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Quantum theory of coupled parametric down-conversion and up-conversion with simultaneous phase matching*

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A two-step nonlinear optical interaction consisting of (i) a spontaneous frequency down-conversion process coupled with (ii) a frequency up-conversion process is treated quantum mechanically in the experimentally observed case of simultaneous collinear phase matching of both processes. The temporal behavior of the output signals is found to depend on which process, (i) or (ii), predominates. The nature of the photon statistics for each of the output modes is found to be "super-Poisson," regardless of which process, (i) or (ii), is dominant, and the zero-point fluctuations characteristic of parametric amplification are exhibited.

I. INTRODUCTION

The quantum theory of the nonlinear optical processes of parametric amplification and frequency conversion has been widely discussed, based on the model proposed by Louisell, Yariv, and Siegman.[1] In their model each of these two processes is described by an effective macroscopic Hamiltonian bilinear in the photon creation and annihilation operators. The model is suitable for investigating the photon statistics of parametric amplification and frequency conversion in the case of a large (classical) pump field. Mollow and Glauber[2] employed such a model Hamiltonian in a treatment of the quantum theory of parametric amplification. The quantum theory of frequency conversion has been discussed by Tucker and Walls,[3] and, in the more general case of time-dependent pump amplitude and phase, by one of the present authors.[4] The photon statistics obtained for these two processes differ markedly, with the parametric amplifier exhibiting a zero-point fluctuation not found in the case of the frequency converter.[1] Such previous treatments were limited to a single-step nonlinear optical process, either parametric amplification or frequency conversion, corresponding to the normal experimental conditions under which only one of these two processes is favored.

Recently, a two-step nonlinear optical interaction has been experimentally observed.[5] The observed interaction consisted of a spontaneous frequency down-conversion process (parametric amplification) coupled with an up-conversion process (frequency conversion). In the experiment, a sample nonlinear crystal was illuminated by a laser pump at frequency ω_p with equal ordinary (o) and extraordinary (e) polarization components. The experimental arrangement was such that both the down-conversion and the up-conversion processes were favorable; in fact, simultaneous collinear phase matching (SCPM) of both processes was obtained.

Under these conditions, the two-step interaction proceeds as follows. The extraordinary component of the pump generates in the crystal a signal mode at frequency ω_1^e of extraordinary polarization and an idler mode at frequency ω_i^o of ordinary polarization, by means of the spontaneous down-conversion process,

$$\omega_p^e \rightarrow \omega_i^o + \omega_1^e . \qquad (1)$$

The ordinary component of the pump, coupled with the idler mode generated in the down-conversion process (1), generates a signal mode at frequency ω_2^e of extraordinary polarization by means of the up-conversion process,

$$\omega_p^o + \omega_i^o \rightarrow \omega_2^e . \qquad (2)$$

In this paper we investigate the quantum theory of the two-step interaction described above in the case of simultaneous collinear phase matching. We adopt a generalization of the model of Ref. 1, where each step (down-conversion followed by up-conversion) in the two-step process is represented by a Hamiltonian bilinear in the appropriate photon creation and annihilation operators. The model was rendered exactly solvable with the help of an operator transformation which transforms the total Hamiltonian for the two-step process into a Hamiltonian describing a single-step process. Working in the new basis defined by this simplifying transformation, we calculate the moments of the photon number operator for each of the given modes. We find that the photon statistics in each of the two signal modes and the idler mode are identical and exhibit the zero-point fluctuations characteristic of parametric amplification.[1] In particular, the normalized variance of the number of photons in each mode takes the form.

$$(\Delta N)^2/\langle N\rangle = \langle N\rangle + 1\,, \tag{3}$$

where $\langle N\rangle$ is the expectation value of the number of photons in that mode. Thus the photon statistics are "super-Poisson." Such statistics are characteristic of the radiation field emitted from a Dicke laser in a highly inverted state, where the quantum effects of spontaneous emission predominate.[6]

The output in all three modes is either an exponentially increasing or an oscillating function of time, depending on whether the down-conversion process Eq. (1) or the up-conversion process Eq. (2), respectively, is dominant. Our calculations agree with the reported experimental results[5] as follows. First, the common idler signal is present only for a nonvanishing extraordinary pump. Second, the up-converted signal is present only for a pump with both polarization components. Third, the up-converted signal satisfies a square-law dependence on power with the pump.

The organization of the paper is as follows. In Sec. II the model Hamiltonian for the two-step interaction is introduced. A disentangling transformation for the time-dependent operator, which transforms the two-step interaction into a single-step interaction, is presented in Sec. III, along with the corresponding transformed states and operators. In Sec. IV the operator dynamics in the transformed basis are obtained. The photon statistics are given in Sec. V, and a discussion of our results, in Sec. VI. The disentangling theorem is developed in the Appendix.

II. THE MODEL HAMILTONIAN

To describe the two-step nonlinear interaction represented by Eqs. (1) and (2), we adopt a generalization of the quantum-mechanical model of parametric interactions proposed in Ref. 1. The pump field at frequency ω_p, because of the high intensity of the laser, is treated classically in this model. The radiation field at the down-converted frequency ω_1, at the common idler frequency ω_i, and at the up-converted frequency ω_2 is treated quantum mechanically. The free-field Hamiltonian $H^{(0)}$ for the three quantum modes is given as

$$H^{(0)} = \hbar\omega_1 a_1^\dagger a_1 + \hbar\omega_i b^\dagger b + \hbar\omega_2 a_2^\dagger a_2\,, \tag{4}$$

where $a_1^\dagger$ (a_1) is the photon creation (annihilation) operator for the down-converted mode, $b^\dagger$ (b) is the photon creation (annihilation) operator for the common idler mode, and $a_2^\dagger$ (a_2) is the photon creation (annihilation) operator for the up-converted mode. Under resonant conditions, these are the only quantum modes contributing to the two-step process.

The interaction Hamiltonian for the down-conversion process (1) is given as[1]

$$H_1^{(1)} = \hbar\kappa_e(a_1^\dagger b^\dagger e^{-i\omega_p t} + a_1 b e^{i\omega_p t})\,, \tag{5}$$

for collinear-phase-matching conditions. The frequencies of the interacting modes in this case satisfy the relation,

$$\omega_p = \omega_i + \omega_1\,. \tag{6}$$

The interaction parameter κ_e is proportional to the extraordinary component of the pump amplitude. The interaction Hamiltonian for the up-conversion process (2) is given as[1]

$$H_2^{(1)} = \hbar\kappa_o(b a_2^\dagger e^{-i\omega_p t} + b^\dagger a_2 e^{i\omega_p t})\,, \tag{7}$$

again for collinear-phase-matching conditions. The frequencies of the interacting modes in this case satisfy the relation,

$$\omega_p + \omega_i = \omega_2\,. \tag{8}$$

The interaction parameter κ_o is proportional to the ordinary component of the pump amplitude. The total interaction Hamiltonian $H^{(1)}$ for the two-step process described by Eqs. (1) and (2), under simultaneous-collinear-phase-matching conditions, is then the sum of (5) and (7),

$$H^{(1)} = H_1^{(1)} + H_2^{(1)}\,. \tag{9}$$

The noninteraction time-development operator is given as

$$U^{(0)}(t) = \exp[-iH^{(0)}t/\hbar]\,, \tag{10}$$

with $H^{(0)}$ given by Eq. (4). Transforming to the interaction representation by means of the transformation (10), the total interaction Hamiltonian $H_I^{(1)}$ in the interaction representation is given as

$$\begin{aligned} H_I^{(1)} &\equiv U^{(0)\dagger}(t) H^{(1)} U^{(0)}(t) \\ &= \hbar\kappa_e(a_1^\dagger b^\dagger + a_1 b) + \hbar\kappa_o(b a_2^\dagger + b^\dagger a_2)\,. \end{aligned} \tag{11}$$

Since the Hamiltonian (11) is time-independent, the interaction time-development operator in the interaction picture is

$$U(t) \equiv \exp[-iH_I^{(1)} t/\hbar]\,. \tag{12}$$

For an initial state $|\psi(0)\rangle$ of the field, the interaction state of the system in the interaction representation is given by

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle\,. \tag{13}$$

The complicated form of the interaction Hamiltonian (11) would make any calculations difficult.

In Sec. III we show how the problem may be simplified by introducing an operator transformation which transforms the Hamiltonian (11) describing a two-step process into a Hamiltonian describing a single-step process.

III. UNITARY TRANSFORMATION

The time-development operator Eq. (12) for the two-step process is given as the exponential of the sum of two terms, one involving the Hamiltonian for the down-conversion process Eq. (5), the other involving the Hamiltonian for the up-conversion process Eq. (7). Since these two single-step Hamiltonians neither commute with each other, nor commute with their commutator, the time-development operator cannot be simply factorized. However, utilizing the formal group properties of these operators, we were able to factorize[7] (see Appendix) the time-development operator into the product of three exponential operators as

$$U(t)=S^\dagger \hat{U}(t)S. \tag{14}$$

Depending on the relative magnitude of the interaction parameters κ_e and κ_o, the time-dependent operator $U(t)$ takes the form,

$$\begin{aligned}\hat{U}(t)&=\exp[-it(\kappa_e^2-\kappa_o^2)^{1/2}(ba_1+b^\dagger a_1^\dagger)], \quad \text{for } \kappa_e>\kappa_o \\ &=\exp[-it(\kappa_o^2-\kappa_e^2)^{1/2}(ba_2^\dagger+b^\dagger a_2)], \quad \text{for } \kappa_o>\kappa_e .\end{aligned} \tag{15}$$

The time-independent unitary operator S is given by

$$S=\exp[(\ln\kappa)(a_1a_2-a_1^\dagger a_2^\dagger)], \tag{16}$$

where we have defined the parameter,

$$\kappa\equiv\left(\frac{\kappa_>-\kappa_<}{\kappa_>+\kappa_<}\right)^{1/2}, \tag{17}$$

where $\kappa_>$ and $\kappa_<$ are the greater and lesser of κ_e and κ_o.

Now using the factorization (14), the interaction state Eq. (13) may be rewritten as

$$|\psi(t)\rangle=S^\dagger\hat{U}(t)S|\psi(0)\rangle . \tag{18}$$

If we now transform to a new basis under the unitary operator S, the interaction state in this new basis,

$$|\hat{\psi}(t)\rangle\equiv S|\psi(t)\rangle , \tag{19}$$

is given from Eq. (18) as

$$|\hat{\psi}(t)\rangle=\hat{U}(t)|\hat{\psi}(0)\rangle . \tag{20}$$

The photon operators transform under S as

$$\begin{aligned}\hat{a}_1^\dagger&\equiv Sa_1^\dagger S^\dagger=a_1^\dagger\cosh(\ln\kappa)+a_2\sinh(\ln\kappa), \\ \hat{a}_2^\dagger&\equiv Sa_2^\dagger S^\dagger=a_2^\dagger\cosh(\ln\kappa)+a_1\sinh(\ln\kappa), \\ \hat{b}^\dagger&\equiv Sb^\dagger S^\dagger=b^\dagger .\end{aligned} \tag{21}$$

These transformations (21) may be used to directly verify Eq. (14).

The operator S is identical to the operator described by one of us[8] which transforms a two-mode photon state to a mixed-mode "new coherent" state. Defining the mixed-mode operators,

$$C_\pm\equiv(1/\sqrt{2})(a_1\pm a_2), \tag{22}$$

it follows that these operators also obey the boson commutation relations,

$$[C_\pm, C_\pm^\dagger]=1, \tag{23}$$

and that the two mixed-mode operators are decoupled, that is,

$$[C_\pm, C_\mp^\dagger]=0. \tag{24}$$

All other commutators of these operators also vanish.

In terms of the mixed-mode operators (22), the operator S itself decouples as

$$S=S_+(\ln\kappa)S_-(-\ln\kappa), \tag{25}$$

where we have defined the decoupled operators,

$$\begin{aligned}S_+(\theta)&\equiv\exp[\tfrac{1}{2}\theta(C_+^2-C_+^{\dagger 2})], \\ S_-(\theta)&\equiv\exp[\tfrac{1}{2}\theta(C_-^2-C_-^{\dagger 2})].\end{aligned} \tag{26}$$

Now these unitary operators (26) transform the operators $C_\pm$ as

$$\begin{aligned}C_\pm(\theta)&\equiv S_\pm(\theta)C_\pm S_\pm^\dagger(\theta) \\ &=C_\pm\cosh\theta+C_\pm^\dagger\sinh\theta .\end{aligned} \tag{27}$$

The transformed operators (27) so defined also obey the boson commutation relations,

$$[C_\pm(\theta), C_\pm^\dagger(\theta)]=1. \tag{28}$$

A two-mode Glauber coherent state[9] $|\alpha_1\rangle_1|\alpha_2\rangle_2$, with

$$\begin{aligned}a_1|\alpha_1\rangle_1|\alpha_2\rangle_2&=\alpha_1|\alpha_1\rangle_1|\alpha_2\rangle_2, \\ a_2|\alpha_1\rangle_1|\alpha_2\rangle_2&=\alpha_2|\alpha_1\rangle_1|\alpha_2\rangle_2,\end{aligned} \tag{29}$$

is also a coherent state in the $C_\pm$ basis, since, using Eq. (22),

$$\begin{aligned}C_\pm|\alpha_1\rangle_1|\alpha_2\rangle_2&=(1/\sqrt{2})(a_1\pm a_2)|\alpha_1\rangle_1|\alpha_2\rangle_2 \\ &=(1/\sqrt{2})(\alpha_1\pm\alpha_2)|\alpha_1\rangle_1|\alpha_2\rangle_2 .\end{aligned} \tag{30}$$

Thus we have the equivalence,

$$|\alpha_1\rangle_1|\alpha_2\rangle_2\rightarrow|\gamma_+\rangle_+|\gamma_-\rangle_- , \tag{31}$$

of the two-mode coherent state and the mixed-mode coherent state, with the eigenvalue relation,

$$\gamma_{\pm} = (1/\sqrt{2})(\alpha_1 \pm \alpha_2). \tag{32}$$

The operator S acting on the coherent state (31) generates the "new-coherent" state,[8]

$$\begin{aligned} |\gamma_+; \theta\rangle_+ |\gamma_-; -\theta\rangle_- &\equiv S|\gamma_+\rangle_+ |\gamma_-\rangle_- \\ &= S_+(\theta)S_-(-\theta)|\gamma_+\rangle_+ |\gamma_-\rangle_- \end{aligned} \tag{33}$$

with $\theta = \ln\kappa$. Using Eq. (30), we have the eigenvalue equations,

$$\begin{aligned} C_+(\theta)|\gamma_+; \theta\rangle_+ &= S_+(\theta)C_+S_+^\dagger(\theta)S_+(\theta)|\gamma_+\rangle_+ \\ &= S_+(\theta)C_+|\gamma_+\rangle_+ \\ &= \gamma_+|\gamma_+; \theta\rangle_+ , \\ C_-(-\theta)|\gamma_-; -\theta\rangle_- &= \gamma_-|\gamma_-; -\theta\rangle \ . \end{aligned} \tag{34}$$

So the transformed states (33) a e again "coherent" states, in the $C_+(\theta), C_-(-\theta)$ bas .

Thus we have found that unde :he transformation S, the two-step time-devel ment operator $U(t)$ has been decoupled, Eq. (1 , while the states have been mixed, Eq. (33). Sol ng the two-step interaction problem is equivale to solving the problem of either parametric d n-conversion or up-conversion with mixed "new oherent" initial states. Next we show how to calculate expectation values of the photon-number operators using the transformation introduced in this section.

IV. FIELD-OPERATOR DYNAMICS

In this section we find the dynamical behavior of the field operators in the mixed-mode "new coherent" basis introduced in Sec. III. In the interaction picture, the general time-dependent operator $a(t)$ is given as

$$a(t) = U^\dagger(t)aU(t), \tag{35}$$

where $U(t)$ is the time-development operator (12) for the two-step process. Using Eqs. (14) and (21), the operator $a(t)$ may be written as

$$\begin{aligned} a(t) &= S^\dagger \hat{U}^\dagger(t)SaS^\dagger\hat{U}(t)S \\ &= S^\dagger\hat{U}^\dagger(t)\hat{a}\hat{U}(t)S. \end{aligned} \tag{36}$$

So in the mixed-mode "new coherent" basis the time-dependent operator $\hat{a}(t)$ is given as

$$\begin{aligned} \hat{a}(t) &= Sa(t)S^\dagger \\ &= \hat{U}^\dagger(t)\hat{a}\hat{U}(t). \end{aligned} \tag{37}$$

The form of $\hat{a}$ for each of the three modes is given by Eq. (21). The time-development operator $\hat{U}(t)$ is given by Eq. (15).

We now express the single-mode operators in terms of the mixed-mode "new coherent" operators. Inverting Eq. (27) to express the mixed-mode operators (22) in terms of the mixed-mode "new coherent" operators $C_\pm(\pm\theta)$, we have

$$C_{\pm} = C_{\pm}(\pm\theta)\cosh\theta \mp C_{\pm}^\dagger(\pm\theta)\sinh\theta. \tag{38}$$

Then using the definition (22) of the $C_\pm$ operators, we find the desired relations,

$$\begin{aligned} a_1 &= (1/\sqrt{2})\{[C_+(\theta) + C_-(-\theta)]\cosh\theta \\ &\qquad - [C_+^\dagger(\theta) - C_-^\dagger(-\theta)]\sinh\theta\}, \\ a_2 &= (1/\sqrt{2})\{[C_+(\theta) - C_-(-\theta)]\cosh\theta \\ &\qquad - [C_+^\dagger(\theta) + C_-^\dagger(-\theta)]\sinh\theta\}. \end{aligned} \tag{39}$$

Now for $\kappa_e > \kappa_o$, the time-development operator (15) is of the form,

$$\hat{U}(t) = \exp[-i\tau(ba_1 + b^\dagger a_1^\dagger)], \tag{40}$$

where we have defined the variable,

$$\tau \equiv t(\kappa_e^2 - \kappa_o^2)^{1/2}. \tag{41}$$

The field operators transform under $\hat{U}(t)$ as

$$\begin{aligned} \hat{U}^\dagger(t)a_1^\dagger\hat{U}(t) &= a_1^\dagger\cosh\tau + ib\sinh\tau, \\ \hat{U}^\dagger(t)b^\dagger\hat{U}(t) &= b^\dagger\cosh\tau + ia_1\sinh\tau, \\ \hat{U}^\dagger(t)a_2^\dagger\hat{U}(t) &= a_2^\dagger . \end{aligned} \tag{42}$$

Applying these transformations (42) to the S transformed operators (21), the time-dependent field operators (37) in the mixed-mode "new coherent" state basis are given as

$$\begin{aligned} \hat{a}_1^\dagger(\tau) &= a_1^\dagger\cosh\theta\cosh\tau + ib\cosh\theta\sinh\tau + a_2\sinh\theta, \\ \hat{a}_2^\dagger(\tau) &= a_2^\dagger\cosh\theta + a_1\cosh\tau\sinh\theta - ib^\dagger\sinh\tau\sinh\theta, \\ \hat{b}^\dagger(\tau) &= b^\dagger\cosh\tau + ia_1\sinh\tau. \end{aligned} \tag{43}$$

Then using relations (39), these operators (43) may be written in terms of the mixed-mode "new coherent" operators as

$$\begin{aligned} \hat{a}_1^\dagger(\tau) &= (1/\sqrt{2})\{[C_+^\dagger(\theta) + C_-^\dagger(-\theta)](\cosh^2\theta\cosh\tau - \sinh^2\theta) \\ &\qquad + [C_+(\theta) - C_-(-\theta)]\cosh\theta\sinh\theta(1 - \cosh\tau)\} \\ &\qquad + ib\sinh\tau\cosh\theta, \\ \hat{a}_2^\dagger(\tau) &= (1/\sqrt{2})\{[C_+^\dagger(\theta) - C_-^\dagger(-\theta)](\cosh^2\theta - \sinh^2\theta\cosh\tau) \\ &\qquad - [C_+(\theta) + C_-(-\tau)]\cosh\theta\sinh\theta(1 - \cosh\tau)\} \\ &\qquad - ib^\dagger\sinh\tau\sinh\theta, \\ \hat{b}^\dagger(\tau) &= (i/\sqrt{2})\{[C_+(\theta) + C_-(-\theta)]\cosh\theta\sinh\tau \\ &\qquad - [C_+^\dagger(\theta) - C_-^\dagger(-\theta)]\sinh\theta\sinh\tau\} \\ &\qquad + b^\dagger\cosh\tau. \end{aligned}$$

In the case $\kappa_o > \kappa_e$, the time-development operator (15) is of the form,

$$\hat{U}(t) = \exp[-it(\kappa_o^2 - \kappa_e^2)^{1/2}(ba_2^\dagger + b^\dagger a_2)]. \tag{45}$$

We could in this case go through a procedure completely analogous to that given above for the previous case $(\kappa_e > \kappa_o)$ to obtain expressions corresponding to Eq. (44). But when we proceed to calculate matrix elements, the matrix elements in this case $(\kappa_o > \kappa_e)$ may be obtained from the corresponding matrix elements in the case $(\kappa_e > \kappa_o)$ by simply taking $\kappa_o > \kappa_e$ in those expressions. In Sec. V we calculate matrix elements of the photon-number operators using the results of this section.

V. PHOTON STATISTICS

In this section we find the expectation value of the photon-number operator and its variance for each of the three modes produced in the two-step process, Eqs. (1) and (2), with simultaneous-collinear-phase-matching conditions. The experimentally pertinent initial state of the system is the three-mode product vacuum state,

$$|\psi(0)\rangle = |0\rangle_1 |0\rangle_2 |0\rangle_i \, . \tag{46}$$

Using the equivalence relations Eqs. (31) and (32), the initial vacuum state (46) is also a vacuum state,

$$|\psi(0)\rangle = |0\rangle_+ |0\rangle_- |0\rangle_i \, , \tag{47}$$

in the mixed-mode basis Eq. (22). Transforming the initial state (47) to the mixed-mode "new coherent" state basis, the resulting initial state,

$$|\hat{\psi}(0)\rangle = S|\psi(0)\rangle = S_+(\theta) S_-(-\theta) |0\rangle_+ |0\rangle_- |0\rangle_i \, , \tag{48}$$

using Eq. (34), is given as

$$|\hat{\psi}(0)\rangle = |0;\theta\rangle_+ |0;-\theta\rangle_- |0\rangle_i \, , \tag{49}$$

which is again a product vacuum state.

Using the expressions Eq. (44) for the time-dependent field operators in the mixed-mode "new-coherent" state basis, we can calculate the expectation value of the photon-number operator in each of the three modes for the case $(\kappa_e > \kappa_o)$ as

$$\begin{aligned}\langle N(t)\rangle &= \langle\psi(t)|a^\dagger a|\psi(t)\rangle \\ &= \langle\psi(0)|a^\dagger(t) a(t)|\psi(0)\rangle \\ &= \langle\hat{\psi}(0)|\hat{a}^\dagger(t)\hat{a}(t)|\hat{\psi}(0)\rangle \, .\end{aligned} \tag{50}$$

We obtain the results,

$$\begin{aligned}\langle N_1(t)\rangle &= \langle\psi(t)|a_1^\dagger a_1|\psi(t)\rangle \\ &= \cosh^2\theta \sinh^2\theta [1-\cosh\tau]^2 \\ &\quad + \sinh^2\tau \cosh^2\theta \, , \\ \langle N_2(t)\rangle &= \langle\psi(t)|a_2^\dagger a_2|\psi(t)\rangle \\ &= \cosh^2\theta \sinh^2\theta (1-\cosh\tau)^2 \, , \\ \langle N_i(t)\rangle &= \langle\psi(t)|b^\dagger b|\psi(t)\rangle \\ &= \cosh^2\theta \sinh^2(\tau) \, .\end{aligned} \tag{51}$$

Thus we see that the expectation value of the number of photons in each mode is for this case $(\kappa_e > \kappa_o)$ an exponentially increasing function of time for long times.

From the explicit form of the parameter $\theta = \ln\kappa$ from Eq. (17) in this case, we find that

$$\begin{aligned}\cosh\theta &= \frac{\kappa_e}{(\kappa_e^2 - \kappa_o^2)^{1/2}} \, , \\ \sinh\theta &= \frac{-\kappa_o}{(\kappa_e^2 - \kappa_o^2)^{1/2}} \, .\end{aligned} \tag{52}$$

Then Eq. (51) may be written explicitly in terms of the interaction parameters κ_e and κ_o as

$$\begin{aligned}\langle N_1(t)\rangle &= \frac{\kappa_e^2 \kappa_o^2}{(\kappa_e^2 - \kappa_o^2)^2} \{1 - \cosh[t(\kappa_e^2 - \kappa_o^2)^{1/2}]\}^2 \\ &\quad + \frac{\kappa_e^2}{\kappa_e^2 - \kappa_o^2} \sinh^2[t(\kappa_e^2 - \kappa_o^2)^{1/2}] \, , \\ \langle N_2(t)\rangle &= \frac{\kappa_e^2 \kappa_o^2}{(\kappa_e^2 - \kappa_o^2)^2} \{1 - \cosh[t(\kappa_e^2 - \kappa_o^2)^{1/2}]\}^2 \, , \\ \langle N_i(t)\rangle &= \frac{\kappa_e^2}{\kappa_e^2 - \kappa_o^2} \sinh^2[t(\kappa_e^2 - \kappa_o^2)^{1/2}] \, .\end{aligned} \tag{53}$$

To find the corresponding expectation values in the case $(\kappa_o > \kappa_e)$, we need simply to take $\kappa_o > \kappa_e$ in the above expressions Eq. (53). The results are

$$\begin{aligned}\langle N_1(t)\rangle &= \frac{\kappa_e^2 \kappa_o^2}{(\kappa_o^2 - \kappa_e^2)^2} \{1 - \cos[t(\kappa_o^2 - \kappa_e^2)^{1/2}]\}^2 \\ &\quad + \frac{\kappa_e^2}{\kappa_o^2 - \kappa_e^2} \sin^2[t(\kappa_o^2 - \kappa_e^2)^{1/2}] \, , \\ \langle N_2(t)\rangle &= \frac{\kappa_e^2 \kappa_o^2}{(\kappa_o^2 - \kappa_e^2)^2} \{1 - \cos[t(\kappa_o^2 - \kappa_e^2)^{1/2}]\}^2 \, , \\ \langle N_i(t)\rangle &= \frac{\kappa_e^2}{\kappa_o^2 - \kappa_e^2} \sin^2[t(\kappa_o^2 - \kappa_e^2)^{1/2}] \, .\end{aligned} \tag{54}$$

Thus the expectation value of the number of photons in each mode is in this case $(\kappa_o > \kappa_e)$ an oscillating function of time.

Note that the relation,

$$\langle N_2(t)\rangle = \langle N_1(t)\rangle - \langle N_i(t)\rangle \, , \tag{55}$$

holds for both cases $(\kappa_e > \kappa_o)$ and $(\kappa_o > \kappa_e)$ as can be seen from Eqs. (53) and (54). The expectation value of the number of photons in the up-converted mode is equal to the expectation value of the number of photons in the down-converted mode minus the expectation value of the number of photons in the common idler mode.

The expectation value of the square of the number of photons in each mode may be calculated in the same manner as was done above for the expectation value of the number of photons. What is found is that the expectation value of the square of the number of photons in each of the three modes has the *general* form,

$$\langle N^2(t)\rangle = 2\langle N(t)\rangle^2 + \langle N(t)\rangle\,, \tag{56}$$

in terms of the corresponding $\langle N(t)\rangle$ given in Eq. (53) for the case $(\kappa_e > \kappa_o)$ and in Eq. (54) for the case $(\kappa_o > \kappa_e)$, for an initial three-mode vacuum state. Then the *general* form of the normalized variance for such an initial state is

$$[\Delta N(t)]^2/\langle N(t)\rangle = \langle N(t)\rangle + 1 \geq 1\,. \tag{57}$$

Thus the photon statistics in each of the three modes is "super-Poisson," regardless of which is greater, κ_e or κ_o.

For $\kappa_e > \kappa_o$, the normalized variance (57) increases exponentially for long times, and for $\kappa_o > \kappa_e$, the normalized variance is a periodic function of time. From Eqs. (55) and (57) we obtain the relation,

$$\frac{[\Delta N_2(t)]^2}{\langle N_2(t)\rangle} + \frac{[\Delta N_i(t)]^2}{\langle N_i(t)\rangle} - \frac{[\Delta N_1(t)]^2}{\langle N_1(t)\rangle} = 1\,, \tag{58}$$

between the normalized variances for the three modes.

VI. CONCLUSIONS

We have treated the problem of a two-step parametric interaction consisting of a spontaneous down-conversion process Eq. (1) followed by an up-conversion process Eq. (2) with simultaneous collinear phase matching. We found that the temporal behavior of the output signals depends on whether the down-conversion (1) or the up-conversion (2) process predominates. When the down-conversion process (1) predominates $(\kappa_e > \kappa_o)$, the output signals Eq. (53) are exponentially increasing functions of time for long times. When the up-conversion process (2) predominates $(\kappa_o > \kappa_e)$ the output signals Eq. (54) are oscillating functions of time.

In contrast to the temporal behavior of the output signals, the nature of the photon statistics is the same regardless of which process predominates. We found that the photon statistics in each of the three modes is "super-Poisson" Eq. (57) and exhibits the zero-point fluctuations characteristic of parametric amplification.[1]

The results we obtained for the expectation value of the photon-number operator Eqs. (53) and (54) agree with the reported experimental checks[5] as follows. First, the common idler signal is present only for a nonvanishing extraordinary component of the pump. Second, the up-converted signal is present only for a pump with both polarization components. Third, the up-converted signal satisfies a square-law dependence on power with the pump, for short times.

APPENDIX

In this appendix we show how the time-development operator Eq. (12) may be disentangled to obtain Eq. (14). The method we adopt follows that of Arecchi *et al.*,[9] where they employ the lowest-dimensional operator representation to obtain a BCH-type formula for angular momentum operators.

Our interaction Hamiltonian in the interaction picture, Eq. (11), may be written as

$$H_I^{(1)} = \hbar\kappa_e h_1 + \hbar\kappa_o h_2\,, \tag{A1}$$

where we have defined the operators,

$$\begin{aligned} h_1 &\equiv a_1^\dagger b^\dagger + a_1 b\,,\\ h_2 &\equiv a_2^\dagger b + a_2 b^\dagger\,. \end{aligned} \tag{A2}$$

The operators h_1 and h_2 are noncommuting with

$$[h_1, h_2] = h_3\,, \tag{A3}$$

where we have defined the operator,

$$h_3 \equiv a_1 a_2 - a_1^\dagger a_2^\dagger\,. \tag{A4}$$

The operators h_1 and h_2 do not commute with their commutator h_3 but satisfy the relations,

$$\begin{aligned} [h_3, h_1] &= h_2\,,\\ [h_3, h_2] &= h_1\,. \end{aligned} \tag{A5}$$

So the set of operators (h_1, h_2, h_3) constitute a closed algebra.

We then write these operators in terms of the angular momentum operators (l_x, l_y, l_z) as

$$\begin{aligned} h_1 &= i l_y\,,\\ h_2 &= l_x\,,\\ h_3 &= l_z\,. \end{aligned} \tag{A6}$$

This transformation (A6) preserves the algebraic properties of the operators (h_1, h_2, h_3) as given in Eqs. (A3) and (A5), but not their hermiticity properties. However, in what follows, we will not make use of the hermiticity properties of the operators (h_1, h_2, h_3), so that the transformation (A6) is justifiable.

The operators (h_1, h_2, h_3) may then be represented on the Pauli-spin matrices as

$$\begin{aligned} h_1 &= \tfrac{1}{2}\sigma_y = \frac{1}{2}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\\ h_2 &= \tfrac{1}{2}\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\\ h_3 &= \tfrac{1}{2}\sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}. \end{aligned} \tag{A7}$$

Now the time-development operator Eq. (12) may be written as

$$U(t)=\exp(\theta_1 h_1+\theta_2 h_2)\,, \tag{A8}$$

where the parameters,

$$\theta_1\equiv -i\kappa_e t\,,\quad \theta_2\equiv -i\kappa_o t\,. \tag{A9}$$

Using the representation (A7), the time-development operator (A8) may be exponentiated to give

$$U(t)=\begin{pmatrix}\cosh(\theta_+\theta_-)^{1/2} & (\theta_+/\theta_-)^{1/2}\sinh(\theta_+\theta_-)^{1/2}\\ (\theta_-/\theta_+)^{1/2}\sinh(\theta_+\theta_-)^{1/2} & \cosh(\theta_+\theta_-)^{1/2}\end{pmatrix}, \tag{A10}$$

where

$$\theta_+=\tfrac{1}{2}(\theta_2+\theta_1)\,,\quad \theta_-=\tfrac{1}{2}(\theta_2-\theta_1)\,. \tag{A11}$$

We are now in a position to examine various operator products which take the form (A10) in the representation (A7). First we consider the operator product

$$\exp[\ln(\tau_1)h_3]\exp(\phi h_1)\exp[\ln(\tau_2)h_3]\,, \tag{A12}$$

where τ_1, τ_2, and ϕ are parameters to be determined. In the representation (A7), this operator product (A12) exponentiates to give the matrix

$$\begin{pmatrix}(\tau_1\tau_2)^{1/2}\cos(\phi/2) & (\tau_1/\tau_2)^{1/2}\sin(\phi/2)\\ -(\tau_2/\tau_1)^{1/2}\sin(\phi/2) & (\tau_1\tau_2)^{-1/2}\cos(\phi/2)\end{pmatrix}. \tag{A13}$$

Equating elements of this matrix (A13) with corresponding elements of the matrix representation (A10) of $U(t)$, we find that a solution exists for the case $\kappa_e>\kappa_o$. In particular, we find the following solution for the parameters

$$\tau_2=\frac{1}{\tau_1}=\left(\frac{\kappa_e-\kappa_o}{\kappa_e+\kappa_o}\right)^{1/2},\quad \phi=-it(\kappa_e^2-\kappa_o^2)^{1/2}\,. \tag{A14}$$

Then we obtain the factorization

$$U(t)=S^\dagger\hat{U}(t)S, \tag{A15}$$

as given in Eqs. (15)–(17) for the case $\kappa_e>\kappa_o$.

Next we examine the operator product

$$\exp[\ln(\tau_1)h_3]\exp(\phi h_2)\exp[\ln(\tau_2)h_3]\,, \tag{A16}$$

which, using the representation (A7), exponentiates to give the matrix,

$$\begin{pmatrix}(\tau_1\tau_2)^{1/2}\cosh(\phi/2) & (\tau_1/\tau_2)^{1/2}\sinh(\phi/2)\\ (\tau_2/\tau_1)^{1/2}\sinh(\phi/2) & (\tau_1\tau_2)^{-1/2}\cosh(\phi/2)\end{pmatrix}. \tag{A17}$$

We find that this matrix (A17) can be set equal to the matrix representation (A10) of $U(t)$ in the case $\kappa_o>\kappa_e$ with the following solution for the parameters,

$$\tau_2=\frac{1}{\tau_1}=\left(\frac{\kappa_o-\kappa_e}{\kappa_o+\kappa_e}\right)^{1/2},\quad \phi=-it(\kappa_o^2-\kappa_e^2)^{1/2}\,. \tag{A18}$$

Thus we obtain the factorization Eq. (14) as given by Eqs. (15)–(17) for the case $\kappa_o>\kappa_e$.

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[7]This is another example of the fact that boson operators often close among themselves and form a closed algebra. We previously (Ref. 4) made use of the fact that in a single-step frequency converter problem, the boson operators there form the angular momentum algebra SU(2). In the case of parametric amplification, the closed algebra is that of SU(1, 1) [A. Colavita, thesis (Washington University, 1974) (unpublished)]. In general, for a large number of boson processes, the boson operators form suitable closed algebras [E. Y. C. Lu (unpublished)].

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