PHYSICAL REVIEW A VOLUME 10, NUMBER 5 NOVEMBER 1974

Scaling variables and dimensions

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(Received 26 December 1973)

The concept of scaling dimensions is important in the physics of large systems, in particular, in the statistical mechanics of critical phenomena. The main purpose of this article is to explain and formulate this important concept in a more transparent and precise fashion. Discussion is in the framework of the n-component classical spin model and the renormalization group. It is emphasized that not every quantity, but only special ones, called *scaling variables*, have well-defined scaling dimensions. In general these variables can be derived by making use of certain parameters which are the *scaling fields* of Wegner. The dimensions are simply related to the exponents associated with the renormalization group. We hope to extract a fairly concrete picture by a general formulation followed by explicit determination of the scaling variables in the large-n limit. Dimensions are obtained to $O(1/n)$.

I. INTRODUCTION AND SUMMARY

We are interested in some special random variables in the statistical mechanics of classical n-component fields. These random variables, which will be called "scaling variables," transform simply under the Kadanoff scale transformations, or more precisely, the renormalization group—Wilson version.[1-3] These scaling variables are useful in studying critical phenomena. In formulating the reduction hypothesis,[4] Kadanoff constructed a set of variables in the framework of the Ising model. What is presented here is simply a more general study of that set of variables, and the associated concept of scaling dimensions.[5]

The basic idea of scaling variables is very elementary indeed and is best illustrated by a simple analog.

In quantum mechanics, states and operators are often classified according to properties under various transformations. To be specific, let us speak of the set of transformations $\{R(\theta), 0<\theta<2\pi\}$. $R(\theta)$ rotates the coordinate system about the z axis through an angle θ. Under $R(\theta)$, an operator A transforms to A'. There are special operators T_i, called "tensor operators," having the simple transformation rule under $R(\theta)$:

$$T_i \to T_i' = T_i e^{-i\theta m_i}, \tag{1.1}$$

where m_i is a quantum number characterizing the operator T_i. The angular momentum components $J_x \pm i J_y$, J_z are simple examples of tensor operators with $m_i = \pm 1, 0$, respectively. An arbitrary operator in general does not have the simple property (1.1) but can be expanded as a linear combination of the tensor operators. Experience tells us that the classification and construction of tensor operators are of fundamental importance.

Now consider a statistical mechanical system and the renormalization group $\{R_s, 1\le s<\infty\}$. R_s increases the unit of length by a factor of s and does other things to be specified later. (The rule $R_s R_{s'} = R_{ss'}$ is obeyed, but unlike the rotations, the inverse of R_s is not defined.) Under R_s, a random variable A transforms to A'. There are special random variables $\mathfrak{D}_i$ having the simple transformation rule, analogous to (1.1),

$$\mathfrak{D}_i \to \mathfrak{D}_i' = \mathfrak{D}_i s^{y_i}, \tag{1.2}$$

where the constant $-y_i$, which characterizes the variable $\mathfrak{D}_i$, will be called the *scaling dimension*, or simply the *dimension* of $\mathfrak{D}_i$. These special random variables will be called *scaling variables*.

If R_s were just a change of length scale and nothing else, then *any* random variable would have property (1.2), and $-y_i$ would be just the usual dimension of the variable in units of *inverse length*. This usual dimension will be referred to as the *naive dimension*. Then there would be nothing we could add to the results of the usual dimensional analysis. Of course, R_s does more than change the scale. As a result, the dimension defined by (1.2) in general differs from the naive dimension and only the scaling variables, *not* every random variable, have a definite dimension. Hopefully, one can construct a complete set of scaling variables so that any random variable can be expressed as a linear combination of scaling variables.

The important ingredient of R_s, which makes R_s different from a straight change of scale, is an averaging process explained briefly in Sec. II. A more detailed explanation can be found in Refs. 1–3. However, despite the extra ingredient of R_s, the naive dimensional analysis does provide a qualitative guide. The study of the scaling variables will tell more about the extent to which the naive dimensional analysis is applicable.

The tensor operators of rotation are useful in analyzing matrix elements. Analogously, the scaling variables will help study certain behaviors of correlation functions. We have in mind especially correlation functions for a system near its critical point. In fact, the physics of critical phenomena is characterized by the scaling variables of lowest dimensions.

We now outline the content of this paper briefly and summarize the main results.

Our discussion will be within the framework of a model of an n-component classical "spin-vector" field $\phi(x)=(\phi_\sigma(x), \sigma=1,2,\ldots,n)$ in a d-dimensional space $(2<d<4)$. The Fourier components of $\phi_\sigma(x)$ are denoted by $\phi_{\sigma k}$, with k smaller than a cutoff Λ. The quantities $\phi_{\sigma k}$ or $\phi_\sigma(x)$ are our basic random variables. All other variables are sums and products of them. The probability distribution P is specified by a set of parameters $\mu=(u_1, u_2, u_3, \ldots)$. We write

$$P\propto \exp[-\mathcal{H}(\mu,\phi)], \quad \mathcal{H}=\int d^dx\, H(\mu,\phi(x)). \tag{1.3}$$

Here we assume that the interactions are of a short-range nature, so that H only depends on $\phi(x)$ and its derivatives at x. The transformation R_s transforms μ to $\mu'=(u_1', u_2', \ldots)=R_s\mu$, which specifies a new probability distribution $P'\propto\exp[-\mathcal{H}(\mu',\phi)]$. The rules of transformation of random variables under R_s are obtained from the condition that the averages of transformed random variables over P' are the same as the corresponding averages of the original variables over P.

To determine scaling variables, we shall make use of a set of special parameters g_i, the "scaling fields" of Wegner.[6] These parameters have the simple property that R_s takes g_i to g_i' according to

$$g_i'=g_i s^{y_i}, \tag{1.4}$$

where y_i are constants. In other words, we can get a set of more convenient parameters g_i by making combinations of $u_1, u_2, u_3, \ldots$. It will become obvious that once the scaling fields g_i are found, the scaling variables can be constructed by differentiating $\exp(-\mathcal{H})$:

$$\mathfrak{D}_i e^{-\mathcal{H}}=-\frac{\partial}{\partial g_i}e^{-\mathcal{H}}, \quad \mathfrak{D}_i=\frac{\partial\mathcal{H}}{\partial g_i}. \tag{1.5}$$

The random variable $\mathfrak{D}_i$ satisfies (1.2) and has the dimension $-y_i$. The reason for the appearance of $e^{-\mathcal{H}}$ is that the transformation of random variables under R_s will be defined in terms of average values of products. The factor $e^{-\mathcal{H}}$ of course always appears when average values are calculated. Note that in (1.5) $\mathfrak{D}_i$ in general depends on the scaling fields. Thus, definition (1.2) of the scaling variables is now generalized to

$$\mathfrak{D}_i(g)\to[\mathfrak{D}_i(g)]'=s^{y_i}\mathfrak{D}_i(g'). \tag{1.5$'$}$$

It will become clear later that this is a very natural and convenient generalization. One might think that $\mathfrak{D}_i\mathfrak{D}_j$ and $\partial^2\mathcal{H}/\partial g_i\partial g_j$ should be scaling variables of dimension $-y_i-y_j$, but in general neither of them is. In fact, it turns out that only the combination, called "scaling product,"

$$\{\mathfrak{D}_i\mathfrak{D}_j\}\equiv\mathfrak{D}_i\mathfrak{D}_j-\frac{\partial^2\mathcal{H}}{\partial g_i\partial g_j} \tag{1.6}$$

is a scaling variable of dimension $-y_i-y_j$. Equation (1.6) is of course obtained from differentiating $\exp(-\mathcal{H})$ twice. In view of (1.3), we have the notion of *local scaling variable* $D_i(x)$:

$$\mathfrak{D}_i=\int d^dx\, D_i(x). \tag{1.7}$$

$D_i(x)$ will have the dimension $d-y_i$. The local version of (1.6) is

$$\{D_i(x)D_j(y)\}=D_i(x)D_j(y)-\delta(x-y)D_{ij}(x), \tag{1.7$'$}$$

where $D_{ij}=\partial^2H/\partial g_i\partial g_j$. The spatial resolution of $\delta(x-y)$ is of $O(\Lambda^{-1})$, owing to the cutoff Λ in wave vector space. Correlation functions with simple transformation properties will be defined and their universal properties will be discussed. All these general discussions will be included in Secs. II and III.

General arguments will not go very far. Our understanding of the renormalization group is still at a qualitative and intuitive level. There are no practical rules to find the scaling fields. To hunt for a more concrete picture, we shall turn to the special case of $n\to\infty$. This is a simple but still nontrivial case. The scaling fields can be determined and we shall obtain practically all scaling variables explicitly. This explicit illustration will provide an over-all view. Many of the qualitative features are expected to remain when n is not large. Of course, only a small subset of the scaling variables are of interest to critical phenomena. These are the ones with lowest dimensions. Section IV is devoted to the study of the large-n case. Once the scaling variables are determined to the zeroth order in $1/n$, their dimensions can be calculated to the next order, i.e., to $O(1/n)$. In Sec. V, such a calculation is carried out. Let us go through some main features of these sections.

In the large-n limit, we take H [defined by (1.3)] of the form

$$H=(\nabla\phi)^2+U(\phi^2), \tag{1.8}$$

where

$$\phi^2 \equiv \tfrac{1}{2}\sum_{\sigma=1}^{n} \phi_\sigma^2(x), \quad (\nabla\phi)^2 \equiv \tfrac{1}{2}\sum_{\sigma=1}^{n}[\nabla\phi_\sigma(x)]^2, \qquad (1.9)$$

and $U(\phi^2)$ is regarded as a power series in ϕ^2. Let

$$t(\phi^2) \equiv \frac{\partial U}{\partial \phi^2}. \qquad (1.10)$$

The set of parameters μ can be thought of as coefficients in the power series $t(\phi^2)$. Apart from an additive constant, H is specified by these parameters. We determine in Sec. IV a set of scaling fields g_l as combinations of these parameters and then find the corresponding set of local scaling variables. The results are simple:

$$D_l(x) = \frac{\partial H}{\partial g_l} = [t(\phi^2)]^l, \qquad (1.11)$$

with dimensions $d - y_l = 2l$,

$$y_l = d - 2l, \quad l = 1, 2, 3, \ldots . \qquad (1.12)$$

According to (1.10), t, and hence D_l, depends on the details of the interaction. Since t is a power series in ϕ^2, we can write any power series in ϕ^2 as a power series in t, i.e., a linear combination of D_l. Thus, the set of scaling variables t^l forms a *complete basis set* for the space of all power series of ϕ^2. Obviously, it does not make sense to speak of the dimension of ϕ^2 or $(\phi^2)^2$, for example. One has to say that ϕ^2 or $(\phi^2)^2$ is a sum of terms each of which has a definite dimension.

Of course there are more scaling variables than t^l. There are variables involving gradients and those which are not invariant under rotations in spin space or in coordinate space. These additional scaling variables are found by adding to (1.8) a small term W containing variables in general, and then examining the renormalization group. The results are more complicated. To give some rough ideas, we plot the dimensions of some scaling variables in Fig. 1. This is like an energy level diagram for an atom. The levels are classified according to the symmetry properties of the scaling variables under rotations in spin space and in coordinate space. The symbols $I, \phi, \tau, T, \Upsilon$ respectively stand for invariants, spin vectors, spin tensors, tensors, and spin-tensor tensors. One can include more if necessary.

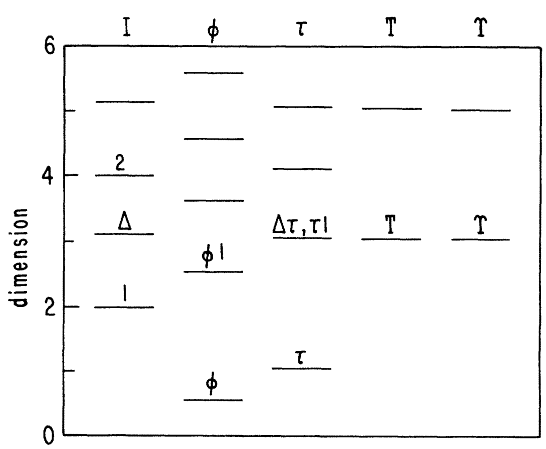

FIG. 1. Lowest dimensions in the large-n limit for d slightly larger than 3.

Let us list the scaling variables of lowest dimensions. The qualitative pattern should be general. We shall denote these variables by the following names:

invariants $D_1, D_\Delta, D_2, \ldots,$

spin vectors $D_\phi, D_{\phi 1}, \ldots,$

spin tensors $D_\tau, D_{\Delta\tau}, D_{\tau 1}, \ldots,$

tensors $D_T, \ldots,$

spin-tensor tensors $D_\Upsilon, \ldots .$

Their dimensions are $d - y_i$. We list y_i, correct to $O(1/n)$:

$$\begin{aligned}
&y_1 = 1/\nu = d - 2 + 8d^{-1}(d-1)(d-2)S_d/n,\\
&y_\Delta = 0,\\
&y_2 = d - 4 + 8d^{-1}(4-d)(d-1)^2 S_d/n,\\
&y_\phi = \tfrac{1}{2}(d + 2 - \eta),\\
&y_{\phi 1} = \tfrac{1}{2}(d - 2 + \eta),\\
&y_\tau = 2 - 8d^{-1}S_d/n,\\
&y_{\Delta\tau} = 0,\\
&y_{\tau 1} = -8(d - 3 + 4d^{-1})S_d/n,\\
&y_T = 0,\\
&y_\Upsilon = -16(4/d - 1)(d+2)^{-1}S_d/n.
\end{aligned} \qquad (1.13)$$

The exponent η is given by $4(4/d - 1)S_d/n$ and

$$S_d \equiv \frac{\sin\pi(\frac{1}{2}d - 1)}{\pi(\frac{1}{2}d - 1)B(\frac{1}{2}d - 1, \frac{1}{2}d - 1)}, \qquad (1.14)$$

where B is the beta function.

It should be noted that the spatial derivatives of any local scaling variable are also local scaling variables, for example, $\nabla_a D_i$, $\nabla^2 D_i$, $(\nabla_a\nabla_b - \delta_{ab}\nabla^2)D_i$, etc. The dimension of the derivative increases by one for each additional ∇ applied. These derivatives must be included in the complete set of local scaling variables on the same footing as D_i themselves.

The explicit expressions for the scaling variables, to the leading order in $1/n$ only, are

$$
\begin{aligned}
&D_1 = t\,,\\
&D_\Delta = (\nabla\phi)^2 - M_1(t),\\
&D_2 = t^2 - a_2\nabla^2 t\,,\\
&D_{\phi,\sigma} = \phi_\sigma,\\
&D_{\phi 1,\sigma} = t\phi_\sigma - \nabla^2\phi_\sigma = \frac{\partial H}{\partial\phi_\sigma}\,,\\
&D_{\tau,\sigma\sigma'} = \tau_{\sigma\sigma'} \equiv \phi_\sigma\phi_{\sigma'} - (2/n)\phi^2\delta_{\sigma\sigma'}\,,\\
&D_{\Delta\tau,\sigma\sigma'} = \lambda_{\sigma\sigma'} + t\tau_{\sigma\sigma'},\\
&D_{\tau 1,\sigma\sigma'} = [(d-4)/d(d-2)]\lambda_{\sigma\sigma'} + t\tau_{\sigma\sigma'}\,,\\
&D_{T,ab} = T_{ab} \equiv \sum_{\sigma=1}^{n} \nabla_a\phi_\sigma\nabla_b\phi_\sigma - (2/d)(\nabla\phi)^2\delta_{ab}\,,\\
&D_{\mathrm{T},ab\sigma\sigma'} = \nabla_a\phi_\sigma\nabla_b\phi_{\sigma'} - (1/d)\delta_{ab}\lambda_{\sigma\sigma'}\\
&\qquad - (1/n)\delta_{\sigma\sigma'}T_{ab} - (2/nd)(\nabla\phi)^2\delta_{ab}\delta_{\sigma\sigma'}\,,
\end{aligned}
\tag{1.15}
$$

where the spin tensor $\lambda_{\sigma\sigma'}$ is defined as

$$\lambda_{\sigma\sigma'} = \nabla\phi_\sigma\cdot\nabla\phi_{\sigma'} - (2/n)(\nabla\phi)^2\delta_{\sigma\sigma'}\,, \tag{1.16}$$

and $\nabla_a \equiv \partial/\partial x_a$; a, b run from 1 to d. Of course, $t \equiv \partial U/\partial\phi^2$, where ϕ^2 and $(\nabla\phi)^2$ are defined by (1.9). The subtraction term $M_1(t)$ in D_Δ will be explained in Sec. IV, and the coefficient a_2 is a constant given by (5.96). To the lowest order in $1/n$, $\lambda_{\sigma\sigma'}$ and $t\tau_{\sigma\sigma'}$ are both scaling variables but with the same dimension, namely, d. This situation of *degeneracy* is the same as that in first-order perturbation theory of elementary quantum mechanics. Namely, when a perturbation is turned on, the degenerate zeroth order eigenstates in general cease to be the right eigenstates to the lowest order. One has to take linear combinations of them. Here, when $O(1/n)$ contributions to the dimensions are considered, one finds that $\lambda_{\sigma\sigma'}$ and $t\tau_{\sigma\sigma'}$ are no longer scaling variables; their combinations, as given by $D_{\Delta\tau}$ and $D_{\tau 1}$ in (1.15), are. The same situation arises in the case of D_2 and the case of $D_{\phi 1}$. For $n\to\infty$, the dimension four is shared by t^2 and $\nabla^2 t$, and the dimension $\frac{1}{2}d+1$ is shared by $t\phi_\sigma$ and $\nabla^2\phi_\sigma$. Of course, this consideration is not needed for variables of different symmetry.

In general, the local scaling variables are linear combinations of products of ϕ, $\nabla\phi$, etc., with the same rotational symmetry. The above-listed results show that the combinations are nontrivial and interaction dependent. For large n, the detailed dependence on interaction comes through $t \equiv \partial U/\partial\phi^2$. For not-large n, more complicated relationships are expected. Unfortunately, we have not found better rules of determining scaling variables than (1.5). The best way of determination would probably depend on the details of H.

Further remarks are included in Sec. VI. We shall comment on simple features of exponents, physical interpretation of scaling variables, and universality. Special attention is paid to the variables $D_{\phi 1}$ and $D_{\Delta\tau}$, which are in some sense trivial.

II. PRELIMINARY

In this section we define our notation and review briefly the basic aspect of the renormalization group. A detailed but elementary introduction can be found in Ref. 3.

A. Definitions

Let us imagine an n-component classical "spin" field $(\phi_\sigma(x), \sigma = 1, 2, 3, \ldots, n) = \phi(x)$ in a d-dimensional cubic volume L^d with periodic boundary condition. Let $\phi_{\sigma k}$ be the Fourier components of $\phi_\sigma(x)$:

$$\phi_\sigma(x) = L^{-d/2}\sum_{k<\Lambda}\phi_{\sigma k}e^{ik\cdot x}\,, \tag{2.1}$$

where Λ is a cutoff and Fourier components with $k>\Lambda$ are always excluded in our model. The variables $\phi_{\sigma k}$ or $\phi_\sigma(x)$ are the basic random variables. Their probability distribution P will be written in the form (1.3). The transformation R_s introduced in Sec. I transforms μ to $\mu' = R_s\mu$ according to the rule

$$P' \propto e^{-\mathcal{H}(\mu',\phi)} = \left(\prod_{\sigma,\Lambda/s<k'<\Lambda}\int d\phi_{\sigma k'}\,e^{-\mathcal{H}(\mu,\phi)}\right)_{\phi_k\to s^y\phi_{sk}}\,, \tag{2.2}$$

where sk means s times k, and the constant y will be specified shortly. A few remarks are in order.

(a) R_s effectively represents a transformation of a probability distribution P to another P'. The multiple integral in (2.2) is a "coarse graining" procedure which smears out variations of ϕ of wave vectors k' greater than Λ/s. The substitution $\phi_k\to s^y\phi_{sk}$ for the unintegrated variables is a change of length scale by a factor s.

(b) P' describes a system in a volume $L'^d \equiv s^{-d}L^d$, as a result of scale change, but the cutoff Λ remains the same as before (the effect of scale change on Λ compensates that of the coarse graining).

(c) In terms of $\phi(x)$, the substitution $\phi_k\to s^y\phi_{sk}$ means

$$\phi(x)\to s^{-d/2+y}\phi(x/s), \tag{2.3}$$

where it is understood that variations of wave vectors higher than Λ/s in $\phi(x)$ have been washed off.

(d) The transformation R_s does not change the short-range nature of the interaction. We have

$$\mathcal{H}(\mu', \phi) = \int d^d x\, s^{-d} H(\mu', \phi(x/s))$$

$$= \int d^d x' \, H(\mu', \phi(x')), \qquad (2.4)$$

where the x' integral is taken over a volume $L'^d = s^{-d}L^d$.

(e) For random variables ϕ_k with $k<\Lambda/s$, P and P' are equivalent in the sense that

$$\langle \phi_{k_1}\phi_{k_2}\cdots\phi_{k_m}\rangle_P = s^{my}\langle \phi_{sk_1}\phi_{sk_2}\cdots\phi_{sk_m}\rangle_{P'}\,. \qquad (2.5)$$

The average $\langle\cdots\rangle_P$ is given by

$$\langle \phi_{k_1}\cdots\phi_{k_m}\rangle_P = \int \delta\phi\, e^{-\mathcal{H}}\phi_{k_1}\cdots\phi_{k_m} \Big/ \int \delta\phi\, e^{-\mathcal{H}}, \qquad (2.6)$$

where

$$\int \delta\phi \equiv \prod_{\sigma, k<\Lambda} \int d\phi_{\sigma k}\,. \qquad (2.7)$$

(f) A *fixed point* μ^* has the property that

$$R_s\mu^* = \mu^* \qquad (2.8)$$

if an appropriate value is chosen for the exponent y. This value will be denoted by

$$y = 1 - \tfrac{1}{2}\eta. \qquad (2.9)$$

B. Scaling fields

The fact that $R_s R_{s'} = R_{ss'}$ suggests that we should be able to choose a proper set of parameters $\{g_i\}$, where i runs through a set of labels. Each of them is a function of the old parameters in the set μ. Under R_s, g_i will transform simply into $g_i s^{y_i}$, where y_i are constants, which we call "exponents." These parameters are called "scaling fields" by Wegner.[6] The way to determine them as functions of old parameters would depend on the details of R_s, which we shall not elaborate.

Let us note some qualitative points. If the probability distribution is fixed at the physical distribution, i.e., the canonical ensemble, then all the parameters, and hence the scaling fields, will assume values determined by the temperature T and microscopic coupling constants. It is important to note that g_i will depend on the physical parameters *smoothly* because they describe interactions over a short distance Λ^{-1} (of course g_i depends on Λ in general). The whole theory of critical phenomena is to explain how singular large-scale collective behavior comes out of smooth short-range interactions.

We can define the scaling fields in such a way that the fixed point μ^* of interest corresponds to $g_i = 0$ for all i. The system is said to be at its "critical point" if all those g_i with $y_i > 0$ vanish. This condition requires in particular that $T = T_c$. In the absence of external fields, we assume there is only one such scaling field, g_1, with $y_1 > 0$. Thus we can expand

$$g_1(T) = A(T - T_c) + B(T - T_c)^2 + \cdots. \qquad (2.10)$$

We shall give the name $1/\nu$ to y_1, and define

$$\xi \equiv |g_1(T)|^{-\nu}. \qquad (2.11)$$

For $T - T_c \to 0$, we have $g_1(T) \propto T - T_c$, assuming $A \neq 0$ in (2.10).

C. Free energy

We define the free energy per unit volume $F(\mu)$ as

$$e^{-F(\mu)L^d} = \int \delta\phi\, e^{-\mathcal{H}(\mu,\phi)}. \qquad (2.12)$$

It is easy to show that

$$F(\mu) = s^{-d}F(\mu') + \Delta F(\mu), \qquad (2.13)$$

where $\Delta F(\mu)$ is the part of the free energy contributed from the part of $\mathcal{H}$ involving only $\phi_{k'}$, with $k' > \Lambda/s$. In obtaining $\mathcal{H}(\mu', \phi)$ from (2.4), $\Delta F(\mu)$ has been thrown away as an additive constant.

III. TRANSFORMATION OF RANDOM VARIABLES, SCALING VARIABLES, AND PRODUCTS

Under R_s, the probability distribution P is transformed into P'. The transformation of a random variable may be defined by the criterion that the average of transformed variables over the transformed probability distribution is the same as that of the untransformed ones over the untransformed probability. For example

$$\phi_k \to (\phi_k)' = s^{1-\eta/2}\phi_{sk}\,, \qquad (3.1)$$

$$\phi(x) \to (\phi(x))' = s^{-d/2+1-\eta/2}\phi(x/s), \qquad (3.2)$$

according to (2.5) with $y = 1 - \frac{1}{2}\eta$. We define the transformation of any random variable (sum of products of the basic variables ϕ) $A \to A'$ under R_s by the criterion

$$\langle A\phi_{k_1}\cdots\phi_{k_m}\rangle_P = \langle A'\phi_{sk_1}\cdots\phi_{sk_m}\rangle_{P'}\, s^{m(1-\eta/2)} \qquad (3.3)$$

for arbitrary m and $k_1, \ldots, k_m < \Lambda/s$. The transformation rule for ϕ is very simple because ϕ_k for $k < \Lambda/s$ is not involved in the coarse-graining procedure of R_s. However, in general a variable A would contain products of ϕ_k with all k, greater as well as smaller than Λ/s. Such variables may be called "composite" as opposed to the basic variable ϕ which is "elementary." The transformation rule for a composite variable is in general very complicated. Fortunately, through the use of the scaling fields, variables with simple transformation rules similar to (3.2) can be constructed easily.

A. Scaling variables

Let us use the scaling fields g_i to label our probability distribution and write

$$\mathcal{H} = \mathcal{H}(g_i),\qquad \mathcal{H}' = \mathcal{H}(g_i') = \mathcal{H}(g_i s^{y_i}). \tag{3.4}$$

We define a variable $\mathfrak{D}_i(g)$ for each scaling field g_i by differentiating $e^{-\mathcal{H}}$:

$$-\frac{\partial}{\partial g_i} e^{-\mathcal{H}} = \mathfrak{D}_i(g)\, e^{-\mathcal{H}},\qquad \mathfrak{D}_i(g) \equiv \frac{\partial \mathcal{H}}{\partial g_i}. \tag{3.5}$$

Now let us differentiate both sides of (2.5) with respect to g_i. From the left-hand side, we get

$$-\frac{\partial}{\partial g_i}\left[\int \delta\phi e^{-\mathcal{H}} \phi_{k_1}\cdots\phi_{k_m} \Big/ \int \delta\phi\, e^{-\mathcal{H}}\right] = \langle [\mathfrak{D}_i(g) - \langle \mathfrak{D}_i(g)\rangle_P]\, \phi_{k_1}\cdots\phi_{k_m}\rangle_P, \tag{3.6}$$

where the term $-\langle \mathfrak{D}_i(g)\rangle_P$ of course comes from differentiating the denominator. Similarly, from the right-hand side we get

$$s^{y_i}\langle [\mathfrak{D}_i(g') - \langle \mathfrak{D}_i(g')\rangle_{P'}]\, \phi_{sk_1}\cdots\phi_{sk_m}\rangle_{P'}\, s^{m(d+2-\eta)/2}, \tag{3.7}$$

where $\mathfrak{D}_i(g')$ is of course $\partial\mathcal{H}(g')/\partial g_i'$. By our criterion (3.3), we have the transformation rule

$$\mathfrak{D}_i(g) - \langle \mathfrak{D}_i(g)\rangle_P \to s^{y_i}[\mathfrak{D}_i(g') - \langle \mathfrak{D}_i(g')\rangle_{P'}], \tag{3.8}$$

which is similar to (3.1) and (3.2). Thus $\mathfrak{D}_i(g)$ with its average subtracted off is a scaling variable of dimension $-y_i$, in the spirit of (1.5′). It would be nicer if we did not need to subtract the average of $\mathfrak{D}_i$ in (3.8). Indeed we do not need to if

$$\langle \mathfrak{D}_i(g)\rangle_P = s^{y_i}\langle \mathfrak{D}_i(g')\rangle_{P'}. \tag{3.9}$$

This is almost true. We know, by differentiating (2.12), that

$$\langle \mathfrak{D}_i(g)\rangle_P = -\frac{\partial}{\partial g_i} F(g) L^d, \tag{3.10}$$

$$\langle \mathfrak{D}_i(g')\rangle_{P'} = -\frac{\partial}{\partial g_i'} F(g') L'^d. \tag{3.11}$$

According to (2.13), we have

$$\langle \mathfrak{D}_i(g)\rangle_P = s^{y_i}\langle \mathfrak{D}_i(g')\rangle_{P'} - \frac{\partial}{\partial g_i}\Delta F(g) L^d. \tag{3.12}$$

Thus, (3.9) is true apart from the last term of (3.12). Since the average value of variables like $\mathfrak{D}_i$ will almost always drop out automatically in our later formulas, we shall for simplicity regard $\mathfrak{D}_i$ as a scaling field of dimension y_i. One must return to (3.12) when the average of $\mathfrak{D}_i$ does play a role.

B. Products of scaling variables

Similarly to (3.5), let us differentiate $e^{-\mathcal{H}}$ once more to obtain

$$\frac{\partial^2}{\partial g_i \partial g_j} e^{-\mathcal{H}} = \{\mathfrak{D}_i(g)\mathfrak{D}_j(g)\}\, e^{-\mathcal{H}}, \tag{3.13}$$

$$\{\mathfrak{D}_i(g)\mathfrak{D}_j(g)\} \equiv \mathfrak{D}_i(g)\mathfrak{D}_j(g) - \mathfrak{D}_{ij}(g), \tag{3.14}$$

$$\mathfrak{D}_{ij}(g) \equiv \frac{\partial^2\mathcal{H}}{\partial g_i \partial g_j}. \tag{3.15}$$

Following previous arguments for $\mathfrak{D}_i$, one easily shows that $\{\mathfrak{D}_i\, \mathfrak{D}_j\}$ is a scaling variable of dimension $-y_i - y_j$, i.e.,

$$\{\mathfrak{D}_i(g)\mathfrak{D}_j(g)\} \to s^{y_i+y_j}\{\mathfrak{D}_i(g')\mathfrak{D}_j(g')\}. \tag{3.16}$$

Previous comments concerning the average value below (3.12) also apply. Note that $\mathfrak{D}_{ij}$ must be subtracted from $\mathfrak{D}_i\, \mathfrak{D}_j$ to get a scaling variable. Neither $\mathfrak{D}_i\, \mathfrak{D}_j$ nor $\mathfrak{D}_{ij}$ is a scaling variable. Let us call $\{\mathfrak{D}_i\, \mathfrak{D}_j\}$ the "scaling product" of $\mathfrak{D}_i$ and $\mathfrak{D}_j$. Obviously, scaling products of any number of $\mathfrak{D}_i$'s can be constructed by differentiating $e^{-\mathcal{H}}$, and we have the transformation rule

$$\{\mathfrak{D}_i(g)\mathfrak{D}_j(g)\cdots\mathfrak{D}_l(g)\} \to s^{y_i+y_j+\cdots+y_l}\{\mathfrak{D}_i(g')\cdots\mathfrak{D}_l(g')\}. \tag{3.17}$$

C. Local random variables

It follows from (1.3) and (3.5) that

$$\mathfrak{D}_i(g) = \int d^d x\, D_i(g,x), \tag{3.18}$$

$$D_i(g,x) \equiv \frac{\partial H(g,\phi(x))}{\partial g_i}, \tag{3.19}$$

where D_i is a "local variable." From the transformation rule for $\mathfrak{D}_i$ and (2.4), we expect that

$$D_i(g,x) \to s^{-d+y_i} D_i(g', x/s), \tag{3.20}$$

i.e., D_i is a scaling variable of dimension $d - y_i$. However, this is correct only if D_i is integrated over a slowly varying function of x, i.e., only the Fourier components of D_i with wave vectors much less than Λ satisfy (3.20). Again this is due to the coarse-graining operation in R_s, and must be kept in mind.

The local variable D_{ij} is defined as

$$D_{ij}(g,x)\equiv \frac{\partial^2 H}{\partial g_i \partial g_j}\ . \tag{3.21}$$

The local versions of (3.14) and (3.16) are, respectively,

$$\{D_i(x)D_j(y)\}=D_i(x)D_j(y)-D_{ij}(x)\delta(x-y), \tag{3.22}$$

$$\{D_i(g,x)D_j(g,y)\} \to s^{-2d+y_i+y_j}\{D_i(g',x/s)D_j(g',y/s)\}. \tag{3.23}$$

Similar conclusions follow for scaling products of more than two local variables.

The basic variables $\phi_\sigma(x)$ are specially simple local scaling variables. They can be expressed as a derivative of the form (3.19) when a term

$$\sum_{\sigma=1}^{n} h_\sigma \phi_\sigma(x) \tag{3.24}$$

is included in H and h can be included in the set of g_i's. For any scaling variable D_i, $\phi_k D_i$ as well as $\{\phi_k D_i\}$ are scaling variables ($k<\Lambda/s$), although they might not be equal.

More local scaling variables can be constructed by applying the gradient operator ∇ to D_i one or more times. The dimension increases by one every time a ∇ is applied. Scaling products involving gradients can be defined easily.

D. Correlation functions

We introduce the following notation for correlation functions:

$$G_A(p_1,p_2\cdots p_m;g) \equiv L^{-d}\int d^dx_1\cdots d^dx_m\, e^{-ip_1\cdot x_1-\cdots-ip_m\cdot x_m} \times\langle A(x_1)\phi(x_2)\cdots\phi(x_m)\rangle_P. \tag{3.25a}$$

From this definition, we generate the following names of other correlation functions. Let "$A\to\phi$" mean "replacing A by ϕ in (3.25a)." Then we define

$G(p_1\cdots p_m;g)$ for $A(x_1)\to\phi(x_1)$, (3.25b)

G_{AB} for $\phi(x_2)\to B(x_2)$,

G_{ABC} for $\phi(x_2)\to B(x_2)$, $\phi(x_3)\to C(x_3)$, etc., (3.25c)

$G_i\equiv G_{D_i}$, $G_{ij}\equiv G_{D_iD_j}$, etc., (3.25d)

$\tilde G_{ij}$ for $A(x_1)\phi(x_2)\to\{D_i(x_1)D_j(x_2)\}$,

$\tilde G_{ijl}$ for $A(x_1)\phi(x_2)\phi(x_3)\to\{D_i(x_1)D_j(x_2)D_l(x_3)\}$, etc. (3.25e)

The component labels σ for the ϕ's are dropped for simplicity. It is understood that in (3.25) $p_1+p_2+\cdots+p_m=0$ but no subset of these p's sums to zero. If we write $G(0,0\cdots)$ we mean the limit $p_1\to0$, $p_2\to0,\ldots,$ not $p_1=0$, $p_2=0,\ldots.$ It is also understood that all p's are very small compared to Λ.

It follows from the transformation rules of scaling variables and scaling products that

$$G_i(p_1\cdots p_m;g) = s^{-d+y_i+(m-1)(d+2-\eta)/2}G_i(sp_1\cdots sp_m;g'),$$
$$\tilde G_{ij}(p_1\cdots p_m;g) = s^{-d+y_i+y_j+(m-2)(d+2-\eta)/2}\tilde G_{ij}(sp_1\cdots sp_m;g'), \tag{3.26}$$

and similarly for $\tilde G_{ijl}$, etc. Of course, G_{ij}, G_{ijl}, etc., do not satisfy equations like (3.26). Equation (3.26) simply states that we can apply the rules of the naive dimensional analysis provided that the variables are scaling variables and products are scaling products. From (3.26) it is easy to deduce behaviors of G_i, $\tilde G_{ij}$, etc., for $T-T_c\to0$. Let us list a few formulas for future references. Define

$$G(k)=G(k,-k;g), \tag{3.27}$$

$$G(0)=r^{-1}. \tag{3.28}$$

Setting $s=\xi$ [see (2.11) for ξ] in (3.26), we obtain, for example,

$$1/r\propto\xi^{2-\eta}, \tag{3.29}$$

$$G_i(0,0,0;g(T))\propto r^{-y_i/(2-\eta)-1}, \tag{3.30}$$

$$\tilde G_{ij}(0,0,0,0;g(T))\propto r^{-(y_i+y_j)/(2-\eta)-1}, \tag{3.31}$$

$$\left(\frac{\partial G_i(0,k,-k;g(T))}{\partial k^2}\right)_{k=0}\propto r^{-(y_i+\eta)/(2-\eta)-2} \tag{3.32}$$

for very small $T-T_c$. For $T=T_c$, we set $s=k^{-1}$ to obtain from (3.26), for example,

$$G(k)\propto k^{-2+\eta},$$

$$G_i(0,k,-k;g(T_c))\propto k^{-y_i-2+\eta}, \tag{3.33}$$

$$\tilde G_{ij}(k,-k,0,0;g(T_c))\propto k^{-y_i-y_j-2+\eta}, \tag{3.34}$$

for very small k, provided that they do not vanish.

We can write any local random variable $A(x)$ as a linear combination of local scaling variables:

$$A(x)=\sum_i a_i D_i(x). \tag{3.35}$$

Note that the complete set of D_i in the summation should include also those which are total spatial derivatives of other scaling variables. Of course, a_i would depend on the details of the model since D_i in general do. They are all functions of the scaling fields even if A is not. It follows that

$$G_A = \sum_i a_i G_i\,, \tag{3.36}$$

$$G_{AB} = \sum_{i,j} a_i b_j G_{ij}\,. \tag{3.37}$$

We have to write G_{ij} in terms of $\tilde{G}_{ij}$ following the definition (3.22). Then we have to expand $D_{ij}(x)$ in terms of scaling variables. That is to say,

$$G_{ij}(p_1 \cdots p_m; g) = \tilde{G}_{ij} + G_{D_{ij}}(p_1 + p_2, p_3 \cdots p_m; g),$$

$$G_{D_{ij}} = \sum_l d_{ijl} G_l\,. \tag{3.38}$$

Once the expansion coefficients are known, the critical behavior of G_A, G_{AB}, etc., can be deduced easily from that of G_i, $\tilde{G}_{ij}$, etc.

There is a set of Ward identities which follows trivially from our definitions of scaling variables and scaling products. We get

$$-\frac{\partial}{\partial g_i} G(p_2 \cdots p_m; g) = G_i(0, p_2 \cdots p_m; g),$$

$$-\frac{\partial}{\partial g_i} G_j(p_2 \cdots p_m; g) = \tilde{G}_{ij}(0, p_2 \cdots p_m; g),$$

$$-\frac{\partial}{\partial g_i} \tilde{G}_{jl}(p_2 \cdots p_m; g) = \tilde{G}_{ijl}(0, p_2 \cdots p_m; g),$$

$$\text{etc.} \tag{3.39}$$

The motivation of Ward identities is to add an extra variable into a correlation function by differentiation. It must be noticed that in (3.39), apart from the first identity, $-\partial/\partial g_i$ does more than adding a variable. For example, the right-hand side of the second one is $\tilde{G}_{ij}$, *not* the G_{ij} obtained by adding D_i into G_j.

E. Universality of correlation functions

The transformation rules (3.26) for the correlation functions G_i, $\tilde{G}_{ij}$, $\tilde{G}_{ijl}$, etc., are the final objectives of the general renormalization-group analysis. They are analogous to the transformation rules of matrix elements of tensor operators following a rotation-group analysis. The factors s^{y_i} correspond to the irreducible representations. The exponents y_i reflect the geometrical properties of R_s. They are "universal" in the sense that they are functions of n and d only.

The scaling variables in general depend on all of the parameters g_i. However, such dependence is precisely that needed to achieve the simple transformation rules (3.26). In the event when the wave vectors $p_1 \cdots p_m \sim p$ and $T - T_c$ approach zero but $p\xi$ remains finite, we can drop terms of $O(p/\Lambda)$, $O(\xi^{-1}/\Lambda)$. The dependence on parameters other than g_i disappears as can be seen by choosing $s = \xi$ in (3.26). Thus, in this limit, each of G_i, $\tilde{G}_{ij}$, etc., is expected to be, apart from a multiplicative factor, a function of $p_1\xi, p_2\xi, \ldots, p_m\xi$ and nothing else except n and d, i.e., a universal function. In principle, these functions can be calculated once and for all. The remaining work is to calculate the expansion coefficients a_i of (3.35) for the random variable of interest.

To sum up, the scaling variables and the related concepts of dimension and scaling products are designed in such a way that those correlation functions defined as averages of scaling products of scaling variables will transform under R_s by the rules of the naive dimensional analysis, with scaling dimensions replacing naive dimensions. Any random variables can be expanded as a linear combination of scaling variables. Only the first few terms with the lowest dimensions are of interest when the system is near its critical point. The concepts of scaling variables and dimensions introduced here are not restricted to the fixed point. This is in contrast to some previous work in the literature.

IV. SCALING FIELDS AND SCALING VARIABLES IN THE LARGE-n CASE

So far our discussion has been formal. We now turn to the large-n case, where explicit construction of scaling variables can be carried out, as a concrete illustration.

A. Renormalization group and scaling fields

The renormalization group in the case of large n has been studied extensively.[3] We shall summarize the basic features briefly and cast some results into a form which is most convenient for our purpose here.

We start with H and t defined by (1.8) and (1.10). The transformation R_s takes $t(\phi^2)$ to another power series $t'(\phi^2)$. The formula is given by [see (4.35) of Ref. 3]

$$t'(\phi^2) = s^2 t(N),$$

$$N = s^{-d+2}\phi^2 + \tfrac{1}{2} n K_d \int_{\Lambda/s}^{\Lambda} dp\, p^{d-1}/(t'/s^2 + p^2). \tag{4.1}$$

Given t, these two equations can be solved simultaneously for t'. The quantity K_d is $(2\pi)^{-d} \times$ (the area of a unit sphere in a d-dimensional space):

$$K_d \equiv 2^{-d+1}\pi^{-d/2}/\Gamma(\tfrac{1}{2}d). \tag{4.2}$$

We shall write (4.1) in a more convenient form. Let us introduce the Legendre transformation of U:

$$\Omega(t) \equiv U - \phi^2 t, \tag{4.3}$$

which is regarded as a function of t. It follows that

$$\phi^2 = \frac{-\partial \Omega}{\partial t}. \tag{4.4}$$

Similarly, we have $\Omega'(t') = U' - \phi^2 t'$, and

$$\phi^2 = \frac{-\partial \Omega'}{\partial t'}. \tag{4.5}$$

Thus, the transformation R_s brings Ω to Ω'. It is easy to verify that (4.1) is equivalent to

$$\Omega'(t') = s^d \Omega(t's^{-2}) - \tfrac{1}{2} n K_d \int_\Lambda^{\Lambda s} dp\, p^{d-1} \ln(t' + p^2) \tag{4.6}$$

and (4.5). Now we write $\Omega(t)$ as a power series:

$$\Omega(t) = \sum_{m=1}^{\infty} a_m t^m, \tag{4.7}$$

and then expand (4.6) in powers of t':

$$\Omega'(t') = \sum_{m=1}^{\infty} [(a_m - a_m^*) s^{d-2m} t'^m + a_m^* t'^m], \tag{4.8}$$

where

$$a_m^* \equiv \frac{(-)^m}{m} \frac{\Lambda^{d-2m}}{d-2m} \left(\frac{nK_d}{2}\right). \tag{4.9}$$

Now let

$$g_m = a_m - a_m^*, \tag{4.10}$$

$$y_m = d - 2m, \quad m = 1, 2, 3, \ldots. \tag{4.11}$$

Then (4.7) and (4.8) become, respectively,

$$\Omega(t) = \sum_{m=1}^{\infty} (g_m + a_m^*) t^m, \tag{4.12}$$

$$\Omega'(t') = \sum_{m=1}^{\infty} (g'_m + a_m^*) t'^m, \tag{4.13}$$

where

$$g'_m = g_m s^{y_m}. \tag{4.14}$$

From (4.4) and (4.5) we can solve for $t(\phi^2)$ and $t'(\phi^2)$, which are now parametrized by g_m and g'_m, respectively. By our definition of scaling fields and their exponents, (4.10) and (4.11) are the desired ones.

B. Scaling variables

By the definition (3.19) and by (1.8) and (4.12), we obtain the local scaling random variables

$$D_m(x) = \frac{\partial U}{\partial g_m} = \frac{\partial \Omega}{\partial g_m} = [t(\phi^2(x))]^m, \quad m = 1, 2, 3, \ldots \tag{4.15}$$

which have the dimensions

$$d - y_m = 2m. \tag{4.16}$$

These are very simple results. Note that we can obtain D_m directly as t^m. There is no need to determine explicit expressions for g_m in terms of parameters originally given by μ. The scaling products of these variables are easily worked out. For example, (3.21) gives

$$\begin{aligned} D_{mn}(x) &= \frac{\partial}{\partial g_m} t^n \\ &= nt^{n-1} \frac{\partial}{\partial g_m} \frac{\partial U}{\partial \phi^2} \\ &= nt^{n-1} \frac{\partial}{\partial \phi^2} t^m \\ &= mnt^{m+n-2} \frac{\partial t}{\partial \phi^2} \end{aligned} \tag{4.17}$$

and $\{D_m(x) D_n(y)\}$ follows from (3.22) and (4.17). Scaling products of more variables can be worked out in the same manner. Only powers of t and its derivatives with respect to ϕ^2 appear in the products.

C. Completeness and determination of other scaling variables by perturbation

So far we have determined the scaling fields and thereby obtained local scaling variables $D_m = t^m$ of dimension $2m$, $m = 1, 2, 3, \ldots$. Write

$$t = t_0 + u\phi^2 + v(\phi^2)^2 + w(\phi^2)^3 + \cdots \tag{4.18}$$

and assume $u \neq 0$. Then it is evident that any power series in ϕ^2 can be written as a power series in t. Therefore, any local random variable which is a power series in ϕ^2 can be expanded as a linear combination of the scaling variables t^m. From this linear combination, various properties of correlation functions involving this local variable can be deduced easily as discussed in Sec. III. The set of $D_m = t^m$ is *complete* as far as random variables which are powers of ϕ^2 are concerned. Such random variables are of course the most important ones. However, there are other variables of interest, for example, $(\nabla^2\phi)^2$, $\phi^2(\nabla\phi)^2$, $\phi_\sigma \phi^2$, $\frac{1}{2}\phi_\sigma \phi_{\sigma'} - (1/n)\phi^2 \delta_{\sigma\sigma'}$, ... They are of interest to the extent of their correlation functions over our old probability distribution. To be more precise, we are interested in random variables of various structure and symmetries, but their correlation functions are to be calculated with a simple, completely symmetric probability distribution like (1.8).

Therefore, for the sake of generating new scaling variables, we shall attempt to find small perturbation terms of the form $g_\nu D_\nu$ and add them to H of (1.8). Here g_ν and D_ν are the new scaling fields and scaling random variables to be determined and g_ν are taken to be small. If we differentiate $H+\sum_{\nu'} g_{\nu'} D_{\nu'}$ with respect to a given g_ν and, after the differentiation, we set all $g_{\nu'}=0$, then g_ν will disappear and we get our scaling variable D_ν. Of course, if one wants to find scaling products of two new scaling variables, it is necessary to include second-order terms in g_ν in addition to the first-order terms. After differentiating twice, one sets $g_\nu=0$. We now proceed to the explicit construction of the perturbation terms.

We add to H of (1.8) a small term W and write the Legendre transform as

$$\Omega=\Omega_0(t)+W(t,(\nabla\phi)^2,(\nabla^2\phi)^2,\ldots), \tag{4.19}$$

where $\Omega_0(t)$ is given by (4.3). Now we must define

$$t=\frac{\partial U}{\partial\phi^2}+\frac{\partial W}{\partial\phi^2}. \tag{4.20}$$

W is of course a function of ϕ^2 through t. Equations (4.4) and (4.5) hold for the full Ω given by (4.19). Let us simplify the notation by writing

$$z_l\equiv(\nabla^l\phi)^2,\qquad W=W(t,z_l). \tag{4.21}$$

To first order in W, one finds that

$$\Omega'=\Omega_0'+s^d W(t's^{-2},s^{-d+2-2l}(z_l+\zeta_l)), \tag{4.22}$$

$$\zeta_l\equiv\tfrac{1}{2}nK_d\int_\Lambda^{\Lambda s}dp\,p^{d-1+2l}(t'+p^2)^{-1}, \tag{4.23}$$

where Ω_0' is (4.6) with Ω',Ω carrying the subscript 0. We now expand (4.23) in powers of t' to find

$$(z_l+\zeta_l)s^{2-2l-d}=[z_l-M_l(t')]s^{2-2l-d}+M_l(t's^{-2}), \tag{4.24}$$

$$M_l(t)\equiv\tfrac{1}{2}nK_d\Lambda^{d+2l-2}\sum_{m=0}^{\infty}(-t/\Lambda^2)^m/(d+2l-2-2m). \tag{4.25}$$

It is clear now from (4.24) that in (4.22) W can be regarded as a function of $[z_l-M_l(t')]s^{2-2l-d}$ and $t's^{-2}$. We write

$$W=W(t,\bar z_l), \tag{4.26}$$

$$\Omega'=\Omega_0'+s^dW(t's^{-2},\bar z_l s^{2-2l-d}), \tag{4.27}$$

where the new symbol $\bar z_l$ is the subtracted gradient:

$$\bar z_l\equiv z_l-M_l(t)\equiv(\nabla^l\phi)^2-M_l(t). \tag{4.28}$$

Of course, t' replaces t in the $\bar z_l$ appearing in (4.27). Now write (4.26) as a power series

$$W=\sum_m\sum_{\nu,\nu_1,\nu_2,\ldots,\nu_m}t^\nu\bar z_1^{\nu_1}\bar z_2^{\nu_2}\cdots\bar z_m^{\nu_m}g_{\nu\nu_1\nu_2\cdots\nu_m}. \tag{4.29}$$

In view of (4.27), $g_{\nu\nu_1\cdots\nu_m}$ are scaling fields with

$$y_{\nu\nu_1\cdots\nu_m}=d-2\nu+\sum_{l=1}^m\nu_l(2-2l-d). \tag{4.30}$$

The corresponding local scaling variables are

$$D_{\nu\nu_1\cdots\nu_m}=t^\nu\bar z_1^{\nu_1}\cdots\bar z_m^{\nu_m}, \tag{4.31}$$

with dimensions $d-y_{\nu\nu_1\cdots\nu_m}$.

The local scaling variables (4.15) and (4.31) are invariant under rotation in spin space and under rotation in the coordinate space. Generalization to vector and tensor variables is straightforward. The simplest spin vector is just $\phi_\sigma(x)$, which can be generated by adding $W=\sum_\sigma h_\sigma\phi_\sigma$ to H as stated before [see (3.24)]. We can extend the coefficients of ϕ_σ to functions of t and $\bar z_l$, so that

$$W=\sum_\sigma\phi_\sigma W_\sigma(t,\bar z_l)=\sum_m\sum_{\nu\nu_1\cdots\nu_m}D_{\sigma,\nu\nu_1\cdots\nu_m}h_{\sigma,\nu\nu_1\cdots\nu_m}, \tag{4.32}$$

$$D_{\sigma,\nu\nu_1\cdots\nu_m}\equiv\phi_\sigma D_{\nu\nu_1\cdots\nu_m}, \tag{4.33}$$

$$\Omega'=\Omega_0'+s^{-d/2+1}s^d\sum_\sigma\phi_\sigma W_\sigma(t's^{-2},\bar z_l s^{2-2l-d}) \tag{4.34}$$

corresponding to (4.26), (4.29), and (4.27). The scaling fields are $h_{\sigma,\nu\nu_1\cdots\nu_m}$ and the scaling variables are given by (4.33) with dimensions $\frac{1}{2}d-1+d-y_{\nu\nu_1\cdots\nu_m}$.

Results for rank-two traceless spin tensors follow in a similar fashion. Let $\tau_{\sigma\sigma'}$ be such a tensor of the lowest dimension. In this case

$$\tau_{\sigma\sigma'}=\phi_\sigma\phi_{\sigma'}-(2/n)\phi^2\delta_{\sigma\sigma'} \tag{4.35}$$

with the dimension $d-2$. Ones with higher dimensions are generated by

$$W=\sum_{\sigma\sigma'}\tau_{\sigma\sigma'}W_{\sigma\sigma'}(t,\bar z_l). \tag{4.36}$$

We get the scaling variables

$$D_{\sigma\sigma',\nu\nu_1\cdots\nu_m}=\tau_{\sigma\sigma'}D_{\nu\nu_1\cdots\nu_m} \tag{4.37}$$

with dimensions $d-2+(d-y_{\nu\nu_1\cdots\nu_m})$. Similar arguments apply if we start with

$$\lambda_{\sigma\sigma'}=\nabla\phi_\sigma\cdot\nabla\phi_{\sigma'}-(2/n)(\nabla\phi)^2\delta_{\sigma\sigma'}, \tag{4.38}$$

instead of (4.35). Scaling variables which are vectors and tensors in the coordinate space can be obtained easily also. For example,

$$T_{ab}=\sum_\sigma (\nabla_a \phi_\sigma)(\nabla_b\phi_\sigma)-(2/d)(\nabla\phi)^2\delta_{ab}, \qquad (4.39)$$

$$\Upsilon_{ab\sigma\sigma'}=(\nabla_a\phi_\sigma)(\nabla_b\phi_{\sigma'})-(1/d)\,\delta_{ab}\lambda_{\sigma\sigma'}-(1/n)\delta_{\sigma\sigma'}T_{ab} - (2/nd)(\nabla\phi)^2\delta_{ab}\,\delta_{\sigma\sigma'}. \qquad (4.40)$$

The variables (4.38)–(4.40) are all of dimension d. Variables with higher dimensions can be obtained by multiplying them by $D_{\nu\nu_1\cdots\nu_m}$. Other examples are $\nabla^2\phi_\sigma$, $\nabla^2\tau_{\sigma\sigma'}$. We shall stop our determination of scaling variables here. Further extension is straightforward but is of little interest. A few remarks are in order.

(a) The above results are very easy to remember because they follow the rules of the naive dimensional analysis. The scaling variables are built as powers of t and $\bar{z}_l$ with dimensions 2, $d+2l-2$, $l=1,2,3,\ldots$. To get variables of symmetries other than invariant, just multiply ϕ_σ, $\lambda_{\sigma\sigma'}$, $\tau_{\sigma\sigma'}$, T_{ab}, $\Upsilon_{ab\sigma\sigma'}$, etc., with powers of t and $\bar{z}_l$. The only nontrivial results in the large-n limit are the special important role of $t\equiv\partial U/\partial\phi^2$, and the subtraction terms $M_l(t)$ [see (4.25)].

(b) The subtraction term $M_l(t)$ removes the Λ-dependent terms in $(\nabla^l\phi)^2$ [see (4.28)] so that $\bar{z}_l$ will have the desired behavior in calculating correlation functions [see (5.38)–(5.41)]. As was mentioned before, our local scaling variables make sense only for their Fourier components of wave numbers much smaller than Λ, i.e., corrections of the form $\Lambda^{-2}\nabla^2$, $\Lambda^{-4}\nabla^4,\ldots$ are not taken care of. Such corrections are not important except for local variables involving many powers of ∇, namely, $\bar{z}_l$ with sufficiently large l. Since there will be no further interest for $\bar{z}_l$ with $l>1$, there is no need to consider these correction terms here.

(c) Linear combinations of the above determined scaling variables may be needed in the case of *degeneracy*, i.e., when two or more variables share the same dimension. This point has been discussed in Sec. I.

V. CALCULATION OF y_i TO $O(1/n)$

The exponents y_i can be calculated easily to the first order in $1/n$ knowing the scaling variables D_i to the zeroth order. This section is devoted to such calculations.

The results are already summarized in Sec. I. The details of the algebra are not of vital importance to the ideas which we want to illustrate in this paper. The purpose of this section is to record enough intermediate steps so that the reader can easily repeat the algebra. The technical information furnished here should be useful to researchers working on related problems.

A. Background material for calculations

The background material is all contained in Ref. 7. Here we shall review a few basic formulas for later reference.

The parameters specifying our model are contained in $t(\phi^2)$. Without losing much generality, we shall take a simple $t(\phi^2)$:

$$t(\phi^2)=t_0+u\phi^2 \qquad (5.1)$$

with only two parameters, t_0 and u. The quantities t, ϕ^2/n are taken to be of $O(1)$ for large n, and $u=O(1/n)$. By (1.8) and (1.10) we have

$$H=(\nabla\phi)^2+t_0\phi^2+\tfrac{1}{2}u(\phi^2)^2, \qquad (5.2)$$

which generates the perturbation expansion in powers of u. The values of the scaling fields are, for this special H,

$$g_1=(t_0+uN_c)/u, \qquad (5.3)$$

$$g_2=\tfrac{1}{2}(u^{*-1}-u^{-1}), \qquad (5.4)$$

$$g_m=-a_m^*, \quad \text{for } m\geq 3 \qquad (5.5)$$

where

$$u^*=(2a_2^*)^{-1}, \quad N_c=-a_1^*. \qquad (5.6)$$

[See (4.9) for a_m^*.] We define $G(k)$, r by

$$G(k)\equiv\langle\phi_{\sigma k}\phi_{\sigma -k}\rangle, \qquad (5.7)$$

$$G(0)\equiv 1/r. \qquad (5.8)$$

In the large-n limit, we have

$$G(k)=(r+k^2)^{-1}, \qquad (5.9)$$

$$r=t(N)=t_0+uN=u(g_1+N-N_c), \qquad (5.10)$$

$$N\equiv\langle\phi^2\rangle=\tfrac{1}{2}n(2\pi)^{-d}\int d^dk\,G(k) =\tfrac{1}{2}nK_d\int_0^\Lambda dk\,k^{d-1}/(r+k^2), \qquad (5.11)$$

where K_d is given by (4.2). These equations fix r in terms of t_0 and u. We shall be interested only in cases where $r\ll\Lambda^2$. In these cases,

$$N-N_c=-\tfrac{1}{2}nr^{d/2-1}J/(\tfrac{1}{2}d-1)-\tfrac{1}{2}nr/u^*[1+O(r/\Lambda^2)], \qquad (5.12)$$

where

$$J\equiv\tfrac{1}{2}K_d\pi(\tfrac{1}{2}d-1)\csc\pi(\tfrac{1}{2}d-1). \qquad (5.13)$$

Substituting (5.12) in (5.10), we obtain

$$(2/n)g_1=r^{d/2-1}[J/(\tfrac{1}{2}d-1)-(2/n)g_2r^{2-d/2}] =r^{d/2-1}[J/(\tfrac{1}{2}d-1)+O((r/\Lambda^2)^{2-d/2})], \qquad (5.14)$$

where we have assumed that u is of the same order of magnitude as $u^* \sim \Lambda^{4-d}$ for large Λ, so that $g_2 \sim \Lambda^{d-4}$.

We define

$$\hat{\Pi}(k) = (2\pi)^{-d} \int d^d p \, G(p) G(p+k). \tag{5.15}$$

For $r \ll \Lambda^2$, we get from (5.15)

$$\hat{\Pi}(k) = \Pi(r, k^2) - (2/nu^*)[1 + O(r/\Lambda^2)], \tag{5.16}$$

where $\Pi(r, k^2)$ is given by (5.15) except that the p integral is extended to infinity. We quote the results[8]:

$$\Pi(r, k^2) = \Pi(r/k^2, 1) k^{d-4}, \tag{5.17}$$

$$\begin{aligned}\Pi(r, 1) &= \Pi(0,1)(1+4r)^{(d-3)/2} \\ &\quad + 2J(1 - \tfrac{1}{2}d)^{-1} r^{d/2-1} F(1, \tfrac{1}{2}, \tfrac{1}{2}d; -4r) \\ &= \Pi(0,1)[1 - 2(3-d)r + O(r^2)] \\ &\quad + 2J(1-\tfrac{1}{2}d)^{-1} r^{d/2-1}[1 - (4/d)r + O(r^2)],\end{aligned} \tag{5.18}$$

$$\Pi(0, 1) = JB(\tfrac{1}{2}d - 1, \tfrac{1}{2}d - 1), \tag{5.19}$$

$$\Pi(r, 0) = J r^{d/2-2}, \tag{5.20}$$

where F and B are the hypergeometric function and the beta function, respectively.

A frequently occurring quantity will be

$$\begin{aligned} I(k) &\equiv u/(1 + \tfrac{1}{2} nu \hat{\Pi}(k)) \\ &= \frac{2}{n}\left\{\Pi(r, k^2) + \left(\frac{1}{u} - \frac{1}{u^*}\right)\frac{2}{n}\left[1 + O\left(\frac{k^2}{\Lambda^2}, \frac{r}{\Lambda^2}\right)\right]\right\}^{-1} \\ &= \frac{2}{n}\Pi^{-1}(r, k^2)\left\{1 + O\left[\left(\frac{r}{\Lambda^2}\right)^{2-d/2}, \left(\frac{k}{\Lambda}\right)^{4-d}\right]\right\}. \end{aligned} \tag{5.21}$$

When $O(1/n)$ corrections are included, (5.9) becomes

$$\begin{aligned} G(k) &= [t_0 + k^2 + \Sigma(k, r)]^{-1} \\ &= [G_0^{-1}(k) + \Sigma(k, r) - \Sigma(0, r)]^{-1}, \end{aligned} \tag{5.22}$$

$$G_0^{-1}(k) \equiv r + k^2. \tag{5.23}$$

We shall write the self-energy $\Sigma(k, r)$ as

$$\Sigma(k, r) = u\langle\phi^2\rangle + \Sigma_c(k, r). \tag{5.24}$$

The leading term for Σ_c is shown in Fig. 2(b). Our graph notation is standard: a solid line for G_0, a dashed line for $-u$. Each closed loop contributes a factor n and each u contains a factor $1/n$. A wavy line stands for the geometric sum shown in Fig. 2(a).

B. Calculation of y_l

The scaling variables (4.15) are, for the special t given by (5.1),

$$D_l(x) = [t_0 + u\phi^2(x)]^l + O(1/n), \tag{5.25}$$

where the $O(1/n)$ term is some power series in ϕ^2. The coefficients in this power series are smooth functions of $O(1/n)$ of t_0 and u. What we shall do is to compute the correlation function $G_l(p_1, p_2, p_3)$ for $p_1, p_2, p_3 \to 0$. By (3.30), we have

$$\begin{aligned} G_l(0,0,0) &\sim r^{-y_l/(2-\eta)-1} \\ &\sim r^{-d/2+l-1}(1 + b\ln r). \end{aligned} \tag{5.26}$$

By computing the coefficient b of $\ln r$ in G_l, and with known results of η, y_l can be determined to $O(1/n)$. The $O(1/n)$ term in (5.25) will not contribute to $b \ln r$ to $O(1/n)$ because all correlation functions of powers of ϕ^2 have no logarithmic term to $O(1)$. Since $r = t_0 + \Sigma(0, r)$, we write, using (5.24),

$$t = r - \Sigma_c(0, r) + u(\phi^2 - \langle\phi^2\rangle), \tag{5.27}$$

$$\begin{aligned} D_l(x) &= (r - \Sigma_c)^l + l(r - \Sigma_c)^{l-1} u(\phi^2 - \langle\phi^2\rangle) \\ &\quad + \tfrac{1}{2}l(l-1)(r - \Sigma_c)^{l-2} u^2 (\phi^2 - \langle\phi^2\rangle)^2 \\ &\quad + \cdots. \end{aligned} \tag{5.28}$$

Only the second and the third terms in (5.28) contribute to G_l to $O(1/n)$:

$$r^2 G_l(0,0,0) = l(r - \Sigma_c)^{l-1}\Gamma_1 + \tfrac{1}{2}l(l-1) r^{l-2}\Gamma_2, \tag{5.29}$$

where

$$r^{-2}\Gamma_1 = u\int d^d x \, d^d x' \, e^{-ip\cdot x - ip'\cdot x'} \langle \phi^2(0)\phi_\sigma(x)\phi_\sigma(x')\rangle, \tag{5.30}$$

with p, p' approaching zero. Γ_1 is just $-\partial G^{-1}/\partial t_0$

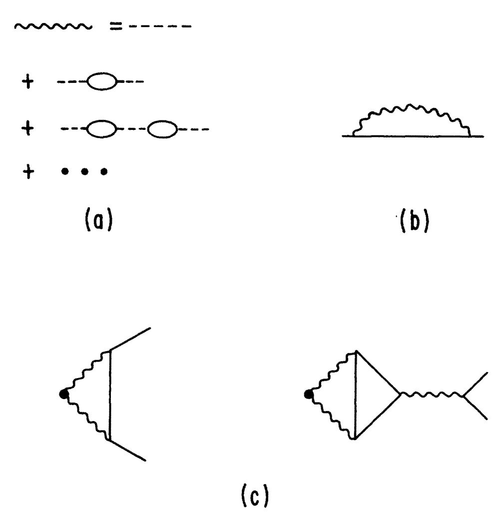

FIG. 2. (a) Wavy line is defined as a geometric sum, and represents a factor $-I$ [see (5.21)]; (b) Σ_c; (c) terms for Γ_2 [see (5.33)].

$\sim r^{1-1/\gamma}$. We simply quote the result to the order of interest:

$$\Gamma_1 = (2/nJ)\, r^{1-1/\gamma}, \tag{5.31}$$

where J is defined by (5.13). The quantity Σ_c is also known [see (C16) in Ref. 7]:

$$\begin{aligned}\Sigma_c &= \text{const} + \text{const}\, r^{d/2-1} \\ &\quad - (K_d/n)\Pi(0,1)^{-1}(5-2d)\, r(\ln r + \text{const}) \\ &\quad + \text{higher orders in } r.\end{aligned} \tag{5.32}$$

Γ_2 is shown in Fig. 2(c). We have

$$\Gamma_2 = K_d \int_0^{\Lambda} dp\, p^{d-1} I(p)^2 \left(\frac{1}{2}\frac{\partial}{\partial r}\, n\hat{\Pi}(p) I(0) + \frac{2}{r+p^2}\right). \tag{5.33}$$

Using the information provided by (5.15)–(5.21), the logarithmic term in (5.33) can be extracted. We obtain

$$\begin{aligned}\Gamma_2 &= (2/n)^2 r^{-d/2+2}(-r\ln r + r\,\text{const}) \\ &\quad \times \tfrac{1}{2}K_d \Pi(0,1)^{-1}\{4[(4-d)^2-1] - 8(3-d)^2\} \\ &\quad + \text{const} + \text{const}\, r^{-d/2+2} \\ &\quad + \text{higher orders in } r.\end{aligned} \tag{5.34}$$

Substituting (5.31), (5.32), and (5.34) in (5.29), and comparing the result with (5.26), we can extract y_l. We find

$$\begin{aligned}y_l &= d - 2l + [4(d-1)/d]\{d - 2 - (l-1)[2 + d(d-4)]\} \\ &\quad \times (2/n)S_d + O(n^{-2}), \quad l = 1, 2, \ldots\end{aligned} \tag{5.35}$$

where

$$S_d = \frac{1}{2} K_d \Pi(0,1)^{-1} = \frac{\sin\pi(\frac{1}{2}d-1)}{\pi(\frac{1}{2}d-1)B(\frac{1}{2}d-1, \frac{1}{2}d-1)}. \tag{5.36}$$

We have used the expressions[7]

$$\begin{aligned}\gamma^{-1} &= (\tfrac{1}{2}d-1)(1+6S_d/n), \\ \eta &= 4(4/d-1)S_d/n\end{aligned} \tag{5.37}$$

in arriving at (5.35).

C. Calculations of y_Δ

First, let us look at the correlation function $G_{\bar l}(0,0,0)$ for a general l. Our notation here is that

$$D_{\bar l} \equiv \bar{z}_l = (\nabla^l \phi)^2 - M_l(t), \tag{5.38}$$

with M_l defined by (4.25). To the leading order in $1/n$, we have

$$\begin{aligned}\langle \bar{z}_l \rangle &= \langle (\nabla^l\phi)^2\rangle - \langle M(t)\rangle \\ &= \tfrac{1}{2} n K_d \int_0^{\Lambda} dp\, p^{d-1+2l}(r+p^2)^{-1} - M_l(r) \\ &= -(-r)^l\, \tfrac{1}{2} n J r^{d/2-1}/(\tfrac{1}{2}d-1).\end{aligned} \tag{5.39}$$

The leading contribution to $G_{\bar l}(0,0,0)$ is shown in Fig. 3(a). We have [see (5.21) for $I(p)$]

$$\begin{aligned}G_{\bar l}(0,0,0)\, r^2 &= -\frac{\partial}{\partial r}\langle(\nabla^l\phi)^2\rangle[-I(0)] - \frac{\partial M_l(r)}{\partial r} I(0) \\ &= -(-r)^l[1 + l/(\tfrac{1}{2}d-1)].\end{aligned} \tag{5.40}$$

Since $G_{\bar l} \sim r^{-y_{\bar l}/(2-\eta)-1}$, we see that

$$y_{\bar l} = 2 - 2l + O(1/n). \tag{5.41}$$

Of course, $\bar{z}_l$ was designed to give this result. Here we only illustrate the role of the subtraction term $M_l(t)$. It is just what is needed to remove all the Λ dependent terms implied by $(\nabla^l\phi)^2$ and to give the right power of r.

We proceed to evaluate the $O(1/n)$ term for $y_{\bar 1} \equiv y_\Delta$, i.e., for $l=1$ only. The graphs are shown in Figs. 3(b)–3(e). The shaded piece is simply the $O(1/n)$ correction to the right-hand part of Fig. 3(a), which is $r^{1-1/\gamma}$. Thus, the sum of graphs in Figs. 3(a) and 3(b) gives

$$r^{1-(\gamma^{-1})_1}[\tfrac{1}{2}d/(\tfrac{1}{2}d-1)], \tag{5.42}$$

where $(\gamma^{-1})_1 = -3(\frac{1}{2}d-1)(2/n)S_d$ is the $O(1/n)$ part of γ^{-1}. The contribution of Fig. 3(c) is

$$\left(\frac{\partial}{\partial r}(2\pi)^{-d}\int d^dq\, T(q)\Pi(q)^{-1} + \Sigma_c(0,r)\frac{\partial^2}{\partial r^2}\langle(\nabla\phi)^2\rangle\right) \times (-J^{-1})\, r^{2-d/2}, \tag{5.43}$$

where

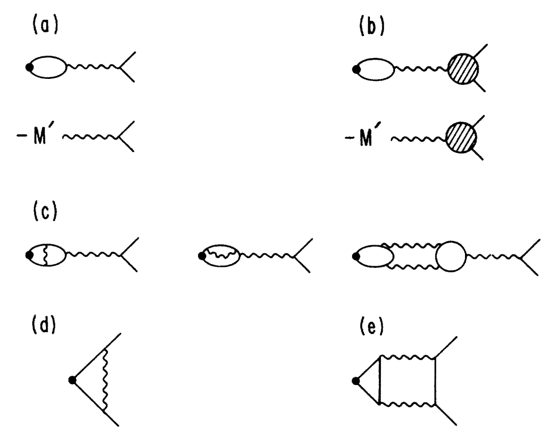

FIG. 3. (a) Leading terms for $G_{\bar l}$ [see (5.40)]; (b) corrections to the left part of (a); (c)–(e) more correction terms.

$$T(q) \equiv (2\pi)^{-d} \int d^d p\, p^2 (r+p^2)^{-2} [r+(p+q)^2]^{-1}$$
$$= \Pi(q) + \tfrac{1}{2} r \frac{\partial \Pi(q)}{\partial r}. \tag{5.44}$$

The first term in large parentheses in (5.43) gives

$$-JS_d \tfrac{1}{2} d(\tfrac{1}{2}d-1)^{-1}(d-2)(d-1) r^{d/2-1} \ln r \tag{5.45}$$

and the second term in large parentheses gives

$$-JS_d \tfrac{1}{2} d(5-2d) r^{d/2-1} \ln r. \tag{5.46}$$

Irrelevant terms are not listed in (5.45), (5.46), and subsequent formulas. Substituting (5.45) and (5.46) in (5.43) and combining the result with (5.42), one finds that the coefficient of $\ln r$ from (5.43) cancels that from $r^{-(\gamma^{-1})_1}$. It remains to evaluate the graphs in Figs. 3(d) and 3(e). Figure 3(d) gives

$$-(2/n) K_d \int dp\, p^{d-1} p^2 (r+p^2)^{-2} \Pi^{-1}(p)$$
$$= -(2/n) S_d\, 2(d-2) r \ln r. \tag{5.47}$$

Figure 3(e) gives

$$2(2/n)(2\pi)^{-d} \int d^d q\, T(q)(r+q^2)^{-1} \Pi(q)^{-2}$$
$$= 2(2/n) S_d (d-2) r \ln r. \tag{5.48}$$

We see that the $\ln r$ terms in these two formulas cancel. Therefore, we conclude that

$$y_\Delta = 0 + O(n^{-2}). \tag{5.49}$$

D. Evaluation of $y_{\phi 1}$

We turn to the scaling variable $D_{\phi 1}$. Consider the correlation function

$$G_{\phi 1}(p) = \int d^d x\, e^{-ip\cdot x} \langle \phi_\sigma(0)\, t(\phi^2(0))\, \phi_\sigma(x) \rangle \tag{5.50}$$

for $p \to 0$. One easily derives the critical behavior

$$G_{\phi 1}(0) r \sim r^{-(y_{\phi 1} - d/2 - 1 + \eta/2)/(2-\eta)}. \tag{5.51}$$

Again, using (5.27), $t = r - \Sigma_c + u(\phi^2 - \langle \phi^2 \rangle)$, we obtain

$$G_{\phi 1}(0) r = (r - \Sigma_c) + \Sigma_c = r, \tag{5.52}$$

where the last Σ_c comes from the graph in Fig. 4(a). By (5.51), we conclude that

$$y_{\phi 1} = \tfrac{1}{2}(d - 2 + \eta) + O(n^{-2}). \tag{5.53}$$

E. Evaluation of y_τ

The scaling variable τ is given by (4.35) $\tau_{\sigma\sigma'} = [\phi_\sigma \phi_{\sigma'} - (2/n)\phi^2 \delta_{\sigma\sigma'}]$. We have

$$G_\tau(0,0,0) r^2 \sim r^{1 - y_\tau/(2-\eta)}. \tag{5.54}$$

The $O(1)$ term simply gives 1 for the left-hand side. The only $O(1/n)$ graph is one that looks like Fig. 3(d). The traceless nature of τ excludes other graphs. The evaluation is easy, and we obtain

$$-(2/n) K_d \int dp\, p^{d-1} (r+p^2)^{-2} \Pi^{-1}(p) = (2/n) S_d \ln r, \tag{5.55}$$

therefore

$$y_\tau = [1 - (2/n) S_d](2-\eta) + O(n^{-2}). \tag{5.56}$$

F. Evaluation of $y_{\Delta\tau}$ and $y_{\tau 1}$

The variables $\lambda_{\sigma\sigma'}$ [see (4.38)] and $t\tau_{\sigma\sigma'}$ must be considered together because they are of identical symmetry under rotations and have the same dimension to the lowest order in $1/n$. As mentioned before, we have to consider linear combinations of $\lambda_{\sigma\sigma'}$ and $t\tau_{\sigma\sigma'}$. Let $D = a\lambda + t\tau$, and

$$G_D \equiv a G_\lambda + G_{t\tau} \tag{5.57}$$

in an obvious notation. For the right choices of the coefficient a, we would get [see (3.30) and (3.32)]

$$G_D(0,0,0) r^2 \sim r^{1-y/(2-\eta)}, \tag{5.58}$$

$$\left(\frac{\partial G_D(0,k,-k)}{\partial k^2} \right)_{k=0} r^2 \sim r^{-(y+\eta)/(2-\eta)}. \tag{5.59}$$

$G_\lambda(0,0,0) r^2$ to $O(1/n)$ comes from a graph like Fig. 3(d) only and gives

$$G_\lambda(0,0,0) r^2 = -2(d-2) S_d (2/n) r \ln r. \tag{5.60}$$

Since $t\tau_{\sigma\sigma'} = (r - \Sigma_c)\tau_{\sigma\sigma'} + \tau_{\sigma\sigma'} u(\phi^2 - \langle \phi^2 \rangle)$, we obtain

$$G_{t\tau}(0,0,0) r^2 = (r - \Sigma_c) G_\tau(0,0,0) r^2 + 2\Sigma_c$$
$$= r[1 + 2(d-2)(\ln r) S_d\, 2/n], \tag{5.61}$$

where the last term $2\Sigma_c$ comes from the graph in Fig. 4(b). Similarly, $\partial G_D / \partial k^2$ can be calculated. We obtain

$$\frac{r^2 \partial G_\lambda}{\partial k^2} = 1 + \tfrac{1}{2} \eta \ln r, \tag{5.62}$$

$$\frac{r^2 \partial G_{t\tau}}{\partial k^2} = -\eta \ln r. \tag{5.63}$$

(a) (b)

FIG. 4. (a) $O(1/n)$ graph for $G_{\phi 1}$ [see (5.52)]; (b) $O(1/n)$ graph for $G_{t\tau}$ [see (5.61)].

Thus, (5.57)–(5.63) give

$$G_D r^2 = a[-2(d-2)S_d 2/n] r \ln r + r + 2(d-2)S_d(2/n) r \ln r = r(1 - \tfrac{1}{2}y \ln r), \tag{5.64}$$

$$\frac{r^2 \partial G}{\partial k^2} = a(1 + \tfrac{1}{2}\eta \ln r) - \eta \ln r = a[1 - \tfrac{1}{2}(y+\eta)\ln r], \tag{5.65}$$

therefore

$$-\tfrac{1}{2}y = (1-a)2(d-2)2S_d/n, \tag{5.66}$$

$$-\tfrac{1}{2}(y+\eta) = \eta(\tfrac{1}{2} - 1/a). \tag{5.67}$$

There are two sets of solutions for y and a:

$$D_{\Delta\tau,\sigma\sigma'} = \lambda_{\sigma\sigma'} + t\tau_{\sigma\sigma'}, \quad y_{\Delta\tau} = 0 + O(n^{-2}), \tag{5.68}$$

for $a = 1$, and

$$D_{\tau 1,\sigma\sigma'} = \frac{d-4}{d(d-2)}\lambda_{\sigma\sigma'} + t\tau_{\sigma\sigma'}, \quad y_{\tau 1} = -8(d-3+4/d)S_d/n + O(n^{-2}), \tag{5.69}$$

for $a = (d-4)d^{-1}(d-2)^{-1}$.

G. Evaluation of y_T

We turn to the traceless tensor variable $T_{ab} = \sum_\sigma \nabla_a\phi_\sigma\nabla_b\phi_\sigma - (2/d)(\nabla\phi)^2\delta_{ab}$. The $O(1/n)$ graphs for G_T are given in Figs. 3(d) and 3(e). The other graphs in Fig. 3 will not contribute due to the traceless condition. For our purpose, it is sufficient to consider the coefficient of $k_a k_b$ for $a \neq b$ in

$$G_T(0,k,-k) r^2 \sim r^{-(y_T+\eta)/(2-\eta)} k_a k_b + O(k^4). \tag{5.70}$$

The trivial term for G_T is simply $k_a k_b$. The contribution of Fig. 3(d) is

$$-\frac{2}{n}(2\pi)^{-d}\int d^d p\,(p+k)_a(p+k)_b[r+(p+k)^2]^{-2}\Pi^{-1}(p) = -\frac{2}{n}S_d\,\frac{(4-d)(d-2)}{d(d+2)}(\ln r)k_a k_b. \tag{5.71}$$

Figure 3(e) is more difficult to evaluate. It gives

$$2(2/n)(2\pi)^{-d}\int d^d q\,\Gamma(q)q_a q_b \Pi^{-2}(q)[r+(q+k)^2]^{-1}, \tag{5.72}$$

where

$$\Gamma(q)q_a q_b \equiv (2\pi)^{-d}\int d^d p\,p_a p_b(r+p^2)^{-2}[r+(p+q)^2]^{-1}. \tag{5.73}$$

A counting of powers of q in (5.72) shows that, to calculate the $(\ln r)k_a k_b$ term, only $\Gamma(q)$ with $r=0$ is required. Once r is set to zero, the integral (5.73) is straightforward. One finds

$$\Gamma(q) = \tfrac{1}{4}(4-d)\Pi(0,1)q^{d-6}, \tag{5.74}$$

which we substitute in (5.72) to obtain

$$-\frac{4(4-d)}{d(d+2)}S_d(2/n)(\ln r)k_a k_b. \tag{5.75}$$

To sum up, we have

$$r^2 G_T(0,k,-k) = k_a k_b[1 - (1/d)(4-d)(2/n)S_d \ln r] = k_a k_b(1 - \tfrac{1}{2}\eta \ln r). \tag{5.76}$$

In view of (5.70), we conclude that

$$y_T = 0 + O(n^{-2}). \tag{5.77}$$

H. Evaluation of y_Υ

Finally, we come to the scaling variable $\Upsilon_{ab\sigma\sigma'}$ given by (4.40). This is again a traceless tensor and we shall evaluate the $k_a k_b (a \neq b)$ term of

$$G_\Upsilon(0,k,-k)r^2 \sim k_a k_b r^{-(y_\Upsilon+\eta)/(2-\eta)} + O(k^4). \tag{5.78}$$

The $O(1)$ term is just $k_a k_b$. The $O(1/n)$ term is again given by Fig. 3(d) and the answer given by (5.71). We get

$$y_\Upsilon = -8(4/d-1)(d+2)^{-1}S_d 2/n + O(n^{-2}). \tag{5.79}$$

I. Determination of the gradient term in $D_{\phi 1}$

In the $n\to\infty$ limit, the variables $t\phi_\sigma$ and $\nabla^2\phi_\sigma$ have the same dimension, $\frac{1}{2}d+1$. Thus we expect that

$$D_{\phi 1} = t\phi_\sigma - a\nabla^2\phi_\sigma, \tag{5.80}$$

in the same fashion as the case of degeneracy discussed in Sec. V F. Note that $\nabla^2\phi_\sigma$ needs no modification. The gradient of a scaling variable is always a scaling variable. We need to determine the constant a. The gradient term in (5.80) does not contribute to the correlation function calculation of Sec. V D, which was done at $k=0$. To determine the constant a, let us take $k\neq 0$, $r=0$. If $G_{\phi 1}(k)\neq 0$ at $r=0$, then we expect that

$$G_{\phi 1}(k)G(k)^{-1} \sim k^{-y_{\phi 1}+d/2+1-\eta/2} \sim k^{2-\eta}, \tag{5.81}$$

where we have used (5.53) for $y_{\phi 1}$. The calculation of $G_{\phi 1}(k)$ is trivial. For $r=0$, we have $t = -\Sigma_c + u(\phi^2 - \langle\phi^2\rangle)$. Figure 4(a) gives the $O(1/n)$ contribution:

$$G_{\phi 1}(k)G(k)^{-1} = -\Sigma_c + \Sigma_c(k) + ak^2 = -\eta k^2 \ln k + ak^2. \tag{5.82}$$

Clearly, the answer is $a=1$.

J. Gradient term in D_2

Here is another case of degeneracy at $n\to\infty$. The variables t^2 and $\nabla^2 t$ both have the dimension 4 for

$n\to\infty$. Again we write

$$D_2 = t^2 - a_2\nabla^2 t. \tag{5.83}$$

As in the last case, the calculation of y_2 was done at $k=0$ and the gradient term was not needed. To determine a_2, we shall calculate

$$\Gamma(k) \equiv G_2(-k, \tfrac{1}{2}k, \tfrac{1}{2}k)\,G(\tfrac{1}{2}k)^{-2} = \Gamma_2(k) + a_2 k^2 \Gamma_1(k), \tag{5.84}$$

at $r=0, k\neq 0$. For our purpose,

$$\Gamma_1(k) = (2/n)\Pi(0,1)^{-1}k^{-y_1+2-\eta}, \tag{5.85}$$

and Γ_2 is the contribution of Fig. 2(c). We must have

$$\Gamma(k) \sim k^{-y_2+2-\eta} \tag{5.86}$$

if it is not zero at $r=0$. If Γ_2 has a term proportional to $\ln k$, i.e.,

$$\Gamma_2 = c(2/n)\Pi(0,1)^{-1}k^{6-d}\ln k + \cdots, \tag{5.87}$$

then, in order that (5.84)–(5.86) are consistent to $O(1/n)$, we must have $-y_1 + 2 + c/a_2 = -y_2$, namely,

$$a_2 = c/(y_1 - y_2 - 2). \tag{5.88}$$

To find c, we write down Γ_2 following Fig. 2(c):

$$\Gamma_2 = (2/n)^2(2\pi)^{-d}\int d^dq\,\Pi(q)^{-1}\Pi(q+k)^{-1} \times[-2T(k,q)\Pi(k)^{-1} + 2(q+\tfrac{1}{2}k)^{-2}], \tag{5.89}$$

where $T(k,q)$ is the solid-line triangle [not (5.44)] in the first graph. It has been studied in Ref. 7 using the Mellin transform [see (5.14)–(5.18) of Ref. 7]. It was shown there that

$$T(k,q) = k^{d-6}\,T(\hat{k}, q/k), \tag{5.90}$$

$$T(\hat{k},q) = \Pi(0,1)q^{-2} + Aq^{d-6} + Bq^{-3} + Cq^{d-7} + Dq^{-4} + Eq^{d-8} + \cdots, \tag{5.91}$$

where $\hat{k}$ is the unit vector along k, and $A, B, \ldots$ are angular integrals. Replace q in (5.89) by kq; then expand $\Pi(q+\hat{k})^{-1}$ in powers of $1/q$. Substituting (5.91) in (5.89), we collect terms proportional to dq/q:

$$\Gamma_2(k) = k^{6-d}(2/n)^2\Pi(0,1)^{-3}(-2)K_d \times\int^{1/k}(dq/q)\{E + (2-\tfrac{1}{2}d)[1 + (2-d)\langle(\hat{q}\cdot\hat{k})^2\rangle]A\}, \tag{5.92}$$

where $\langle(\hat{q}\cdot\hat{k})^2\rangle = 1/d$ stands for the angular average. All other terms are of no interest. The coefficient A is given by (5.29) of Ref. 7:

$$A = -\pi\csc\pi(6-d)K_{d-1}2^{2-d}(d-3)\sin\tfrac{1}{2}\pi(5-d). \tag{5.93}$$

The coefficient E is the residue of the pole at $s = 8-d$ of (5.21) of Ref. 7:

$$E = -\pi\csc\pi(8-d)\frac{K_{d-1}}{2\pi} \times\int_0^\pi d\phi\,\sin^{d-2}\phi\,\frac{\sin(7-d)\phi\,\sin3\theta}{\sin\phi\,\sin\theta}. \tag{5.94}$$

Replacing $\sin3\theta/\sin\theta$ by $-1+4\langle\cos^2\theta\rangle = -1+4/d$, the ϕ integral can be done. Substituting the result and (5.93) in (5.92), we obtain the constant c defined by (5.87):

$$c = -2(d-3)(4-d)(6-d)d^{-1}(2/n)S_d. \tag{5.95}$$

By (5.88), and the values of $y_1 y_2$ given in (1.14), we obtain

$$a_2 = -\frac{(d-3)(4-d)(6-d)}{2(d-1)[2-d(4-d)]}. \tag{5.96}$$

Within the range $2<d<4$, a_2 vanishes at $d=3$. It blows up at $d=2+\sqrt{2}$, where the degeneracy $y_1 = y_2 + 2$ remains to $O(1/n)$. This degeneracy may remain to higher orders with a slightly different d. At this special d, $D_2 \propto (\nabla^2 D_1 +$ terms which vanish at the fixed point).

VI. DISCUSSION

Our results have been summarized in some detail in Sec. I. Here we add a few remarks.

A. Exponents

The exponents y_i calculated in the previous section are summarized in (1.13). The dimensions of the corresponding local scaling variables are given by $d - y_i$. An over-all qualitative view is given by Fig. 1. The exponents $y_1 = 1/\nu$ and $y_\phi = \frac{1}{2}(d+2-\eta)$ are familiar. Some of the other exponents have been discussed by many authors. Kawasaki discussed the strain tensor and y_T.[9] In connection with the anisotropic perturbation, y_T was examined by Riedel and Wegner,[10] Fisher and Pfeuty,[11] and Wallace.[12] Wegner made more general classifications of perturbations to the symmetric n-vector model.[13] The exponent y_2 gives an estimate of how fast the fixed point is approached under R_s, and was examined by many authors.[2,3,13–15]

Some of the results listed in (1.13) are expected to be general, i.e., they should hold when n is not large. One expects that $y_\Delta = y_T = 0$ should hold in general also in view of the argument that components of the strain tensor, which is dimensionless, should appear as scaling fields g_Δ (the trace, i.e., the dilatation) and $g_{T,ab}$ (the traceless part of the strain tensor). One also expects that the result for $y_{\phi 1}$ is general. (Of course, y_ϕ defines η.) Checking (1.15), one sees that the corresponding scaling variable $D_{\phi 1,\sigma}=\partial H/\partial\phi_\sigma$ can be interpreted as the fluctuation of the local "magnetic field." The exponent $d-y_\phi$ is what one expects for such a magnetic field. The scaling field $g_{\phi 1,\sigma}$ coupled to it is thus a shift in the spin variable ϕ_σ. The scaling variable $D_{\Delta T,\sigma\sigma'}$ may be viewed as a local "stress" in the spin space, and the scaling field $g_{\Delta T,\sigma\sigma'}$ is a "strain" in the spin space. Thus, the exponent $y_{\Delta T}$ should vanish. A shift or a strain in the spin space is a change of variable. We shall discuss this point in more detail shortly.

It would be very desirable to derive more scaling laws to relate one exponent to others. Note that the question whether all the exponents can be expressed in terms of two fundamental exponents η and ν needs more qualification. In principle we can express d and n in terms of η and ν and thereby find all exponents as functions of η and ν, since all exponents are functions of d and n. This is, of course, an extreme statement, but the fact that there are only two variables n and d implies many relationships among the many exponents. Since only ν and η seem to be needed in studying the thermodynamics and the leading behavior of simplest correlation functions near the critical point, a better understanding of the additional exponents would demand more detailed analysis of more correlation functions. The reduction hypothesis seems to be a natural next step.

B. Variables $D_{\phi 1}$ and $D_{\Delta T}$

A perturbation term $g_\sigma D_{\phi 1,\sigma}$ added to H can be generated by a change of variable

$$\phi_\sigma \to \phi_\sigma + g_\sigma \tag{6.1}$$

in H, since $D_{\phi 1,\sigma}=\partial H/\partial\phi_\sigma$. As a consequence, the free energy will not depend on the scaling field g_σ since it is unchanged under (6.1).[16] Wegner calls such scaling fields, which can be removed from the free energy by a change of variable, "redundant."[17] It is warned in the last part of the Appendix of Ref. 2 (Wilson and Kogut) that the scaling variables associated with some scaling fields are "equivalent to zero" in some sense. An equivalent comment is made in Ref. 17. Let us clarify this point.

The correlation function $G_{\phi 1}(p_1\cdots p_m;g)$ is the Fourier transform [see (3.25)] of

$$\langle D_{\phi 1,\sigma}(x_1)\phi(x_2)\cdots\phi(x_m)\rangle_P \propto \int\delta\phi\left(-\frac{\delta}{\delta\phi_\sigma(x_1)}e^{-\mathcal{H}}\right)\phi(x_2)\cdots\phi(x_m)=0, \tag{6.2}$$

unless x_1 = one or more of $x_2, x_3, \ldots, x_m$. We have integrated by parts and made use of the fact that

$$D_{\phi 1,\sigma}(x_1)e^{-\mathcal{H}} = -\frac{\delta}{\delta\phi_\sigma(x_1)}e^{-\mathcal{H}}, \tag{6.3}$$

$$\frac{\delta}{\delta\phi_\sigma(x)}\phi_{\sigma'}(x')=\delta_{\sigma\sigma'}\delta(x-x'). \tag{6.4}$$

Of course, our $\delta(x-x')$ has a spatial resolution $\sim\Lambda^{-1}$. In the sense of (6.2), $D_{\phi 1}$ is equivalent to zero, or shows no long-range correlation even at T_c.

Similarly, a term $g_{\sigma\sigma'}D_{\Delta T,\sigma\sigma'}$ added to H can be generated by a change of variable

$$\phi_\sigma \to \phi_\sigma + g_{\sigma\sigma'}\phi_{\sigma'}, \tag{6.5}$$

where $\sigma\neq\sigma'$. Thus $g_{\sigma\sigma'}$ is "redundant" also. In view of the expression for $D_{\Delta T}$ in (1.15), we see that

$$D_{\Delta T,\sigma\sigma'} = \frac{\delta\mathcal{H}}{\delta\phi_\sigma}\phi_{\sigma'}. \tag{6.6}$$

By the same arguments as above, we conclude that

$$\langle D_{\Delta T,\sigma\sigma'}(x_1)\phi(x_2)\cdots\phi(x_m)\rangle_P \propto \int\delta\phi\left(-\frac{\delta}{\delta\phi_\sigma(x_1)}e^{-\mathcal{H}}\right)\phi_{\sigma'}(x_1)\phi(x_2)\cdots\phi(x_m)=0, \tag{6.7}$$

unless x_1 = one of $x_2,\ldots,x_m$. The above argument runs parallel to some of the development in Sec. 2 of Ref. 17.

The results (6.2) and (6.7) are consistent with our previous general conclusion on correlation functions in Sec. III. This consistency is possible only if the exponents $y_{\phi 1}$ and $y_{\Delta T}$ assume the values given by (1.13). For example, since $y_{\Delta T}=0$, (3.33) implies that

$$G_{\Delta T}(0,k,-k;g(T_c))\sim k^{-2+\eta}, \tag{6.8}$$

which is the behavior for $G(k)$. In other words, $D_{\Delta T}$ acts as a constant attached to one of ϕ's. Another example is given by (5.81), which says that

$$G_{\phi 1}(k)\sim 1 \tag{6.9}$$

at T_c, i.e., ~Fourier transform of $\delta(x)$.

In spite of the trivial features of these redundant scaling variables, they are still needed in forming a complete set of scaling variables. Precisely how important they are is a question to be studied further.

C. Physical meaning of scaling variables

The variable ϕ_σ is the order parameter and we have indicated above that $D_{\phi 1}$ can be interpreted as the local fluctuation of "magnetic field." It is also reasonable to identify D_Δ and $D_{T,ab}$ respectively as the trace and the traceless part of the stress tensor.

It would be very nice to associate each of the rest of the scaling variables listed in (1.15) with a familiar physical quantity, too. We feel that, for best results of such effort, the association should be made individually for each model of interest. A general association may often be pointless. For example, the most important variable D_1 has been often interpreted as the energy density fluctuation in model systems. Would it be useful to think of D_1 this way in general? Since $D_1 = \partial H/\partial g_1$ and $g_1 \propto (T - T_c)$ for $T - T_c \to 0$ [see (2.15)], then near T_c we have $D_1 \propto \partial H/\partial T$. Since H may be thought of as the "local free energy," then D_1 is proportional to the local entropy density fluctuation. From the relation $E = F + TS$, D_1 also accounts for the fluctuation of local energy density (the free energy does not fluctuate as much). Clearly, what is more convenient is not to think of D_1 as the energy density or entropy density, but to do it the other way around. Namely, the energy density, entropy density, and perhaps other density fluctuations are all dominated by the scaling variable D_1. More physical significance should be attached to the scaling variables, rather than to the familiar quantities. More precise and useful interpretation can be made when the details of the model are taken into consideration. For example, let us take our model as a model of $\frac{1}{2}n$-component Bose system. Then ϕ^2 is the density of Bosons. For large n, $D_1 = t = \partial U/\partial \phi^2$ is thus the effective local potential seen by a Boson. In this case, the Hartree approximation is a good one. The local potential plays a dominating role and turns out to be precisely the leading scaling variable D_1.

D. Universality

Although scaling variables depend on the details of H, certain correlation functions of them have universal (i.e., independent of the details of H) features, as mentioned in Sec. III. In fact, the scaling variables are so structured to ensure this universality. These correlation functions are very important and are analogous to the matrix elements of tensor operators of the rotation group. We have not made a detailed study of the universal correlation functions.[18] The exponents y_i are also universal. They are analogous to the quantum numbers specifying irreducible representations of the rotation group. All the universal features seem to be *geometrical* in nature, in that n and d appear as the only parameters. However, we know that an averaging process, which is *statistical* in nature, is a vital part of R_s. Namely, R_s is not a purely geometrical operation. Which of the universal features are statistical in nature? That is an intriguing and important question.

E. Conclusion

The scaling variables should be the basic set of variables for studying general problems in critical phenomena or any subject where the renormalization group is a dominating feature. They are entirely analogous to the tensor operators of the rotation group in atomic or nuclear physics. By the use of scaling fields, we have shown that the concepts of scaling dimension and scaling variables can be clearly defined away from the fixed point as well as at the fixed point. We hope the above study has made this point transparent, and has provided a fairly concrete qualitative picture as well as some useful quantitative information for the case of large n.

ACKNOWLEDGMENT

I thank Professor K. M. Watson and other members of the Berkeley Physics Department for hospitality .

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*Alfred P. Sloan Foundation Fellow; supported in part by the National Science Foundation under Grant No. GP-38627X; supported in part at University of California at Berkeley by the Air Force Office of Scientific Research under Contract No. F44620-70-C-0028.
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