Re}n')\langle\alpha\rangle-C\,\mathrm{Re}(n'^2)\\&=-C[(\mathrm{Im}n'/\mathrm{Re}n')\,\mathrm{Im}(n'^2)+\mathrm{Re}(n'^2)]\\&=-C[2(\mathrm{Im}n')^2+(\mathrm{Re}n')^2-(\mathrm{Im}n')^2]\\&=-C|n'|^2.\end{aligned} \tag{67}$$

Now, the derivative of cn$u$ vanishes for real u only when u is a multiple of $2K$, where $4K$ is the full period of the elliptic functions on the real line:

$$K=\int_0^1\frac{dz}{(1-z^2)(1-k^2z^2)^{1/2}}. \tag{68}$$

The second derivative of cn$u$ is equal to $-$cn$u$ at all multiples of $2K$. It follows that if we use (56), then on account of (50) and (65) we must have

$$\eta(0)=\eta(L), \tag{69}$$

which contradicts (66) and (67). Therefore (56) is ruled out, and (57) is correct.

The derivative of dn$u$ vanishes at all multiples of K. The second derivative of dn$u$ has the value $-k^2\,\mathrm{dn}u$ at even multiples of K, and $+k^2\,\mathrm{dn}u$ at odd multiples. Applying these facts to (57), and using (65)–(67) with (50), we have

$$\rho L=(2s-1)K, \tag{70}$$

where s may be any positive integer, and

$$k\rho=2|n'|, \tag{71}$$

$$l=L. \tag{72}$$

We shall confine our attention for the time being to the *principal solution* obtained by setting $s=1$. For this solution, the combination (70) and (71) yields

$$kK=2|n'|L, \tag{73}$$

which together with (68) determines k.

From (58) we obtain

$$\int\mathrm{dn}u\,du=\int\frac{d\,\mathrm{sn}u}{\mathrm{cn}u}=\sin^{-1}(\mathrm{sn}u)+\mathrm{const.} \tag{74}$$

Thus, comparing (10) with (57), we find

$$\cos\phi=\mathrm{sn}\rho(x+L), \tag{75}$$

where we use the cosine instead of the sine in order to make $\phi(0)=0$. By combining (75) with (57) and using (58) again, we find

$$\frac{d\phi}{dx}=\psi=\pm\frac{2|n'|}{k}(1-k^2\cos^2\phi)^{1/2}, \tag{76}$$

which is a useful way to express the dependence of ϕ on x.

The sign of ψ is still undetermined. We shall show at the end of Sec. VIII how to choose it so as to make N_2 positive. Two rather surprising observations may be made. The first is that in view of (57) and (73) the whole form of $\phi(x)$ is independent of the argument of n', depending only on its absolute magnitude. The second is that regardless of the value of k we have

$$\phi(L)=\pm\pi/2 \tag{77}$$

on account of (75).

VIII. DEPENDENCE ON PATH LENGTH

In this section we shall write N_2 or $N_2(L)$ or $N_2(k)$ for the circular Stokes parameter obtained from the principal solution of (54); that is, N_2 is given by (22) with (76), where k is determined by (73) and (68). In order to compute N_2 as a function of L or k, we first study its derivative with respect to L, for fixed n and n'.

If L is replaced by $L+dL$, N_2 may be affected in two ways. First, a path length dL is added at the end. Second, ψ is slightly altered for $0<x<L$ since L enters into (57) through k, ρ, and l. But the second of these changes has no effect on N_2 to first order, since ψ has been chosen so as to make N_2 stationary. Hence, referring to (22), we have

$$\begin{aligned}\frac{dN_2}{dL}&=\mathrm{Tr}\sigma_2\frac{dM}{dx}\bigg|_{x=L}=\langle p(L)|\frac{d}{dx}|m(x)\rangle_{x=L}\\&=\langle p|-\lambda+\alpha-\psi\beta|m\rangle=-\lambda N_2+\langle\alpha\rangle,\end{aligned} \tag{78}$$

where we have made use of (15) and (42).

We shall now relate $\langle\alpha\rangle$ to N_2 by studying the quantity

$$|M|=M_0^2-M_1^2-M_2^2-M_3^2 \tag{79}$$

as a function of x. From (1) we have

$$M^cM=\sigma_0|M|, \tag{80}$$

where

$$M^c=\sigma_0 \operatorname{Tr}M - M = M_0\sigma_0 - M_1\sigma_1 - M_2\sigma_2 - M_3\sigma_3. \tag{81}$$

Let us define

$$A^c=\sigma_0 \operatorname{Tr}A - A \tag{82}$$

for arbitrary 2×2 matrices A. Then it is easily seen, by writing A as $A_0\sigma_0+\vec{A}\cdot\vec{\sigma}$, that

$$(AB)^c=B^cA^c \tag{83}$$

for any A, B.

Now we combine (80) with (14) to obtain

$$\begin{aligned}\sigma_0\frac{d}{dx}|M| &= \frac{d}{dx}(M^cM)\\ &= i[M^cHM - M^cMH^\dagger + (HM)^cM - (MH^\dagger)^cM]\\ &= i[M^c(H+H^c)M - M^cMH^\dagger - (H^\dagger)^cM^cM]\\ &= i[M^c(\operatorname{Tr}H)M - M^cMH^\dagger - (H^\dagger)^cM^cM]\\ &= i\sigma_0|M|\operatorname{Tr}(H-H^\dagger)\\ &= -2\lambda\sigma_0|M|\end{aligned} \tag{84}$$

and therefore, using (19),

$$4|M|=e^{-2\lambda x}. \tag{85}$$

At $x=L$ we can apply (21) to obtain

$$\begin{aligned}2M_0 &= \langle\Sigma_2^L\rangle=\langle\Sigma_2^R\rangle=\tfrac{1}{2}\langle\beta'\rangle,\\ 2M_1 &= -i\langle\Sigma_3^L\rangle=i\langle\Sigma_3^R\rangle=-\frac{1}{2\operatorname{Re}n'}\langle\alpha\rangle,\\ 2M_2 &= \langle I\rangle=N_2,\\ 2M_3 &= i\langle\Sigma_1^L\rangle=-i\langle\Sigma_1^R\rangle=\frac{1}{2\operatorname{Re}n'}\langle\gamma\rangle.\end{aligned} \tag{86}$$

Substituting from (44), (46), and (47), we find

$$4|M|=Q\langle\alpha\rangle^2-N_2^2, \tag{87}$$

where

$$\begin{aligned}Q &= \frac{1}{4}\left(\frac{\psi(L)^2}{\operatorname{Im}(n'^2)^2}-\frac{1}{(\operatorname{Re}n')^2}\right)\\ &= \frac{1}{\operatorname{Im}(n'^2)^2}\left[\tfrac{1}{4}\psi(L)^2-(\operatorname{Im}n')^2\right]\\ &= \frac{1}{\operatorname{Im}(n'^2)^2}\left[\tfrac{1}{4}\rho^2-(\operatorname{Im}n')^2\right]\\ &= \frac{1}{\operatorname{Im}(n'^2)^2}\left[\frac{|n'^2|}{k^2}-(\operatorname{Im}n')^2\right].\end{aligned} \tag{88}$$

Here we have used (57) and (71), remembering that $\operatorname{dn}2K=1$.

Comparing (87) with (85) for $x=L$, we have

$$Q\langle\alpha\rangle^2=N_2^2+e^{-2\lambda L}. \tag{89}$$

But if we write

$$N_2=\mathfrak{F}(L)e^{-\lambda L} \tag{90}$$

and substitute in both (89) and (78), we find

$$\frac{d\mathfrak{F}}{dL}=\langle\alpha\rangle e^{\lambda L}=[Q^{-1}(1+\mathfrak{F}^2)]^{1/2} \tag{91}$$

and hence

$$\mathfrak{F}=\sinh y, \tag{92}$$

where

$$\frac{dy}{dL}=\frac{1}{Q^{1/2}}. \tag{93}$$

The sign of y is that of $Q^{1/2}$, which by (91) is that of $\langle\alpha\rangle$. Thus, $y, \mathfrak{F}, N_2$ will be positive if $\langle\alpha\rangle$ is. That means, by (46) and (47), that we should give ψ the same sign as $\langle\beta'\rangle\operatorname{Im}(n'^2)$. The sign of ψ is the same for all x, and $\langle\beta'\rangle$ must be positive at $x=L$ by (86). Therefore the sign in (57) should be taken as that of $\operatorname{Im}(n'^2)$ or of $\operatorname{Re}n'\operatorname{Im}n'$, if N_2 is to be positive. The physical sense of this may be inferred from (11) and (22) with (2).

IX. ELLIPTIC INTEGRAL

Let us now write

$$n'=|n'|e^{i\chi} \tag{94}$$

and

$$q=(1-k^2\sin^2\chi)^{1/2}. \tag{95}$$

Then (88) becomes

$$Q=q^2/k^2|n'^2|\sin^2 2\chi, \tag{96}$$

which with (93) and (73) gives, taking ψ positive,

$$\frac{dy}{dk}=\frac{\sin 2\chi}{2}\,\frac{k}{q}\,\frac{d}{dk}(kK). \tag{97}$$

It will now be prudent to replace (68) by a contour integral:

$$K=\tfrac{1}{4}\oint\frac{dz}{(1-z^2)(1-k^2z^2)^{1/2}}, \tag{98}$$

where the contour runs counterclockwise around the cut from -1 to 1, and the square roots are both positive when z is negative imaginary. We then have

$$y=\frac{\sin 2\chi}{8}\oint\frac{dz}{(1-z^2)^{1/2}}f(k,z), \tag{99}$$

where

$$f(k,z)=\int\frac{k}{q}\,\frac{d}{dk}\,\frac{k}{(1-k^2z^2)^{1/2}}\,dk$$

$$=\int(1-k^2z^2)^{-3/2}\frac{k\,dk}{q}$$

$$=-\frac{1}{\sin^2\chi}\int(1-k^2z^2)^{-3/2}dq$$

$$=-\frac{1}{\sin^2\chi}\int\left(1-\frac{z^2}{\sin^2\chi}+\frac{q^2z^2}{\sin^2\chi}\right)^{-3/2}dq$$

$$=-\frac{1}{\sin^2\chi}\left(1-\frac{z^2}{\sin^2\chi}\right)^{-1}$$

$$\times q\left(1-\frac{z^2}{\sin^2\chi}+\frac{q^2z^2}{\sin^2\chi}\right)^{-1/2}$$

$$=\frac{1}{z^2-\sin^2\chi}\left(\frac{1-k^2\sin^2\chi}{1-k^2z^2}\right)^{1/2}, \tag{100}$$

and so

$$N_2=e^{-\lambda L}\sinh y$$

$$=e^{-\lambda L}\sinh\left(\frac{\sin 2\chi}{8}(1-k^2\sin^2\chi)^{1/2}\times\oint\frac{dz}{(z^2-\sin^2\chi)(1-z^2)^{1/2}(1-k^2z^2)^{1/2}}\right). \tag{101}$$

To show that we have not lost a constant of integration, we remark that when $L\to 0$ we have $k\to 0$, so that the branch points at $\pm 1/k$ recede to ∞. Then the contour can be expanded and we get $y\to 0$, hence $N_2\to 0$ as desired.

X. INEQUALITIES

By comparison with (101), the linear polarization intensity obtained from a constant-θ sample of length L is

$$N_3=e^{-\lambda L}\sinh y_0=e^{-\lambda L}\sinh(2|n'|L\sin\chi) \tag{102}$$

and the circular conversion efficiency is therefore

$$E=\sinh y/\sinh y_0. \tag{103}$$

From (102) and (97) and (73), we find

$$\frac{dy}{dy_0}=\frac{k\cos\chi}{(1-k^2\sin^2\chi)^{1/2}}<1 \tag{104}$$

so that $y<y_0$ and $E<1$, as expected.

We shall now show that the principal solution gives the highest efficiency for a given L. First, we note that if we vary the integer s in (70), the equations of Sec. VIII are completely unaffected. However, instead of (73) we have

$$(2s-1)kK=2|n'|L \tag{105}$$

and this will affect the equations of Sec. IX.

Let us define y_s as the solution of (93) with (88) and (105). Then instead of (97) we have

$$y_s=\tfrac{1}{2}\sin 2\chi\,(2s-1)\,Y(k_s), \tag{106}$$

where Y is the function satisfying

$$\frac{dY(k)}{dk}=\frac{k}{q}\,\frac{d}{dk}Z(k) \tag{107}$$

and $Y(0)=0$, and k_s is the value of k satisfying

$$(2s-1)Z(k_s)=2|n'|L, \tag{108}$$

with the function Z given by

$$Z(k)=kK.$$

It is always understood that K and q depend on k through (68) and (95).

Combining (106) with (108), we have

$$y_s=[Y(k_s)/Z(k_s)]\,|n'|L\sin 2\chi. \tag{109}$$

Since Z is an increasing function of k, we have from (108) the inequality

$$k_1>k_2>k_3>\cdots. \tag{110}$$

From (109), therefore, we obtain

$$y_1>y_2>y_3>\cdots \tag{111}$$

provided that Y/Z is also an increasing function of k.

To prove the latter assertion, we write (107) as

$$\frac{dY}{dZ}=\frac{k}{q}, \tag{112}$$

and considering Z as the independent variable, we apply the "law of the mean" to the interval from 0 to Z. The result is

$$Y/Z=k'/q', \tag{113}$$

where k' lies between 0 and k. Since k/q is obviously an increasing function of k, we may compare (113) with (112) to obtain

$$\frac{dY}{dZ}>\frac{Y}{Z}, \tag{114}$$

which leads easily to the desired inequality,

$$\frac{d}{dk}\left(\frac{Y}{Z}\right)>0 \tag{115}$$

for $0<k<1$. This shows that Y/Z is an increasing function and completes the proof of (111).

Applying (111) to (90) and (92), we see that the efficiency at a given L is highest for the principal solution ($s=1$) and decreases with increasing s.

XI. SHORT-PATH LIMIT

When $|n'|L\ll 1$, we have $k\ll 1$ and $K\approx\pi/2$, and (57) reduces to (taking the positive sign)

$$\psi=\rho=\pi/2L \tag{116}$$

for the principal solution, or

$$\phi(x)=\tfrac{1}{2}\pi x/L. \tag{117}$$

The leading term of the integral in (101) is then

$$\oint \frac{dz}{(z^2-\sin^2\chi)(1-z^2)^{1/2}}\;\frac{k^2z^2}{2} = \frac{k^2}{2}\left(\oint\frac{dz}{(1-z^2)^{1/2}}+\oint\frac{\sin^2\chi\, dz}{(z^2-\sin^2\chi)(1-z^2)^{1/2}}\right) = \pi k^2 \tag{118}$$

and therefore

$$\begin{aligned} N_2 &\simeq e^{-\lambda L} y \simeq \tfrac{1}{8}\pi k^2 \sin 2\chi\, e^{-\lambda L} \\ &= \frac{\pi |n'|^2 L^2 \sin 2\chi}{2(\pi/2)^2}\, e^{-\lambda L} \\ &= (2/\pi)\,|n'|^2 L^2 \sin 2\chi\, e^{-\lambda L} \\ &= (4/\pi)\, L^2 \,\mathrm{Re} n' \,\mathrm{Im} n'\, e^{-\lambda L}\,. \end{aligned} \tag{119}$$

As a check, we shall derive (117) and (119) directly from (14) in the limit

$$g \equiv |n'| L \ll 1. \tag{120}$$

Setting

$$r = x/L \tag{121}$$

and

$$\xi = \frac{d\phi}{dr} \tag{122}$$

we write (14) as

$$\frac{dM}{dr} = -i\,\xi[\sigma_2, M] - \lambda L M + i g (e^{i\chi}\sigma_3 M - e^{-i\chi} M\sigma_3), \tag{123}$$

and using (19) and (22) we obtain (90), where $\mathfrak{F}$ can be expanded in powers of g,

$$\mathfrak{F} = \mathfrak{F}_0 + g\mathfrak{F}_1 + g^2\mathfrak{F}_2 + \cdots. \tag{124}$$

A straightforward iterative solution of (123), with the notation

$$U(r_2, r_1) = \exp\left(-i\sigma_2 \int_{r_1}^{r_2} \xi\, dr\right) = e^{-i\sigma_2[\phi(r_2)-\phi(r_1)]}, \tag{125}$$

yields

$$\mathfrak{F}_0 = \tfrac{1}{2}\,\mathrm{Tr}\{\sigma_2 U(1,0)\sigma_0 U^{-1}(1,0)\} = \tfrac{1}{2}\,\mathrm{Tr}\,\sigma_2 = 0, \tag{126}$$

$$\begin{aligned} \mathfrak{F}_1 &= \tfrac{1}{2}\,\mathrm{Tr}\left\{\sigma_2 \int_0^1 dr_1\, U(1,r_1) i e^{i\chi}\sigma_3 \times U(r_1,0)\sigma_0 U^{-1}(1,0)\right\} + \text{c.c.} \\ &= \mathrm{Re}\, i e^{i\chi} \int_0^1 dr_1\, \mathrm{Tr}\{\sigma_2 U(1,r_1)\sigma_3 U^{-1}(1,r_1)\} \\ &= \mathrm{Re}\, i e^{i\chi}\, \mathrm{Tr}(\sigma_2\sigma_3) = 0, \end{aligned} \tag{127}$$

$$\mathfrak{F}_2 = \int_0^1 dr_1 \int_{r_1}^1 dr_2 [F(r_1,r_2) + F'(r_1,r_2)] + \text{c.c.}, \tag{128}$$

where

$$\begin{aligned} F(r_1,r_2) &= \tfrac{1}{2}\,\mathrm{Tr}\{\sigma_2 U(1,r_2) i e^{i\chi}\sigma_3 U(r_2,r_1) i e^{i\chi}\sigma_3 \times U(r_1,0)\sigma_0 U^{-1}(1,0)\} \\ &= -\tfrac{1}{2} e^{2i\chi}\, \mathrm{Tr}\{\sigma_2\sigma_3 U(r_2,r_1)\sigma_3 U^{-1}(r_2,r_1)\} \\ &= -\tfrac{1}{2} e^{2i\chi}\, \mathrm{Tr}\{\sigma_2 U^{-2}(r_2,r_1)\} \\ &= -i e^{2i\chi} \sin 2[\phi(r_2) - \phi(r_1)] \end{aligned} \tag{129}$$

and

$$\begin{aligned} F'(r_1,r_2) &= \tfrac{1}{2}\,\mathrm{Tr}\{\sigma_2 U(1,r_2) i e^{i\chi}\sigma_3 U(r_2,0)\sigma_0 U^{-1}(r_1,0) \times (-i e^{-i\chi})\sigma_3 U^{-1}(1,r_1)\} \\ &= \tfrac{1}{2}\,\mathrm{Tr}\{\sigma_2\sigma_3 U(r_2,r_1)\sigma_3 U^{-1}(r_2,r_1)\} \\ &= i \sin 2[\phi(r_2) - \phi(r_1)]\,. \end{aligned} \tag{130}$$

Substituting (129) and (130) into (128), we have

$$\mathfrak{F}_2 = 2\sin 2\chi \int_0^1 dr_1 \int_{r_1}^1 dr_2 \sin 2[\phi(r_2) - \phi(r_1)]\,. \tag{131}$$

Our task now is to choose the function $\phi(r)$ so as to maximize the integral in (131). For reasons shortly to appear, we define a function f on the interval $(0, 2\pi)$ by

$$f(\mu) = \tfrac{1}{2}\int_0^1 \delta(|\sin[\tfrac{1}{2}\mu - 2\phi(r)]|)\, dr = \tfrac{1}{4}\sum \left|\frac{d\phi}{dr}\right|^{-1}, \tag{132}$$

where the sum goes over those r satisfying

$$\phi(r) \equiv \tfrac{1}{4}\mu, \quad \mathrm{mod}\,\pi/2. \tag{133}$$

Thus $f(\mu)$ is determined by $\phi(r)$, and satisfies

$$\int_0^{2\pi} f(\mu)\, d\mu = 1 \tag{134}$$

as seen from the first line of (132). Moreover, we have

$$\begin{aligned} \mathfrak{F}_2 &\le 2\sin 2\chi \int_0^1 dr_1 \int_{r_1}^1 dr_2 \,|\sin 2[\phi(r_2) - \phi(r_1)]| \\ &= \sin 2\chi \int_0^{2\pi} f(\mu_1)\, d\mu_1 \int_0^{2\pi} f(\mu_2)\, d\mu_2\, |\sin\tfrac{1}{2}(\mu_2 - \mu_1)|. \end{aligned} \tag{135}$$

On the other hand, given f satisfying (134), we can choose ϕ satisfying (132) so that equality is achieved in (135). This is done by defining $\phi(r)$ through the equation

$$\frac{d\phi}{dr} = 1/4f(4\phi), \tag{136}$$

with $\phi(0)=0$, so that ϕ increases monotonically but never exceeds $\pi/2$, and $\sin 2[\phi(r_2)-\phi(r_1)]$ is never negative.

To maximize (131), therefore, we must choose f so as to maximize the second line of (135) with the restriction (134). Since $2|\sin\frac{1}{2}(\mu_2-\mu_1)|$ is just the length of the chord joining the points μ_1 and μ_2 on the unit circle, we may regard f as the density of matter distributed on the unit circle, and repelling itself with a force whose magnitude between two given point masses is independent of their separation. Then the total mass is 1 by (134), and the total potential energy is proportional to minus the second line of (135). This energy is minimized in the position of stable equilibrium, which intuition may identify as the uniform distribution of matter around the circle.

To confirm this intuition, we write

$$f(\mu)=\frac{1}{2\pi}+\sum_1^\infty (a_m e^{im\mu}+a_m^* e^{-im\mu}) \tag{137}$$

and note that the second line of (135) becomes

$$\sin 2\chi\left(\frac{A_0}{2\pi}+2\sum_1^\infty |a_m|^2 A_m\right), \tag{138}$$

where

$$A_m=\int_0^{2\pi}\sin\tfrac{1}{2}\mu\,\cos m\mu\,d\mu=\frac{1}{m+\frac{1}{2}}-\frac{1}{m-\frac{1}{2}} \tag{139}$$

which is negative for $m>0$. Therefore (138) is maximized by setting all $a_m=0$. Now, using (137) and (136), we get

$$\phi(r)=(\pi/2)\,r, \tag{140}$$

which in view of (121) is the same as (117).

The maximum $\mathfrak{F}_2$ is obtained from (138) and (139):

$$\mathfrak{F}_2=(A_0/2\pi)\sin 2\chi=(2/\pi)\sin 2\chi, \tag{141}$$

and when this is put into (124) and the result combined with (90) and (120), we recover (119) for the leading order in g.

XII. LONG-PATH LIMIT

When $|n'|L\gg 1$, we have $k\to 1$ and $K\sim|n'|L$ for the principal solution. In this limit the integral in (101) blows up, as the contour is pinched between two branch points. But this is expected, as y should behave like y_0 as defined by (102). Combining (101) with (98), and using (73), we have

$$y\frac{(1-k^2\sin^2\chi)^{1/2}}{\cos\chi}-ky_0=\tfrac{1}{4}\sin\chi\left((1-k^2\sin^2\chi)\oint\frac{dz}{(1-z^2)^{1/2}(1-k^2z^2)^{1/2}(z^2-\sin^2\chi)}-k^2\oint\frac{dz}{(1-z^2)^{1/2}(1-k^2z^2)^{1/2}}\right)$$

$$=\tfrac{1}{4}\sin\chi\oint\frac{dz}{z^2-\sin^2\chi}\,\frac{(1-k^2z^2)^{1/2}}{(1-z^2)^{1/2}}, \tag{142}$$

in which both sides approach a finite limit as $k\to 1$.

The quantities y and y_0 grow only logarithmically in $1-k$; therefore $O(y(1-k), y_0(1-k))$ can be neglected and the left side of (142) replaced by $y-y_0$.

On the right side, the factor $(1-k^2z^2)^{1/2}/(1-z^2)^{1/2}$ becomes $+1$ on the negative imaginary side of the cut, and -1 on the positive imaginary side. The limiting form of (142) is therefore

$$\begin{aligned} y-y_0 &\cong \tfrac{1}{4}\sin\chi\, 2P\int_{-1}^{1}\frac{dz}{z^2-\sin^2\chi} \\ &=\tfrac{1}{4}\left(P\int_{-1}^{1}\frac{dz}{z-\sin\chi}-P\int_{-1}^{1}\frac{dz}{z+\sin\chi}\right) \\ &=\tfrac{1}{4}\left(\ln\frac{1-\sin\chi}{1+\sin\chi}-\ln\frac{1+\sin\chi}{1-\sin\chi}\right) \\ &=\tfrac{1}{2}\ln\frac{1-\sin\chi}{1+\sin\chi} \end{aligned} \tag{143}$$

and so, from (103), the limiting efficiency is given by

$$E\to\left(\frac{1-\sin\chi}{1+\sin\chi}\right)^{1/2}=\frac{\cos\chi}{1+\sin\chi}. \tag{144}$$

This efficiency is achieved by the limiting form of (57),

$$\psi=2|n'|\operatorname{sech}2|n'|(L-x), \tag{145}$$

or of (75),

$$\cos\phi=\tanh 2|n'|(L-x). \tag{146}$$

Let us compare (144) with the result of a cruder strategy, in which we put

$$\begin{aligned} \phi(x)&=0, \quad x<L-x_0 \\ &=\phi_0, \quad x>L-x_0. \end{aligned} \tag{147}$$

It is easily found, for fixed x_0 and $L\to\infty$, that

$$E\to 2e^{-2x_0|n'|\sin\chi}\sin\phi_0\cos\phi_0\sin(2x_0|n'|\cos\chi), \tag{148}$$

which is maximized by setting $\phi_0=\pi/4$, $2x_0|n'|\cos\chi=\pi/2-\chi$. We then have

$$E\to\cos\chi\, e^{-(\pi/2-\chi)\tan\chi}. \tag{149}$$

Comparing the crude efficiency E_{cr}, from (149), with the ideal efficiency E_I, from (144), we have

$$E_{cr}/E_I = (1+\sin\chi)e^{-(\pi/2-\chi)\tan\chi}$$
$$\sim 1-(\pi/2-1)\chi,\quad \chi\ll 1 \quad (\mathrm{Im}n' \ll \mathrm{Re}n')$$
$$=(1+\tfrac{1}{2}\sqrt{2})\,e^{-\pi/4}\sim 0.778,\quad \chi=\pi/4 \quad (\mathrm{Im}n'=\mathrm{Re}n')$$
$$\to 2/e\sim 0.736,\quad \chi\to\pi/2 \quad (\mathrm{Im}n' \gg \mathrm{Re}n'). \tag{150}$$

XIII. OPTIMUM LENGTH

To maximize N_2 with respect to L, we differentiate (90) with the help of (92) and (93), obtaining

$$\frac{dN_2}{dL} = -\lambda e^{-\lambda L}\sinh y + \frac{1}{Q^{1/2}}e^{-\lambda L}\cosh y, \tag{151}$$

which vanishes when

$$Q^{1/2}\tanh y = 1/\lambda. \tag{152}$$

If $\lambda \gg |n'|$, we may simplify (152) by using the short-path limit—see (119)—which gives

$$\frac{1}{\lambda}\sim\frac{qy}{k|n'|\sin 2\chi}\sim\frac{\pi k}{8|n'|}\sim\frac{L}{2}, \tag{153}$$

and thus

$$N_2 \simeq \frac{8}{\pi}\left(\frac{|n'|}{\lambda e}\right)^2\sin 2\chi = \frac{4}{\pi e^2}\frac{\mathrm{Re}n'\,\mathrm{Im}n'}{(\mathrm{Im}n)^2} \tag{154}$$

for the maximum circular intensity attainable when the over-all attenuation is much stronger than the polarizing effects.

The opposite limit occurs when one component is not attenuated at all; $\mathrm{Im}(n-n')=0$, or $\lambda = 2|n'|\sin\chi$. Combining (152) with (96), we then have

$$\tanh y = (k/q)\cos\chi, \tag{155}$$

which is satisfied when $k\to 1$, $y\to\infty$. Thus the best result is now obtained with $L\to\infty$. Since the untwisted crystal in this case will give $N_3=\frac{1}{2}$, we have from (144)

$$N_2 \to \cos\chi/2(1+\sin\chi) \tag{156}$$

for the maximum circular intensity in this case.

TABLE I. For eight values of k we tabulate K, and the three quantities y_0, y, E for each of two values of χ ($\tan\chi=\frac{1}{2}, 3$). The tabulated values of y_0 and E are related to the two outer curves in Fig. 1.

		$\mathrm{Im}n'=\frac{1}{2}\mathrm{Re}n'$			$\mathrm{Im}n'=3\,\mathrm{Re}n'$		
k	K	y_0	y	E	y_0	y	E
0.1	1.57	0.070	0.003	0.045	0.149	0.002	0.016
0.5	1.69	0.377	0.088	0.229	0.800	0.070	0.063
0.8	2.00	0.714	0.299	0.390	1.514	0.269	0.126
0.9	2.28	0.918	0.468	0.460	1.947	0.471	0.142
0.94	2.51	1.054	0.591	0.496	2.236	0.644	0.149
0.98	3.02	1.324	0.849	0.546	2.809	1.078	0.157
0.995	3.70	1.645	1.165	0.580	3.490	1.694	0.160
1	∞	∞	∞	0.618	∞	∞	0.162

XIV. NUMERICAL EVALUATION

We seek numerical values of the circular conversion efficiency (see Secs. I and X) given by

$$E = \sinh y/\sinh y_0, \tag{103}$$

where, from (102),

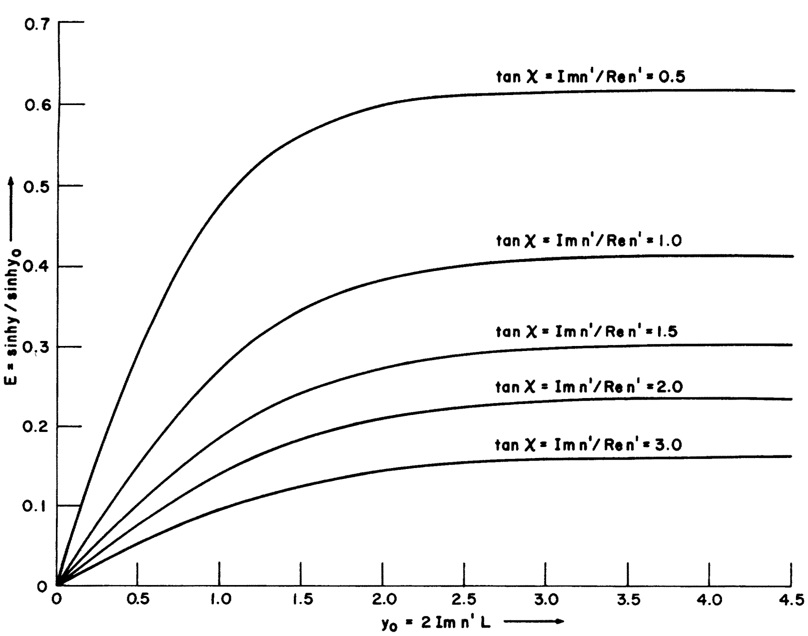

FIG. 1. Maximum possible circular conversion efficiency E (see Sec. X) is graphed against the crystal thickness, measured by the parameter y_0, for five values of $\tan\psi = \mathrm{Im}n'/\mathrm{Re}n'$. The top curve represents strong birefringence; the bottom curve represents strong polarization. The *fractional* linear polarization obtained from an *untwisted* sample is $\tanh y_0 = 0.76$ ($y_0=1$), $=0.96$ ($y_0=2$).

$$y_0 = 2\,\mathrm{Im}n'L = kK\sin\chi \quad (157)$$

and y is given by (97) or (101).

It is easy to find K for a given k by the formulas

$$K=\frac{\pi}{2r_\infty},\quad r_\infty=\lim_{m\to\infty} r_m,\quad r_{m+1}=\tfrac{1}{2}(r_m+s_m),$$
$$s_{m+1}=(r_m s_m)^{1/2},\quad r_0=(1-k^2)^{1/2},\quad s_0=1. \quad (158)$$

This yields y_0 immediately for given k and χ.

The tabulation of y is most easily accomplished by integrating (97) by parts:

$$\frac{d}{dk}\left(\frac{y_0 k\cos\chi}{(1-k^2\sin^2\chi)^{1/2}}-y\right)=\frac{y_0\cos\chi}{(1-k^2\sin^2\chi)^{3/2}}. \quad (159)$$

The right side is easily calculated for given k and χ, and hence it is a suitable integrand for numerical integration. The integrand diverges only logarithmically at $k\to 1$, so that the integral is convergent there. (It should be understood that χ is fixed during the integration which is begun at $y=y_0=k=0$.)

Values of k, y_0, y, and E are tabulated for various χ in Table I. The relation between E and y_0, for fixed χ, is displayed in Fig. 1.

XV. SUMMARY OF RESULTS

We have found that the circular Stokes parameter N_2 is maximized for a given medium and path length when the twisting rate $\psi = d\phi/dx$ is given by

$$\psi(x)=\pm\frac{2|n'|}{k}\,\mathrm{dn}\frac{2|n'|}{k}(x+L), \quad (160)$$

where dn is a standard elliptic function of modulus k, and k is determined by

$$kK = 2|n'|L, \quad (73)$$

where K is the quarter-period associated with k,

$$K=\int_0^1\frac{dz}{(1-z^2)^{1/2}(1-k^2z^2)^{1/2}}, \quad (68)$$

L is the path length, and $2n'$ is the difference in complex wave numbers between the two perpendicular principal axes.

An equivalent form of the solution is the differential equation

$$\frac{d\phi}{dx}=\pm\frac{2|n'|}{k}(1-k^2\cos^2\phi)^{1/2}. \quad (76)$$

The sign of N_2 is the same as that of $\psi\,\mathrm{Re}n'\,\mathrm{Im}n'$. The relation between signs can best be understood by considering the circular polarization produced by the standard 45° configuration, $\psi=\pm\frac{1}{4}\pi\delta(x-x_0)$.

The solution (160) and (73) gives the biggest possible N_2, although the variation of N_2 is also zero if the left side of (73) is multiplied by any odd integer.

The optimum value of N_2 is given by

$$N_2 = e^{-\lambda L}\sinh y, \quad (161)$$

where $\lambda = 2\,\mathrm{Im}n$ is the common attenuation rate (average of two axes) and y is a somewhat complicated function of k and $\chi=\arg n'$, given by (97) or (101).

We compare this with the linear Stokes parameter N_3 produced by an untwisted sample of the same medium with the same length,

$$N_3 = e^{-\lambda}\sinh y_0, \quad (162)$$

where

$$y_0 = 2\,\mathrm{Im}n'L. \quad (163)$$

The ratio N_2/N_3 does not involve λ, and is given by

$$E=\sinh y/\sinh y_0, \quad (103)$$

which is a function of k and χ, or of y_0 and χ. It is plotted in Fig. 1 against y_0 for several values of χ.

When $y_0 \ll 1$, N_2 is maximized by a uniform twisting rate, and is $O[y_0^2]$ so that E is $O[y_0]$. When $y_0 \gg 1$, the optimum strategy puts nearly all the twisting near the exit face, with a hyperbolic secant form for ψ. In this limit, E approaches the constant value $\cos\chi/(1+\sin\chi)$.

In all cases the total twist is 90° ($\phi(L)=\pm\pi/2$) and for a specified $|n'|L$ the twisting function is independent of χ, though E is not.

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[1]H. Uberall, Phys. Rev. 103, 1055 (1956).

[2]W. K. H. Panofsky and A. N. S. Saxena, Phys. Rev. Lett. 2, 219 (1959); O. R. Frisch and D. H. Olson, Phys. Rev. Lett. 3, 141 (1959).

[3]N. Cabibbo, G. Da Prato, G. De Franceschi, and U. Mosco, Nuovo Cimento 27, 979 (1963).