Semiclassical radiation theory and the inverse method

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A relationship between semiclassical radiation theory and the inverse method of solution for nonlinear dispersive waves is developed through two physical examples. The Josephson transmission line is modeled by Maxwell's equations coupled to a phenomenological quantum mechanics. It is shown that this quantum mechanics contains the same linear problem used in the inverse method to solve the sine-Gordon equation, the equation which governs the evolution of the electromagnetic wave. This (nonlinear) wave equation and the linear quantum equations are of equal importance in the physical description of this system. This same relationship exists among the self-induced transparency (SIT) equations of nonlinear optics. This second example, due to Lamb, is discussed in a manner which again displays the precise relationship of the linear problem of the inverse method to the quantum physics. In addition, analogies between SIT and the Josephson transmission line are discussed.

I. INTRODUCTION

Characteristic features of a superconducting Josephson transmission line[1-6] are modeled by the "sine-Gordon" equation. Recently, this nonlinear wave equation has been solved for a large class of initial data by the "inverse method".[7-9] In this method the equation is solved via the introduction of a linear equation. This equation is introduced on an *ad hoc* basis as a computational aid. The primary purpose of this paper is to show that, for the sine-Gordon equation thought of as modeling a Josephson transmission line, the linear equations arise naturally from a phenomenological model of weakly coupled superconductors. Maxwell's equations together with the quantum equations specify the physical system and are of equal importance in the description.

From the point of view of this work the inverse method is a calculational technique used to solve a problem in semiclassical radiation theory. As has been shown by Lamb,[10-12] the same technique can be used in another semiclassical radiation theory problem; that of describing self-induced transparency (SIT). In both these cases the quantum mechanics of the medium in which the electromagnetic wave propagates provides the linear problem for the inverse method. In the appendices Lamb's work is recast into a form which clearly displays the relationship of quantum mechanics to the linear problem used in the inverse method of solution of the SIT equations.[11-14] Also, this appendix shows the precise relationship between the SIT equations and those describing the Josephson transmission line. The possible existence of such an analogy is suggested in Josephson's review article.[1]

The inverse method solves many other nonlinear equations, some without any physical interpretation at this time. Thus, when investigating mathematical classes of partial differential equations, an approach based upon a physical model may not be natural. Certainly to date[7-11] the development of the inverse method has not been motivated by physical reasoning.

Nevertheless, the two examples discussed here display a close relationship between semiclassical radiation theory and the inverse method. They suggest that, when facing another problem in semiclassical radiation theory, one should consider the inverse formalism as a possible method of solution. In such cases the physics should motivate the choice of associated linear problem. Certainly this type of interpretation makes the latest work of Zakharov and Manakov[26] more meaningful.

II. MAXWELL'S EQUATIONS

In this section a review is given of the standard application of Maxwell's equations to a Josephson transmission line. A Josephson transmission line consists of two superconducting metals separated by a thin insulating layer of uniform thickness. The geometry to be considered is depicted in Fig. 1. Attention is restricted to long, thin junctions. The latter restriction, stated more precisely[15] as $W \ll \lambda_J$, permits the electromagnetic wave to be treated as uniform in y.

In this geometry the Josephson transmission line (or wave guide) supports electromagnetic waves traveling in the x direction. To analyze these waves it is sufficient[1] to deal with the field in the insulator, since the fields in the metal are uniquely determined from those in the insulator by a penetration law. In the insulator the fields are governed by Maxwell's equations:

$$\nabla \times \vec{H} = (1/c)\partial_t \vec{D} + (4\pi/c)\vec{j}\,, \qquad (2.1a)$$

$$\nabla \times \vec{E} = -(1/c)\partial_t \vec{B}, \tag{2.1b}$$

and the constitutive relations

$$\vec{D} = \epsilon \vec{E}, \tag{2.1c}$$

$$\vec{B} = \mu \vec{H}, \tag{2.1d}$$

with ϵ, μ constant. Here $\vec{j}$ represents the current that flows from one side of the insulator to the other and is called the Josephson current. The Josephson current arises from the quantum nature of the superconducting state. It will be specified below by a phenomenological quantum mechanics. For the moment it is merely assumed to be in the z direction only, i.e., $\vec{j} = (0, 0, j)$.

If the boundary conditions $\vec{D} = (D_1, 0, D)$ and $\vec{H} = (0, H, 0)$ are imposed and solutions independent of y are sought, Maxwell's equations reduce to

$$\partial_x H = (4\pi/c)j + (1/c)\partial_t D, \tag{2.2a}$$

$$\partial_x E = (d/cl)\partial_t B, \quad d = l + 2\lambda. \tag{2.2b}$$

To obtain (2.2a) consider an integrated form of (2.1b)

$$\int_1^2 dz\,(\nabla \times \vec{E}) = \int_1^2 dz\,[(-1/c)\partial_t B],$$

where the points 1 and 2 have the same x and y coordinates and lie on opposite sides of the insulator, at least a distance λ into the superconductor. By using the facts that (i) E is concentrated between $-\frac{1}{2}l \le z \le \frac{1}{2}l$ and (ii) B penetrates the superconductor between $-(\frac{1}{2}l+\lambda) \le z \le (\frac{1}{2}l+\lambda)$, (2.2b) readily follows. Equation (2.2), together with the constitutive relations (2.1c) and (2.1d), imply

$$(\partial_t + s\partial_x)(\partial_t - s\partial_x) lE = -(4\pi/\epsilon)\partial_t j, \tag{2.3a}$$

$$(\partial_t + s\partial_x)(\partial_t - s\partial_x) dB = -(4\pi c/\epsilon)\partial_x j, \tag{2.3b}$$

where $s = c(l/d\epsilon\mu)^{1/2}$ is the speed of light in the insulator reduced by the penetration of the B field into the superconductor.

The combination $lE + (s/c)dB$ is useful in the sequel. It is easily seen to satisfy

$$(\partial_t + s\partial_x)(\partial_t - s\partial_x)[lE + (s/c)dB] = -(4\pi/\epsilon)(\partial_t + s\partial_x)j. \tag{2.4}$$

The electrodynamics is completely specified by Eq. (2.3) if the Josephson current is known. An expression for j is obtained from the knowledge of the quantum mechanics of weakly coupled superconductors. A phenomenological quantum theory of such systems is presented in the next section.

III. QUANTUM EQUATIONS

The microscopic theory of a Josephson junction in the presence of an external field is best, or at least most easily, understood when the external field is either constant or slowly varying.[1-3] This is particularly true of phenomenological theories.

There are three essential results that a phenomenological theory of the Josephson junction must yield: an expression for the Josephson current as a function of Δ, and the relationships connecting the time and space derivatives of Δ to the electric and magnetic fields ($\vec{E}$ and $\vec{B}$), respectively. Here Δ denotes the phase difference of the superconducting states on opposite sides of the junction. For constant external electric fields Silver[16] and Feynman[17] have given a phenomenological theory that yields these results. Here their theory is generalized and adapted to the transmission-line problem.

To see how this is done it is first necessary to discuss some facts about superconductivity. First recall that the existence of Cooper pairs is fundamental to the superconducting state. Each Cooper pair has charge $2e$ and is described by a quantum-mechanical wave function

$$\psi_k = (\rho_k)^{1/2} e^{i\Delta_k}, \quad k = 1, 2, \ldots, N \ (N \text{ large}). \tag{3.1}$$

Cooper pairs are bosons and, in a superconducting material, essentially all are in the same state. In particular, their wave functions all have a common phase. This feature allows the replacement of the N wave functions by a single macroscopic wave function

$$\psi(x,t) = [\rho(x,t)]^{1/2} e^{i\Delta(x,t)}, \tag{3.2}$$

with $\Delta = \Delta_k$, which is normalized so that $\rho = |\psi|^2$ is the density of Cooper pairs. The wave function ψ is analogous to the Ginzburg-Landau order parameter.[18] In what follows, ψ is interpreted as an "effective" wave function for a Cooper pair.

If two pieces of superconducting material are separated by a perfect barrier, they do not inter-

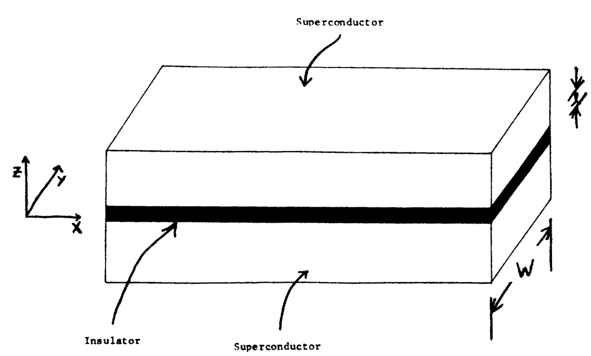

FIG. 1. Basic geometry of the junction. Note: l is the barrier thickness; $\lambda_\pm$ the penetration depths into upper and lower metals; $d \equiv l + \lambda_+ + \lambda_-$; λ_J the Josephson penetration depth into the barrier; W the width of the junction; and $\lambda_J \gg W$.

act. The effective wave functions above and below the barrier can be labeled ψ^+ and ψ^-, respectively. If the barrier is assumed strong, but not perfect, it is meaningful to retain this labeling.

Following Silver[16] and Feynman,[17] if a constant external electric field is impressed across the barrier, the time evolution of the system is governed by

$$i\hbar\partial_t\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}=\begin{pmatrix}2e\int_1^2 E\,dz & T\\ T^* & 2e\int_2^1 E\,dz\end{pmatrix}\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}. \qquad (3.3)$$

Here the interaction across the barrier is modeled through a coupling coefficient T. This coupling coefficient is a function of the barrier. It would be determined empirically. Observe that the magnetic field has been neglected as well as all kinetic-energy terms. The diagonal terms of the Hamiltonian represent the difference in energy of the Cooper pairs above and below the barrier. The off-diagonal terms denote the coupling across the barrier. Complex couplings are allowed, but the Hamiltonian must be Hermitian since, for the moment, dissipation is not being modeled (see Ref. 19). Finally, the points 1 and 2 have the same (x, y) coordinates and lie on opposite sides of the barrier. The quantum mechanics defined by (3.3) is intuitively clear and yet is rich enough to recover the three characteristic features of a Josephson junction. This will be done below.

To use this phenomenology to describe the interaction of electromagnetic waves with the Josephson transmission line, it should first be observed that the Josephson current is very weak.[13,15] In its absence the transmission line supports waves moving with the characteristic velocity s. Selecting waves traveling to the right and treating the Josephson current as a weak perturbation, the waves are essentially a function of $x-st$ only, $w(x-st)$. Thus, under the change of variables

$$x'=x-st, \qquad (3.4a)$$

$$t'=t, \qquad (3.4b)$$

the waves are essentially a function of x' alone, $w(x')$. Hence if the entire problem is done in the (x', t') coordinates defined by (3.4), the above phenomenological quantum theory can be used.

It will be shown that this approach yields a physical interpretation of the associated linear problem for the sine-Gordon equation. Of equal importance it yields, in the (x, t) coordinates, the proper j and the correct relationship between the derivatives of Δ and E and B. That is to say Josephson's original relations.

To use this phenomenological quantum mechanics, it is necessary to transform the $\vec{E}$ and $\vec{B}$ fields to the (x', t') frame. Since $s\simeq 0.05c$ for the Josephson transmission line,[25] it is sufficient to keep terms only through first order in s/c. Hence, denoting quantities in (x', t') coordinates by primes, the following approximations are adequate:

$$\begin{aligned}&E_1'=E_1,\\ &E_2'=0,\\ &E_3'=\gamma[E+(s/c)B]\simeq E+(s/c)B=E';\\ &B_1'=0,\\ &B_2'=\gamma[B+(s/c)E]\simeq[B+(s/c)E]=B',\\ &B_3'=0.\end{aligned} \qquad (3.5a)$$

These are consistent with the coordinate transformations

$$\begin{aligned}&x'=\gamma(x-st)\simeq x-st,\\ &\partial_{x'}=\gamma[\partial_x+(s/c^2)\partial_t]\simeq\partial_x,\\ &y'=y,\quad z'=z,\\ &t'=\gamma(t-xs/c^2)\simeq t,\\ &\partial_{t'}=\gamma(\partial_t+s\partial_x)\simeq\partial_t+s\,\partial x.\end{aligned} \qquad (3.5b)$$

Here $\gamma=[1-(s/c)^2]^{1/2}$. Similar expressions hold for $\vec{D}'$ and $\vec{H}'$. In addition, in Maxwell's equations $j_3=j_3'$.

In (x', t') coordinates the phenomenological quantum mechanics takes the form

$$i\hbar\partial_{t'}\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}=\begin{pmatrix}2e\int_1^2 E'\,dz & T\\ T^* & 2e\int_2^1 E'\,dz\end{pmatrix}\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}. \qquad (3.6)$$

By definition, the Josephson current is

$$j'=2e\,\partial_{t'}(\psi^+\psi^{+*}), \qquad (3.7)$$

evaluated in the insulator. With this definition Eq. (2.4) expressed in terms of (x', t') is

$$(\partial_{t'}-2s\partial_{x'})E'=-(8\pi e/\epsilon)\partial_{t'}(\psi^+\psi^{+*}). \qquad (3.8)$$

Equations (3.6) and (3.8) constitute the interacting nonlinear system to be analyzed.

IV. ANALYSIS OF THE INTERACTING SYSTEM

It is convenient to write Eq. (3.6) in terms of its real and imaginary parts. Defining $\Delta^{\pm}$ and $\rho^{\pm}$ by $\psi^{\pm}=(\rho^{\pm})^{1/2}e^{i\Delta^{\pm}}$, Eq. (3.6) implies

$$\partial_{t'}\rho^+=-\frac{2T^R}{h}(\rho^+\rho^-)^{1/2}\sin\Delta+\frac{2T^I}{h}(\rho^+\rho^-)^{1/2}\cos\Delta, \qquad (4.1a)$$

$$\partial_{t'}\rho^-=\frac{2T^R}{h}(\rho^+\rho^-)^{1/2}\sin\Delta-\frac{2T^I}{h}(\rho^+\rho^-)^{1/2}\cos\Delta, \qquad (4.1b)$$

$$\partial_{t'}\Delta^+ = -\frac{2e}{h}\int_1^2 E'\,dz' - \frac{T^R}{h}\left(\frac{\rho^+}{\rho^-}\right)^{1/2}\cos\Delta - \frac{T^I}{h}\left(\frac{\rho^-}{\rho^+}\right)^{1/2}\sin\Delta, \tag{4.1c}$$

$$\partial_{t'}\Delta^- = -\frac{2e}{h}\int_2^1 E'\,dz - \frac{T^R}{h}\left(\frac{\rho^-}{\rho^+}\right)^{1/2}\cos\Delta - \frac{T^I}{h}\left(\frac{\rho^+}{\rho^-}\right)^{1/2}\sin\Delta, \tag{4.1d}$$

where $T=T^R+iT^I$, $\Delta=\Delta^+-\Delta^-$. From (4.1a) it follows that j' across the barrier is specified in terms of the phase difference Δ as

$$j'_B = -\frac{4e}{h}(\rho^+\rho^-)^{1/2}(T^R\sin\Delta - T^I\cos\Delta). \tag{4.2a}$$

Since the Josephson current is very weak,[1,3,15] it does not appreciably affect the magnitude of $\psi^\pm$. Hence, as is customary, when coupling j back to the wave equation, $(\rho^+\rho^-)^{1/2}$ in (4.2) is replaced by a constant representing an average or initial value for $(\rho^+\rho^-)^{1/2}$. This constant is denoted by $\langle\!\langle\rho^+\rho^-\rangle\!\rangle$, and the source of the Maxwell field is written

$$j' = -(4e/h)\langle\!\langle(\rho^+\rho^-)^{1/2}\rangle\!\rangle(T^R\sin\Delta - T^I\cos\Delta). \tag{4.2b}$$

The replacement of $\rho^+\rho^-$ by its initial or average value $\langle\rho^+\rho^-\rangle$ just states that the Josephson current does not appreciably effect the charge densities of the bulk superconductor. This replacement can be clearly understood through a conservative perturbation argument. If the barrier is strong, the coupling constant T is small, $|T|\ll 1$. Since j is already $O(T)$, from (4.1a) and (4.1b), it follows that any time variation of $\langle\rho^+\rho^-\rangle$ is an $O(T^2)$ deviation and hence negligible.

In addition to giving the correct expression for the Josephson current (4.2a), the phenomenological quantum mechanics contains the equation connecting Δ to the electromagnetic field. This relation is obtained by subtracting (4.1d) from (4.1c):

$$\partial_{t'}\Delta = -\frac{4e}{h}\int_1^2 E'\,dz' + \frac{\rho^+-\rho^-}{h(\rho^+\rho^-)^{1/2}} \times (T^R\cos\Delta - T^I\sin\Delta). \tag{4.3}$$

Recalling again that the Josephson current is weak and the fact that the transmission line is not charged in bulk, it follows from (4.1a) and (4.1b) that $\rho^+-\rho^-$ is $O(T)$. Thus the second term in (4.3) is $O(T^2)$. Neglecting this term, (4.3) becomes

$$\partial_{t'}\Delta = -(4e/h)\int_1^2 E'\,dz'. \tag{4.4}$$

With these approximations the equations which govern the physical system are

$$(\partial_{t'} - 2s\partial_{x'})\int_1^2 E'\,dz' = -\frac{4\pi}{\epsilon}\left(-\frac{4e}{h}\langle\!\langle(\rho^+\rho^-)^{1/2}\rangle\!\rangle \times (T^R\sin\Delta - T^I\cos\Delta)\right), \tag{4.5a}$$

$$\partial_{t'}\Delta = -\frac{4e}{h}\int_1^2 E'\,dz', \tag{4.5b}$$

$$ih\,\partial_{t'}\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix} = \begin{pmatrix}2e\int_1^2 E'\,dz' & T\\ T^* & 2e\int_2^1 E'\,dz'\end{pmatrix}\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}. \tag{4.5c}$$

The system of equations (4.5) is redundant. Equation (4.5c) contains (4.5b). If only the E' field is of interest, (4.5a) and (4.5b) are sufficient. In fact using (4.5b) to eliminate E' from (4.5a) yields

$$(\partial_{t'} - 2s\partial_{x'})\partial_{t'}\Delta = \eta(T^R\sin\Delta - T^I\cos\Delta), \tag{4.6}$$

where

$$\eta = -64\pi e^2\langle\!\langle(\rho^+\rho^-)^{1/2}\rangle\!\rangle/h^2\epsilon.$$

By (4.5b) the time derivative of the solution of the nonlinear equation (4.6) is the field E'. Although it is true in principle that (4.5b) and (4.6) determine E', it is convenient to retain the linear problem (4.5c) as an aid to solving (4.6).

V. RECOVERY OF JOSEPHSON'S RELATIONS

Before discussing the inverse method of solution, it is shown that Josephson's relations, in (x,t) coordinates, can be obtained from (4.5). To do this observe first that Maxwell's equation (2.2b) in (x',t') coordinates is

$$-\partial_{z'}E'_1 + \partial_{x'}E' = \frac{1}{c}\partial_{t'}B'$$
$$\Rightarrow \partial_{x'}\int_1^2 E'\,dz = \partial_{t'}\int_1^2 \frac{B}{c}\,dz. \tag{5.1}$$

Replacing $\int_1^2 E'\,dz$ by (4.5b),

$$\partial_{t'}\partial_{x'}\Delta = -\frac{4e}{hc}\partial_{t'}\int_1^2 B'\,dz'. \tag{5.2}$$

Integrating (5.2) once yields

$$\partial_{x'}\Delta = -\frac{4e}{hc}\int_1^2 B'\,dz'. \tag{5.3}$$

Now in (x,t) coordinates, (4.6) is

$$(\partial_t^2 - s^2\partial_x^2)\Delta = \eta(T^R\sin\Delta - T^I\cos\Delta), \tag{5.4}$$

Eq. (4.5b) is

$$(\partial_t + s\partial_x)\Delta = -\frac{4e}{h}\int_1^2 \left(E + \frac{s}{c}B\right) dz \,, \tag{5.5}$$

and (5.3) is

$$\partial_x\Delta = -\frac{4e}{hc}\int_1^2 \left(B + \frac{s}{c}E\right) dz \,. \tag{5.6}$$

It follows that

$$\begin{aligned}(\partial_t + s\partial_x)\Delta - s\partial_x\Delta = \partial_t\Delta \\ = -\frac{4e}{h}\int_1^2 \left(E + \frac{s}{c}B\right) dz \\ + \frac{4e}{h}\int_1^2 \left(\frac{s}{c}B + \frac{s^2}{c^2}E\right) dz \,.\end{aligned} \tag{5.7}$$

If terms of order s^2/c^2 are neglected, (5.7) yields

$$\partial_t\Delta = -\frac{4e}{h}\int_1^2 E\, dz \,. \tag{5.8}$$

To the same order (5.6) is

$$\partial_x\Delta = -\frac{4e}{hc}\int_1^2 B\, dz \,. \tag{5.9}$$

Equations (5.4), (5.8), and (5.9) are the macroscopic relations Josephson uses to describe weakly coupled superconductors.[1-3] Thus the phenomenological quantum mechanics defined above contains Josephson's relations, complete with both the sine and cosine components of the tunneling current.

VI. CONNECTION WITH THE INVERSE METHOD

To relate the above analysis of the interacting system to the inverse method, recall first that the relevant equations for the junction are

$$(\partial_{t'} - 2s\partial_{x'})\int_1^2 E'\, dz = \eta(T^R \sin\Delta - T^I\cos\Delta)\,, \tag{6.1}$$

$$ih\,\partial_{t'}\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix} = \begin{pmatrix}2e\int_1^2 E'\,dz & T\\ T^* & -2e\int_1^2 E'\,dz\end{pmatrix}\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}, \tag{6.2}$$

$$\partial_{t'}\Delta = -(4e/h)\int_1^2 E'\, dz \,, \tag{6.3}$$

$$\partial_{x'}\Delta = -(4e/hc)\int_1^2 H'\, dz \,. \tag{6.4}$$

The strategy is to solve for Δ and then use (6.3) and (6.4) to find $\int_1^2 E'\,dz$ and $\int_1^2 H'\,dz$. The elimination of $\int_1^2 E'\,dz$ from (6.1) and (6.2) by using (6.3) yields

$$(\partial_{t'} - 2s\partial_{x'})\partial_{t'}\Delta = \eta(T^R\sin\Delta - T^I\cos\Delta)\,, \tag{6.5}$$

$$ih\,\partial_{t'}\begin{pmatrix}\tilde\psi^+\\ \tilde\psi^-\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}h\Delta_{t'} & T\\ T^* & \frac{1}{2}h\Delta_t\end{pmatrix}\begin{pmatrix}\tilde\psi^+\\ \tilde\psi^-\end{pmatrix}. \tag{6.6}$$

It is important to notice that (6.3) is an approximation to (4.3). Because of this, (6.6) is an approximation to the phenomenological quantum mechanics. The approximate wave functions are denoted by $\tilde\psi^\pm$. This approximation is consistent with those made in deriving (6.1), (6.3), and (6.4). It is to be emphasized that while the diagonal terms in the approximate quantum mechanics (6.6) are approximations to those in (6.2), the quantum mechanics retains the same basic structure. It is the approximate quantum mechanics (6.6) that is the key to the inverse method.

The inverse method deals not with the physical system (6.5)–(6.6) as it stands. Rather this method retains the approximate quantum mechanics (6.6), but replaces the nonlinear wave equation (6.5) with a pair of linear equations in $(\tilde\psi^+, \tilde\psi^-)$. This pair is chosen so as to carry the information that specifies the nonlinear wave. This set of four linear equations is integrable (defines the two functions $\tilde\psi^+$ and $\tilde\psi^-$) only if Δ satisfies (6.5). Thus the nonlinear wave equation becomes the integrability condition for a linear Pfaffian system.

More precisely, consider a family of linear equations (indexed by a parameter ζ) having the same structure as the quantum mechanics (6.6),

$$ih\,\partial_{t'}\begin{pmatrix}\tilde\psi^+\\ \tilde\psi^-\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}h\Delta_{t'} & \zeta\\ \zeta & \frac{1}{2}h\Delta_{t'}\end{pmatrix}\begin{pmatrix}\tilde\psi^+\\ \tilde\psi^-\end{pmatrix} = \mathcal{H}\begin{pmatrix}\tilde\psi^+\\ \tilde\psi^-\end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix}-\frac{1}{2}h\Delta_{t'} & \zeta\\ \zeta & \frac{1}{2}h\Delta_{t'}\end{pmatrix}. \tag{6.6'}$$

In the inverse method one assumes that the "x' evolution" of the pair $(\tilde\psi^+, \tilde\psi^-)$ is governed by a linear equation of the form

$$ih\,\partial_{x'}\begin{pmatrix}\tilde\psi^+\\ \tilde\psi^-\end{pmatrix} = \mathcal{B}\begin{pmatrix}\tilde\psi^+\\ \tilde\psi^-\end{pmatrix}, \tag{6.7}$$

where $\mathcal{B}$ is a matrix function of (x', t') and $\Delta(x', t')$. It is to be found. A linear ansatz such as (6.7) seems natural since quantum mechanics in a linear theory and the operator $ih\,\partial_{x'}$ represents the velocity operator in quantum mechanics.

If (6.6) and (6.7) are to hold simultaneously, certain consistency relations must hold. In particular, $\partial_{t'}$ applied to (6.7) must equal $\partial_{x'}$ applied to (6.6). This condition implies the matrix equation

$$ih\,\partial_{t'}\mathcal{B} = ih\,\partial_{x'}\mathcal{H} + (\mathcal{H}\mathcal{B} - \mathcal{B}\mathcal{H})\,. \tag{6.8}$$

If ζ is assumed independent of x' then in components, (6.8) is

$$b_{11,t'} = (i/h)(\zeta^* b_{12} - \zeta b_{21}) - \tfrac{1}{2}h\Delta_{t'x'}, \quad (6.9a)$$

$$b_{22,t'} = -b_{11,t'}, \quad (6.9b)$$

$$b_{12,t'} = i\Delta_{t'} b_{12} + (i\zeta/h)(b_{11} - b_{22}), \quad (6.9c)$$

$$b_{21,t'} = -i\Delta_{t'} b_{21} - (i\zeta^*/h)(b_{11} - b_{22}). \quad (6.9d)$$

It is clearly consistent to seek solutions satisfying

$$b_{11} = -b_{22} = b = b^*, \quad b_{12} = b_{21}^* = c^*. \quad (6.10)$$

If these restrictions are made, (6.9) reduces to

$$b_{t'} = (i/h)(\zeta^* C^* - \zeta C) - \tfrac{1}{2}h\Delta_{t'x'}, \quad (6.11a)$$

$$C_{t'}^* = i\Delta_{t'} C^* + (2i\zeta/h)b, \quad (6.11b)$$

$$C_{t'} = -i\Delta_{t'} C - (2i\zeta^*/h)b. \quad (6.11c)$$

The system (6.11) is solved by observing that Δ satisfies the nonlinear wave equation

$$-\tfrac{1}{2}h\Delta_{t'x'} = -(h/4s)\Delta_{t't'} + (h/4s)F(\Delta);$$
$$F(\Delta) = \eta(T^R \sin\Delta - T^I \cos\Delta), \quad (6.12)$$

and thus (6.11) can be written in the form

$$b_{t'} = \frac{i}{h}(\zeta^* C^* - \zeta C) - \frac{h}{4s}\Delta_{t't'} + \frac{h}{4s}F(\Delta), \quad (6.12a)$$

$$(\zeta^* C^* - \zeta C)_{t'} = i\Delta_{t'}(\zeta^* C^* + \zeta C) + (4i|\zeta|^2/h)b, \quad (6.12b)$$

$$(\zeta^* C^* + \zeta C)_{t'} = i\Delta_{t'}(\zeta^* C^* - \zeta C). \quad (6.12c)$$

It is readily verified that

$$b = -(h/4s)\partial_{t'}\Delta, \quad (6.13a)$$

$$\zeta^* C^* - \zeta C = +(ih^2/4s)F(\Delta), \quad (6.13b)$$

$$\zeta^* C^* + \zeta C = (h^2/4s)F'(\Delta) + s^{-1}|\zeta|^2, \quad (6.13c)$$

provided $F''(\Delta) = -F(\Delta)$, which $F(\Delta)$ does in fact satisfy. It is interesting to note that this condition has arisen previously in a Backlund-transformation approach to the sine-Gordon equation.[20] Equations (6.13) yield the matrix $\mathcal{B}$,

$$b_{11} = -b_{22} = -(h/4s)\partial_{t'}\Delta, \quad (6.14a)$$

$$b_{12} = b_{21}^* = (h^2\eta T^*/8s\zeta^*)e^{i\Delta} + \zeta/2s. \quad (6.14b)$$

In summary, for the nonlinear wave equation

$$(\partial_{t'} - 2s\partial_{x'})\partial_{t'}\Delta = \eta(T^R \sin\Delta - T^I \cos\Delta), \quad (6.15)$$

the linear problem, which arises from the quantum mechanics, is given by

$$ih\partial_{t'}\begin{pmatrix}\tilde\psi^+\\ \tilde\psi^-\end{pmatrix} = \mathcal{H}\begin{pmatrix}\tilde\psi^+\\ \tilde\psi^-\end{pmatrix},$$
$$\mathcal{H} \equiv \begin{pmatrix} -\tfrac{1}{2}h\partial_{t'}\Delta & \zeta \\ \zeta^* & \tfrac{1}{2}h\partial_{t'}\Delta\end{pmatrix}; \quad (6.16)$$

$$ih\partial_{x'}\begin{pmatrix}\tilde\psi^+\\ \tilde\psi^-\end{pmatrix} = \mathcal{B}\begin{pmatrix}\tilde\psi^+\\ \tilde\psi^-\end{pmatrix},$$
$$\mathcal{B} \equiv \begin{pmatrix} -\dfrac{h}{4s}\partial_{t'}\Delta & \dfrac{\zeta}{2s} + \dfrac{h^2\eta T^*}{8s\zeta^*}e^{i\Delta} \\ \dfrac{\zeta^*}{2s} + \dfrac{h^2\eta T^*}{8s\zeta^*}e^{-i\Delta} & \dfrac{h}{4s}\partial_{t'}\Delta \end{pmatrix}. \quad (6.17)$$

No claim is made as to the uniqueness of the matrix $\mathcal{B}$. However, it is claimed that Eqs. (6.16) and (6.17) are consistent for this $\mathcal{B}$ matrix because Δ satisfies the nonlinear wave equation. If Δ did not satisfy the nonlinear wave equation (6.12), the linear problem (6.16) and (6.17) would not be consistent. In this sense the linear equations carry the information contained in the nonlinear wave equation. The inverse method[7-9] exploits this feature and uses Gelfand-Levitan theory to construct Δ from the (known) scattering data of the linear problem.

For completeness (6.16) and (6.17) are expressed in (x, t) coordinates

$$ih\partial_t\tilde\psi^+ = -\tfrac{1}{4}h(\Delta_t + s\Delta_x)\tilde\psi^+ + \frac{1}{2}\left(\zeta - \frac{h^2\eta T^*}{4\zeta^*}e^{i\Delta}\right)\tilde\psi^-, \quad (6.18a)$$

$$ih\partial_t\tilde\psi^- = \frac{1}{2}\left(\zeta^* - \frac{h^2\eta T}{4\zeta}e^{-i\Delta}\right)\tilde\psi^+ + \tfrac{1}{4}h(\Delta_t + s\Delta_x)\tilde\psi^-, \quad (6.18b)$$

$$ih\partial_x\tilde\psi^+ = -\frac{h}{4s}(\Delta_t + s\Delta_x)\tilde\psi^+ + \frac{1}{2s}\left(\zeta + \frac{h^2\eta T^*}{4\zeta^*}e^{i\Delta}\right)\tilde\psi^-, \quad (6.19a)$$

$$ih\partial_x\tilde\psi^- = \frac{1}{2s}\left(\zeta^* + \frac{h^2\eta T}{4\zeta}e^{-i\Delta}\right)\tilde\psi^+ + \frac{h}{4s}(\Delta_t + s\Delta_x)\tilde\psi^-. \quad (6.19b)$$

To compare this linear problem with those appearing in the literature on the inverse method, it is convenient to change dependent variables

$$v^+ = \tilde\psi^+ + \tilde\psi^-, \quad (6.20a)$$

$$v^- = i(\tilde\psi^+ - \tilde\psi^-). \quad (6.20b)$$

The linear problems discussed in the literature correspond to the case of real T so attention is restricted to that case. It should be noted that as long as T is constant, the complex nature of T can be removed from (6.9) by the change of variables $\Delta \to \Delta + c$; $\tilde\psi^+ \to \tilde\psi^{+\prime}e^{ic/2}$; $\tilde\psi^- \to \tilde\psi^{-\prime}e^{-ic/2}$, where the constant c denotes the phase of T. Complex T have been explicitly treated here in order to show that the phenomenological quantum mechanics is rich enough to contain both the sine and cosine components of the Josephson current.[2] It should further be observed that (6.18) is valid for variable T. The restriction to constant T was made when $\mathfrak{B}$ was sought.

Returning to the problem at hand, for real ζ and T and using the transformation (6.20), (6.18) and (6.19) become

$$h\,\partial_t v^{\pm} = \pm\frac{1}{2i}\left(\zeta - \frac{h^2\eta T}{4\zeta}\cos\Delta\right)v^{\pm} \mp \left(-\frac{h}{4}(\Delta_t + s\Delta_x) \pm \frac{ih^2\eta T}{8\zeta}\sin\Delta\right)v^{\mp}, \quad (6.21a)$$

$$h\,\partial_x v^{\pm} = \pm\frac{1}{2is}\left(\zeta + \frac{h^2\eta T}{4\zeta}\cos\Delta\right)v^{\pm} \mp \left(-\frac{h}{4s}(\Delta_t + s\Delta_x) \mp \frac{ih^2\eta T}{8s\zeta}\sin\Delta\right)v^{\mp}. \quad (6.21b)$$

This linear problem agrees with that used by Ablowitz *et al.*[7] to solve the sine-Gordon equation. Thus it has been shown that the associated linear problem used to solve the sine-Gordon equation by the inverse method is not a purely mathematical device which happens to be useful but rather a basic equation of the physical system being analyzed.

In the Appendix a second example illustrating this same feature is given. The Appendix is an exposition of Lamb's work on the SIT equations. There his work is presented in a manner which emphasizes its connection with the linear physical problem. In addition, a precise analogy between the SIT systems and the Josephson transmission line is developed in some detail.

VII. REPRESENTATION IN TERMS OF PHYSICAL QUANTITIES

Equations for physical quantities, such as charge and current, can be derived from the equations for $\psi^{\pm}$ in the preceding sections. To obtain these, use is made of (6.16) and (6.17) with $\zeta = T$. For convenience T is taken to be real.

$$j = -(4e/h)(\rho^+\rho^-)^{1/2}\,T\sin\Delta. \quad (7.1)$$

The surface coupling energy f of the barrier is related to the Josephson current j (Ref. 2) by

$$j = \frac{2e}{h}\frac{\partial f}{\partial\Delta}, \quad (7.2)$$

where f is defined by

$$f = 2(\rho^+\rho^-)^{1/2}T\cos\Delta. \quad (7.3)$$

Thus, in terms of ψ^+ and ψ^-, Eq. (3.6), the densities of physical interest are

$$Q^+ \equiv 2e(\psi^+\psi^{+*} + \psi^-\psi^{-*}), \quad (7.4a)$$

$$Q^- \equiv 2e(\psi^+\psi^{+*} - \psi^-\psi^{-*}), \quad (7.4b)$$

$$j \equiv (2eTi/h)(\psi^+\psi^{-*} - \psi^-\psi^{+*}), \quad (7.4c)$$

$$f \equiv T(\psi^+\psi^{-*} + \psi^-\psi^{+*}), \quad (7.4d)$$

where Q^+ is the total charge on the junction, Q^- is the charge difference across the junction, j is the Josephson current, and f is the barrier energy. In terms of these variables, the basic problem to be solved, Eqs. (3.6) and (3.8), takes the equivalent form

$$(\partial_{t'} - 2s\partial_{x'})\int_1^2 E'\,dz = -(4\pi/\epsilon)j, \quad (7.5)$$

$$\partial_{t'}Q^+ = 0, \quad (7.5a)$$

$$\partial_{t'}Q^- = 2j, \quad (7.5b)$$

$$\partial_{t'}f = \left(-2\int_1^2 E'\,dz\right)j, \quad (7.5c)$$

$$\partial_{t'}j = \left(2e\int_1^2 E'\,dt\right)\left(\frac{4e}{h^2}\right)f - \frac{2T^2}{h^2}Q^-. \quad (7.5d)$$

However, this problem is replaced with the approximation (4.5). Define

$$\tilde{Q}^+ \equiv 2e(\tilde{\psi}^+\tilde{\psi}^{+*} + \tilde{\psi}^-\tilde{\psi}^{-*}), \quad (7.6a)$$

$$\tilde{Q}^- \equiv 2e(\tilde{\psi}^+\tilde{\psi}^{+*} - \tilde{\psi}^-\tilde{\psi}^{-*}), \quad (7.6b)$$

$$\tilde{j} \equiv (2eTi/h)(\tilde{\psi}^+\tilde{\psi}^{-*} - \tilde{\psi}^-\tilde{\psi}^{+*}), \quad (7.6c)$$

$$\tilde{f} \equiv T(\tilde{\psi}^+\tilde{\psi}^{-*} + \tilde{\psi}^-\tilde{\psi}^{+*}). \quad (7.6d)$$

Then, using the dynamics of Eq. (4,5), (7.5) is replaced by the following approximate system:

$$(\partial_{t'} - 2s\partial_{x'})\int_1^2 E'\,dz = \frac{16\pi e}{\epsilon h}n_0 T\sin\Delta, \quad (7.7a)$$

$$\Delta_{t'} = -\frac{4e}{h}\int_1^2 E'\,dz; \quad (7.7b)$$

$$\partial_{t'}\tilde{Q}^+ = 0, \quad (7.8a)$$

$$\partial_{t'}\tilde{Q}^- = 2\tilde{j}, \quad (7.8b)$$

$$\partial_{t'}\tilde{f} = [(h/2e)\Delta_{t'}]\tilde{j}, \quad (7.8c)$$

$$\partial_{t'}\tilde{j} = \left(-\frac{h}{2}\Delta_{t'}\right)\left(\frac{4e}{h^2}\right)\tilde{f} - \frac{2T^2}{h^2}\tilde{Q}^-. \quad (7.8d)$$

The last four equations are redundant, but are equivalent to the linear problem used to solve the nonlinear wave equation. Using equation (6.17) with $\zeta = T$, the x' variation of the physical variables is found to be

$$2s\,\partial_{x'}\tilde{Q}^+ = 0, \quad (7.9a)$$

$$2s\,\partial_{x'}\tilde{Q}^- = 2\tilde{j} + (16\pi e^2/\epsilon T^2)(j\tilde{f} - f\tilde{j}), \quad (7.9b)$$

$$2s\,\partial_{x'}\tilde{f} = \left(\frac{h}{2e}\,\Delta_{t'}\right)\tilde{j} - \frac{4\pi}{\epsilon}\,j\tilde{Q}^{-}\,, \tag{7.9c}$$

$$2s\,\partial_{x'}\tilde{j} = \left(-\frac{h}{2}\,\Delta_{t'}\right)\left(\frac{4e}{h^2}\right)\tilde{f} - \frac{2T^2}{h^2}\,\tilde{Q}^{-} + \frac{16\pi e^2}{\epsilon h^2} f\tilde{Q}^{-}\,. \tag{7.9d}$$

This section concludes with some comments concerning equation (7.7) and (7.8). First, notice that Eqs. (7.8a) and (7.9a) show that Q^{+}, the total charge (density) on the junction, is constant in both space and time. The junction is initially uncharged and remains uncharged. Second, notice that Eq. (7.8b) is merely the definition of the Josephson current. Equations (7.8b) and (7.9b), taken together, would imply that Q^{-} moves at the speed of linear propagation $[Q^{-} = Q^{-}(x - st)]$ if it were not for the second term in (7.9b). This term is a correction arising from the barrier penetration current.

Consider now Eqs. (7.8c) and (7.8d) and (7.9c) and (7.9d). They may be rewritten as

$$\partial_{t'}\tilde{f} = \left(-2\int_1^2 E'\,dz\right)\tilde{j}\,,$$

$$2s\,\partial_{x'}\tilde{f} = -2\left(\int_1^2 E'\,dz\right)\tilde{j} + 2(\tilde{E})j\,,$$

$$\partial_{t'}\tilde{j} = \frac{4e}{h^2}\left(2e\int_1^2 E'\,dz\right)\tilde{f} - \frac{2T^2}{h^2}\,\tilde{Q}^{-}\,,$$

$$2s\,\partial_{x'}\tilde{j} = \frac{4e}{h^2}\left(2e\int_1^2 E'\,dz\right)\tilde{f} - \frac{2T^2}{h^2}\,\tilde{Q}^{-} - \frac{4e}{h^2}\,(2e\tilde{E})f\,,$$

where the additional electric field $\tilde{E}$ due to the charge separation Q^{-} has been defined by

$$\tilde{E} \equiv -2\pi\tilde{Q}^{-}/\epsilon\,.$$

As above, if it were not for the last terms in the expressions for $\partial_{x'}\tilde{f}$ and $\partial_{x'}\tilde{j}$, these equations would imply $\tilde{j} = \tilde{j}(x - st), \tilde{f} = \tilde{f}(x - st)$. However, these last two terms break this translational symmetry.

Notice that the expression $\partial_{t'}\tilde{f}$ represents power and is naturally given by $jV = j(-2dE')$. On the other hand, the "moving derivative" $2s\partial_{x'}\tilde{f}$ also represents a power. Here an additional term appears representing the work done on the Josephson current j by the electric field $\tilde{E}$ which arises from the charge separation $\tilde{Q}^{-}$.

APPENDIX A: GENERAL DESCRIPTION OF SIT

Here a description of ultrashort optical pulses interacting with a two-level quantum system is summarized and presented in a fashion that emphasizes the connection between this quantum mechanics and the linear problem of the inverse method. In addition, this appendix displays concrete analogies between this model of the self-induced transparency (SIT) of optical pulses and the model of the Josephson transmission line discussed in the text. The existence of such an analogy is suggested in Josephson's review article.[1]

Thorough discussions of the physics of SIT may be found in the review article by G. Lamb[10] and the paper by A. Icsevgi and W. E. Lamb.[22] G. Lamb[11,12] was the first to solve the SIT equations by the inverse method; Ablowitz, Newell, and Kaup[13,23] have developed the method and extended its application to SIT considerably; see also the work of Gibbon, Caudrey, Bullough, and Eilbeck.[4] The main point of this appendix is to display the direct connection between physics (quantum mechanics) and the linear inverse method. It begins with a physical description of SIT.

Ultrashort pulses of light can travel in a two-level optical medium as if it were transparent. This effect can be explained as follows. The time interval of an ultrashort pulse (10^{-9}–10^{-12} sec) is less than the phase memory time of the atomic levels in the optical medium; therefore, the induced polarization can retain a definite phase relationship with the incident pulse. The leading edge of the pulse then inverts the atomic population, while the trailing edge returns it to the ground state via stimulated emission. Thus, the energy transferred from the leading edge of the pulse to the quantum system is recaptured by the trailing edge. The result, under proper conditions of coherence and intensity, is a steady pulse profile which propagates without attenuation at a velocity that can be two or three orders of magnitude less than the phase velocity of light in the medium.

In order to model this effect, Maxwell's equations are needed to describe the light wave, and an assembly of quantum (two-level) atoms to describe the medium. The light wave polarizes the atoms which, acting together, become a source of the electromagnetic field.

For the SIT situation[10,22] the electromagnetic field E satisfies

$$\left(\partial_{xx} - \frac{1}{s^2}\,\partial_{tt}\right)E = \frac{4\pi}{c^2}\,\partial_{tt}P\,. \tag{A1}$$

Here E is specialized to a plane wave traveling in the x direction and nonresonant losses have been neglected, s denotes the speed of light in the dielectric medium, and P denotes an additional polarization due to the interaction of the electromagnetic wave with the two-level quantum mechanics of the medium. The quantum-mechanical model for P is now described.

APPENDIX B: THE QUANTUM MECHANICS

Consider the wave function for a typical atom in the medium,

$$\psi(\vec{R}, t; \vec{r})\,. \tag{B1}$$

Here $\vec{R}$ denotes the position of the atom and $\vec{r}$ denotes collectively the N electron coordinates. Neglecting any motion of the atom itself, the Hamiltonian of the system can be assumed to act only on the electron coordinates. In the absence of external fields, the dynamics of the system can be expressed by the Schrödinger equation

$$i\hbar\,\partial_t\psi = \mathcal{H}_{\text{atm}}\psi . \tag{B2}$$

Expanding ψ in terms of the energy eigenfunctions of $\mathcal{H}_{\text{atm}}$ yields

$$\psi(\vec{R}, t; \vec{r}) = a(t, \vec{R})\psi_a(\vec{r}) + b(t, \vec{R})\psi_b(\vec{r}) + \text{(negligible terms)} . \tag{B3}$$

Here the orthonormal eigenfunctions ψ_a and ψ_b satisfy[24]

$$\mathcal{H}_{\text{atm}}\psi_F = \omega_F\psi_F ;$$
$$\langle\psi_F|\psi_{F'}\rangle = \delta_{FF'}; \quad F = a, b; \quad \omega_a > \omega_b . \tag{B4}$$

Only two terms are kept in this energy expansion because of the two-level assumption. For simplicity the following symmetry is assumed:

$$\langle\psi_F|\vec{r}_i|\psi_F\rangle = 0 ,$$
$$\langle\psi_F|\vec{r}_i|\psi_{F'}\rangle = \langle\psi_{F'}|\vec{r}_i|\psi_F\rangle,$$
$$F = a, b; \quad i = 1, 2, \ldots, N . \tag{B5}$$

The external field $\vec{E} = E\hat{z}$ acts as a perturbation on this quantum mechanics, the Hamiltonian taking the form

$$\mathcal{H} = \mathcal{H}_{\text{atm}} + \mathcal{H}_{\text{int}} . \tag{B6}$$

In the dipole approximation the interaction Hamiltonian $\mathcal{H}_{\text{int}}$ takes the form

$$\mathcal{H}_{\text{int}} = -E\hat{\mathcal{P}}, \quad \hat{\mathcal{P}} \equiv -e\sum_{j=1}^{N} z_j , \tag{B7}$$

where $E(\vec{R}, t)$ denotes the electric field at the location of the atom. Thus, the dynamics takes the form

$$i\hbar\,\partial_t\psi = (\mathcal{H}_{\text{atm}} + \mathcal{H}_{\text{int}})\psi , \tag{B8}$$
$$\psi = a\psi_a + b\psi_b .$$

Using the orthogonality of ψ_a and ψ_b, this dynamics may be written in two component form

$$i\hbar\,\partial_t\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}\hbar\omega_a & -Ep\\ -Ep & \hbar\omega_b\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}, \tag{B9}$$

where $p \equiv \langle\psi_a|\hat{\mathcal{P}}|\psi_b\rangle = \langle\psi_b|\hat{\mathcal{P}}|\psi_a\rangle$.

The field $\vec{E}$ perturbs the initial unpolarized states ψ_a and ψ_b into a nonsymmetric state ψ, and thus induces a polarization $\mathcal{P}$. To calculate this polarization, consider the following matrix element:

$$\mathcal{P} \equiv \langle\psi|\hat{\mathcal{P}}|\psi\rangle = p(ab^* + ba^*) . \tag{B10}$$

In order to couple this polarization back to Maxwell's equation (A3), it is necessary to compute $\partial_{tt}\mathcal{P}$. To do this define the quadratic quantities

$$\mathcal{P} \equiv p(ab^* + ba^*) , \tag{B11a}$$
$$T \equiv (aa^* + bb^*) , \tag{B11b}$$
$$N \equiv (aa^* - bb^*) , \tag{B11c}$$
$$Q \equiv ip(ab^* - ba^*) , \tag{B11d}$$

where $\mathcal{P}$ is the polarization, T is the total probability density, and N is the population inversion. The two-level dynamics [Eq. (B9)] may be expressed in an equivalent form in terms of these quadratic quantities:

$$\partial_t T = 0 , \tag{B12a}$$
$$\partial_t N = -(2/\hbar)EQ , \tag{B12b}$$
$$\partial_t\mathcal{P} = \omega Q , \tag{B12c}$$
$$\partial_t Q = \omega\mathcal{P} + (2/\hbar)(pE)N, \quad \omega = \omega_a - \omega_b . \tag{B12d}$$

Using (B12), $\partial_{tt}\mathcal{P}$ is found to satisfy

$$\partial_{tt}\mathcal{P} = -\omega^2\mathcal{P} - [(2p^2/\hbar)E]\omega N . \tag{B13}$$

The macroscopic polarization P is given by $P = \langle\mathcal{P}\rangle$, where $\langle\cdots\rangle$ denotes an averaging process that maps the microscopic quantities into macroscopic quantities. It will be specified more precisely below. Thus, the complete physical system to be solved is given by equations (A1) and (B12) with $\partial_{tt}P = \langle\partial_{tt}\mathcal{P}\rangle$ given by (B13). Equivalently,

$$\left(\partial_{xx} - \frac{1}{s^2}\,\partial_{tt}\right)E = \frac{4\pi}{c^2}\,\langle\partial_{tt}\mathcal{P}\rangle ,$$
$$i\hbar\partial_t\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}\hbar\omega_a & -pE\\ -pE & \hbar\omega_b\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}, \tag{B14}$$

with $\partial_{tt}\mathcal{P}$ satisfying

$$\partial_{tt}\mathcal{P} = -\omega^2\mathcal{P} - (2p^2E/\hbar)\omega N . \tag{B15}$$

It is important to notice that Eqs. (B11), (B12), (B14), and (B15) are directly analogous to Eqs. (7.4), (7.5), (2.4), and (3.7) which describe the Josephson-junction system. The analogies are $E' \approx E$, $Q^- \approx \mathcal{P}$, $\psi^+ \approx \boldsymbol{a} + \boldsymbol{b}$, $\psi^- \approx \boldsymbol{a} - \boldsymbol{b}$.

APPENDIX C: AN APPROXIMATE INTERACTING SYSTEM

Just as in the case of the junction, the full system (B14) is not analyzed but rather is replaced by an approximate system. First Lamb approximates the source $(\partial_{tt}\mathcal{P})$ since, "For the resonant situation being considered, the term $\partial_{tt}\mathcal{P}$ may be replaced by $-\omega_0^2\mathcal{P}$, where ω_0 is the carrier frequency of the incident pulse."[10] This follows from (B13) provided the carrier frequency $\omega_0 \approx \omega$, and provided $\omega \gg (p^2E/h)$,

$$\partial_{tt}\mathcal{P} = -\omega^2\mathcal{P} - (2\,|\mathcal{P}|^2/h)\omega EN,$$
$$\simeq -\omega_0^2\mathcal{P}. \tag{C1}$$

It is to be emphasized that, at this point, the approximation is precisely the opposite of that used in the junction problem. For the junction, the exact source [Eq. (7.5)] satisfies $-2\partial_{t'}j_B$ equal to

$$\partial_{tt}Q^- = -\left(\frac{4T^2}{h}\right)Q^- + \left(\frac{16e^2d}{h^2}\right)E'f. \tag{C2}$$

For the junction, the coupling coefficient T is small. Thus the first term in (C2), being $O(T^2)$, is neglected. This perturbation result is precisely the opposite of the SIT case, where the first term is retained because of the *resonance* situation, while the second is neglected.

To see for the junction that the first term is indeed neglected, recall $f = 2n_0T\cos\Delta$, $\Delta_t = -4edE'/h$. These expressions yield

$$\partial_{t'}2j_B = -\partial_{t't'}Q^-$$
$$= \frac{8n_0eT}{h}\cos(\Delta)\Delta_{t'} + \frac{4T^2}{h^2}Q^-,$$
$$= \frac{8n_0eT}{h}\partial_{t'}(\sin\Delta) + O(T^2). \tag{C2$'$}$$

Upon integration, Eq. (C2$'$) yields the Josephson current after $O(T^2)$ terms have been neglected,

$$j_B = \frac{4n_0eT}{h}\sin\Delta + O(T^2). \tag{C3}$$

The above paragraphs show the precise mathematical differences between this model of the Josephson transmission line and the two-level model for SIT pulses in nonlinear optics. If one considers retaining the first term $(4T^2Q^-/h)$ in Eq. (C2), the model suggests the possibility of SIT pulses on the Josephson transmission line. Such resonances would be in addition to those which involve the "ac Josephson current" interacting with a small-amplitude ac field. It seems of interest to check if these SIT-like resonances, which are present in this phenomenological model, could be realized experimentally.

Returning to the SIT case, the first approximation is the replacement of the source $\partial_{tt}\mathcal{P}$ by $-\omega_0^2\mathcal{P}$, where ω_0 ($\approx\omega$) is the carrier frequency of the electric field. In a related approximation, the electric field is written in terms of the carrier wave and slowly varying envelope and phase functions,

$$E(x,t) = \mathcal{E}(x,t)\cos[k_0x - \omega_0t + \phi(x,t)],$$
$$s^2k_0^2 = \omega_0^2. \tag{C4}$$

It is assumed that the envelope and phase functions vary slowly on the length and time scales of the carrier wave, $\omega_0\mathcal{E} \gg \partial_t\mathcal{E}$ and $k_0\mathcal{E} \gg |\nabla\mathcal{E}|$. Under these assumptions the wave equation reduces to

$$[(\partial_t + s\partial_x)\mathcal{E}]\sin(k_0x - \omega_0t + \phi)$$
$$+ [(\partial_t + s\partial_x)\phi]\mathcal{E}\cos(k_0x - \omega_0t + \phi) = 2\pi\omega_0\langle\mathcal{P}\rangle. \tag{C5}$$

Here $\langle\mathcal{P}\rangle$ denotes the map from microscopic polarization to macroscopic polarization. "In general, it is appropriate to allow for a continuous distribution of transition frequencies about ω":

$$\langle\mathcal{P}\rangle = n_0\int_{-\infty}^{\infty} d\Delta\, g(\Delta)\,\mathcal{P}(\Delta; x, t),$$

where $g(\Delta)$ describes the uncertainty of the energy level ω, and $g(\ )$ is normalized so that

$$\int_{-\infty}^{\infty} d\Delta\, g(\Delta) = 1.$$

These first two approximations determine the form of Maxwell's equation to be used, Eq. (C5). Next, the quantum mechanics [Eq. (B14)] which specifies the source $\langle\mathcal{P}\rangle$ is approximated. This derivation of the approximation differs from Lamb's, but yields equivalent results. This approach has the advantage of providing a direct physical interpretation of the linear problem which is used to solve the SIT equations.

Consider the quantum dynamics governed by equation (B14). It may be written as

$$ih\,\partial_t\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}h\omega_a & 0\\ 0 & h\omega_b\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} + \begin{pmatrix}0 & p\mathcal{E}\cos(k_0x - \omega_0t + \phi)\\ p\mathcal{E}\cos(k_0x - \omega_0t + \phi) & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}. \tag{C6}$$

The E field in this interaction Hamiltonian may be written as

$$\tfrac{1}{2}\mathcal{E}\{\exp[i(k_0x - \omega_0t + \phi)] + \exp[-i(k_0x - \omega_0t + \phi)]\}. \tag{C7}$$

As in the study of paramagnetic resonance, only one component in (C7) contributes significantly in producing level transitions.[21] Thus, we replace the quantum mechanics (C6) by the approximate quantum mechanics

$$i\hbar\,\partial_t\begin{pmatrix}\tilde a\\ \tilde b\end{pmatrix}=\begin{pmatrix}\hbar\omega_a & 0\\ 0 & \hbar\omega_b\end{pmatrix}\begin{pmatrix}\tilde a\\ \tilde b\end{pmatrix}+\begin{pmatrix}0 & -\frac{1}{2}p\mathcal{E}e^{i(k_0x-\omega_0t+\phi)}\\ -\frac{1}{2}p\mathcal{E}e^{-i(k_0x-\omega_0t+\phi)} & 0\end{pmatrix}\begin{pmatrix}\tilde a\\ \tilde b\end{pmatrix}. \tag{C8a}$$

This quantum mechanics, together with Maxwell's equation in the form

$$[(\partial_t+s\partial_x)\mathcal{E}]\sin(k_0x-\omega_0t+\phi)+[(\partial_t+s\partial_x)\phi]\mathcal{E}\cos(k_0x-\omega_0t+\phi)=2\pi\omega_0p\langle\tilde a\tilde b^*+\tilde b\tilde a^*\rangle, \tag{C8b}$$

constitute the approximate interacting system to be studied.

APPENDIX D: ANALYSIS OF THE APPROXIMATE SYSTEMS

In analyzing the quantum mechanics, (C8a), the fast variation $\exp[i(k_0x-\omega_0t)]$ can be removed exactly from the problem by the change of variables

$$\tilde A=e^{-i(k_0x-\omega_0t)/2}\tilde a,\qquad \tilde B=e^{+i(k_0x-\omega_0t)/2}\tilde b, \tag{D1}$$

which reduces this quantum mechanics to

$$\begin{pmatrix}\tilde A\\ \tilde B\end{pmatrix}=\begin{pmatrix}\hbar(\omega_a-\frac{1}{2}\omega_0) & -\frac{1}{2}p\mathcal{E}e^{i\phi}\\ -\frac{1}{2}p\mathcal{E}e^{-i\phi} & \hbar(\omega_b+\frac{1}{2}\omega_0)\end{pmatrix}\begin{pmatrix}\tilde A\\ \tilde B\end{pmatrix}. \tag{D2}$$

After some algebraic manipulation, Maxwell's equation (C8b) takes the form

$$(\partial_t+s\partial_x)\mathcal{E}e^{i\phi}=2\pi\omega_0P_{ab}\langle 2i\tilde A\tilde B^*\rangle. \tag{D3}$$

Defining $\partial_t+s\partial_x\equiv\partial_\xi$ and selecting the zero of the energy such that $(\zeta/\hbar)\equiv(2\omega_a-\omega_0)=(2\omega_b+\omega_0)$, the final form of the SIT equations results:

$$\partial_\xi(\mathcal{E}e^{i\phi})=2\pi\omega_0p\langle 2i\tilde A\tilde B^*\rangle, \tag{D4a}$$

$$i\hbar\,\partial_t\begin{pmatrix}\tilde A\\ \tilde B\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\zeta & p\mathcal{E}e^{i\phi}\\ p\mathcal{E}e^{-i\phi} & -\zeta\end{pmatrix}\begin{pmatrix}\tilde A\\ \tilde B\end{pmatrix}. \tag{D4b}$$

This quantum mechanics (D4b) is identical to the linear eigenvalue problem introduced in Ref. 13 (see also Refs. 11, 12, and 14) to solve the SIT equations by the inverse method. As with the junction, it arises naturally from the quantum mechanics of the medium supporting the nonlinear wave. As discussed above, one now introduces a linear problem which describes evolution in x,

$$i\hbar\,\partial_x\begin{pmatrix}\tilde A\\ \tilde B\end{pmatrix}=\mathcal{B}\begin{pmatrix}\tilde A\\ \tilde B\end{pmatrix}. \tag{D5}$$

The matrix $\mathcal{B}$ is suitably chosen so as to carry the information described by Eq. (D4a). Then Gelfand Levitan theory is applied to construct the $\mathcal{E}$ field. For details, see Ref. 13.

We remark that usually the SIT equations are expressed in a slightly different form than (D4). Define the quadratic quantities by

$$\tilde T=\tilde A\tilde A+\tilde B\tilde B^*,\quad \tilde N=\tilde A\tilde A^*-\tilde B\tilde B^*,$$
$$\lambda'=2ipAB^*,\quad \lambda^*=-2ipA^*B. \tag{D6}$$

In terms of these variables, (D4) can be expressed in more standard form:

$$\begin{aligned}
&\partial_\xi[\mathcal{E}e^{i\phi}]=2\pi\omega_0p\langle\lambda\rangle,\\
&\partial_t\tilde T=0,\\
&\partial_t\tilde N=-(p/2\hbar)[(\mathcal{E}e^{i\phi})\lambda^*+(\mathcal{E}e^{-i\phi})\lambda],\\
&\partial_t\lambda=-(i\zeta/\hbar)\lambda+(p^2\mathcal{E}e^{i\phi}/\hbar)N,\\
&\partial_t\lambda^*=(i\zeta/\hbar)\lambda^*+(p^2\mathcal{E}e^{-i\phi}/\hbar)N.
\end{aligned} \tag{D7}$$

This last form is the starting point in Ref. 13.

In summary, we have gone to some length in these appendices to describe the precise relationship between the linear problem used in the inverse solution of SIT equations and the quantum mechanics of the interacting SIT system. In addition, we have drawn a precise analogy between SIT and the Josephson junction. In both, the interaction of the electromagnetic wave with the medium is described by semiclassical radiation theory. And in both cases the "exact" interacting system is replaced by an approximate interacting system. In the SIT case, the approximation is based upon the resonance interaction between the frequency of the carrier wave and the excitation energy for the two-level medium: for the Josephson junction the approximation is based on the small magnitude of the Josephson current. In both cases the approximation forces the electromagnetic waves to be governed by a nonlinear wave equation which is solved by the inverse method. In both cases the linear problem used in the inverse method is the approximate quantum mechanics. In the case of the junction, this approximate quantum mechanics is related to the exact quantum mechanics by a perturbation theory with respect to the strength of the Josephson current. For the SIT system, this approximate quantum mechanics is obtained from the actual quantum mechanics through a resonance and fast variation argument.

Note added in proof. In Ref. 27 the two-level description of the Josephson junction is derived from a more fundamental theory. There the validity of this two-level description is discussed.

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[1]B. D. Josephson, Advan. Phys. 14, 419 (1965).
[2]B. D. Josephson, in *Superconductivity*, edited by R. D. Parks (Marcel Dekker, New York, 1969).
[3]B. D. Josephson, Rev. Mod. Phys. 36, 216 (1964).
[4]A. C. Scott, *Active and Nonlinear Wave Propagation in Electronics* (Wiley, New York, 1970), Chap. 5.
[5]A. C. Scott, Nuovo Cimento B 69, 241 (1970).
[6]A. Barone, F. Exposito, C. J. Magee, and A. C. Scott, Riv. Nuovo Cimento 1, 227 (1971).
[7]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. 30, 1262 (1973).
[8]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. 31, 125 (1973).
[9]A. C. Scott, F. Chu, and D. W. McLaughlin, Proc. IEEE, 61, 1443 (1973).
[10]G. L. Lamb, Rev. Mod. Phys. 43, 99 (1971).
[11]G. L. Lamb, Phys. Rev. Lett. 31, 196 (1973).
[12]G. L. Lamb, Phys. Rev. A 9, 422 (1974).
[13]M. J. Ablowitz, D. J. Kaup, and A. C. Newell, J. Math. Phys. (to be published).
[14]J. D. Gibbon, P. J. Caudrey, R. K. Bullough, and J. C. Eilbeck, Nuovo Cimento Lett. 8, 775 (1973).
[15]J. R. Waldram, A. B. Pippard, and J. Clarke, Phil. Trans. Roy. Soc. Lond. A 268, 265 (1970).
[16]A. H. Silver (unpublished).
[17]R. P. Feynman, R. B. Leighton, M. Sands, *The Feynman Lectures on Physics* (Addison Wesley, Reading, Mass., 1965), Vol. III, Chap. 21.
[18]E. A. Lynton, *Superconductivity* (Methuen, London, 1969), Chap. V.
[19]The dissipation caused by break up of Cooper pairs into normal electrons can be introduced into this phenomenological quantum mechanics by selecting an appropriate complex "potential" linear in the electromagnetic fields. In this manner one obtains a term Δ_t in the nonlinear wave equation which governs the waves.
[20]D. W. McLaughlin and A. C. Scott, J. Math. Phys. 14, 1817 (1973).
[21]R. H. Dicke and J. P. Wittke, *Quantum Mechanics* (Addison Wesley, Reading, Mass., 1960), p. 200.
[22]A. Icsevgi and W. E. Lamb, Jr., Phys. Rev. 185, 517 (1969).
[23]M. J. Ablowitz, D. J. Kaup, and A. C. Newell, J. Math. Phys. (to be published).
[24]The notation $\langle\psi|\phi\rangle \equiv \int d^3r \cdots d^3r_N\, \psi^*(R,t;\vec{r}_1,\ldots,\vec{r}_N)\, \phi(R,t;\vec{r}_1,\ldots,\vec{r}_N)$.
[25]H. T. Yuan and A. C. Scott, Solid State Electron. 9, 1149 (1966).
[26]V. E. Zakharov and S. V. Manakov, Zh. Eksp. Teor. Fiz. Pis. Red. 18, 413 (1973) [JETP Lett. 18, 243 (1973)].
[27]D. Rogovin and M. O. Scully, Ann. Phys. (N.Y.) (to be published).