Hydrodynamic fluctuations near the convection instability

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Hydrodynamic fluctuations of a horizontal liquid layer heated from below are considered in the vicinity of the point where convection sets in because of buoyancy. It is assumed that convection occurs in the form of nearly two-dimensional rolls. Close to the instability, the hydrodynamics (described in the Boussinesq approximation) is simplified considerably by the appearance of a slow mode which dominates the motion of all hydrodynamic variables. It is described by a slowly varying complex amplitude whose absolute value and phase describe the strength and the position of the convection rolls, respectively. Generalizing previous work by several authors, an approximate equation of motion is derived, which is satisfied by the slow variable. New in this analysis is the inclusion of fluctuating terms, which leads to a Langevin equation. The fluctuations are shown to satisfy a detailed-balance principle. Consequently, a generalized thermodynamic potential can be defined, which was discussed briefly in an earlier paper. It depends as a functional on the slow variable, which thereby assumes the role of an order parameter of the transition. I give a further evaluation of the hydrodynamic fluctuations for a horizontally unbounded liquid layer on both sides of and at the Bénard point by using my potential and applying various approximations. For strictly two-dimensional flows (i.e., independent of one horizontal coordinate) I calculate the time-independent steady-state properties (coherence lengths) without any further approximation by relying on published numerical data obtained for one-dimensional Ginzburg-Landau fields. Dynamic steady-state properties (coherence times) for that case and fluctuations in the three-dimensional case are calculated in a quasilinear approximation which reproduces the time-independent results for two-dimensional flows reasonably well. In the purely heat-conducting region my results contain some earlier results of Zaitsev and Shliomis in lowest order. Large and long-lived fluctuations of velocity and temperature are shown to appear at the critical wave number as the liquid is brought near the convection instability. They are due to the random appearance and disappearance of convection cells. Their size and lifetime at the Bénard point are only limited by nonlinear coupling of the critical modes to other passive modes. In the heat-convection region, the coupling to passive modes stabilizes the amplitude of the convection cells; only slow fluctuations of the positions of the rolls remain (for unbounded layers) and destroy the long-range order of the one-dimensional roll lattice, in agreement with well-known general theorems. If the Bénard point is approached from this side, the stabilizing influence of the passive modes decreases and is efficient only for the large fluctuations at the Bénard point. Approached from either side, the Bénard point resembles the critical point of a Landau phase transition. The width of the region around the Bénard point where the Landau description breaks down is calculated and found to be unobservably small in realistic liquids. An experimental check of these results, though very tedious, seems possible and very worthwhile.

I. INTRODUCTION

A plane horizontal fluid layer at rest, if heated from below, develops a buoyancy force due to volume expansion of the liquid near the bottom. For sufficiently strong heating, the buoyancy force may overcome the viscous shear forces and the fluid starts moving; the liquid undergoes a transition from a state where heat is transported by heat conduction only to a state of combined heat conduction and heat convection. The first experimental investigation of this transition was undertaken by Bénard[1] in 1900, who established that convection occurs in the form of regular convection cells spread over the liquid layer in the form of a regular lattice. The only cell pattern observed by Bénard was a lattice of hexagonal cells.

Lord Rayleigh[2] in 1916 first provided a theory that explains the onset of heat convection. He achieved this goal by means of a linearized perturbation analysis of the hydrodynamic equations around the state at rest. He considered the case of free boundaries at fixed temperatures on the top and on the bottom of the fluid. This case, though physically unrealistic, still contains all qualitative features of the transition and allows for an analytic solution. According to Rayleigh's solution, instability sets in at the critical wave number $k_c = \pi/\sqrt{2}\, l$ as soon as the Rayleigh number $R = g\beta\Delta T\, l^3\rho^2 C_v/\eta\kappa$ is made larger than the critical value $R_c = \frac{27}{4}\pi^4$ (for notation see below). A further result was that for $R > R_c$, convection may occur within a finite band of wave numbers around k_c. More realistic boundary conditions require numerical solutions, and were considered later. Most of these calculations, like Rayleigh's, are based on a set of approximate hydrodynamic equations, suitable for the description of free thermal con-

vection (Boussinesq approximation[3]).

Since then, a large number of experimental and theoretical investigations have been published.[3,4,5] They confirmed the quantitative results for the critical temperature difference very well, but also showed that a great variety of cell patterns is possible, and is observed in experiments with different liquids and boundary conditions.[6] Such cell patterns include plane two-dimensional rolls (i.e., independent of one horizontal Cartesian coordinate), square and rectangular cells and more complicated forms. In the linear-stability theory,[3] they are not distinguished, and correspond to the same critical wave number and Rayleigh number. A further experimental result[7] was the fact that even for $R>R_c$, convection was mainly observed with the wave number k_c. Therefore, the main drawback of the linear theory appears to be its inability to predict a definite cell pattern, and to explain why cells with just the critical wave number occur also for $R>R_c$.

A nonlinear approach to the Bénard problem began to be developed by many authors.[4,8-10] Perhaps the most generally applicable formulation of this approach, which, in a somewhat altered form, will also be used in the present work, was given by Schlüter, Lortz, and Busse.[11] In their approach, the solution of the hydrodynamic equations with given boundary conditions is sought in the form of a power-series expansion of the unknown functions [like the velocity $\vec{v}(\vec{x})$ and the temperature $T(\vec{x})$] with respect to some unknown small parameter ϵ:

$$\vec{v}(\vec{x})=\epsilon[\vec{v}^{(0)}(\vec{x})+\epsilon\vec{v}^{(1)}(\vec{x})+\cdots], \tag{1.1}$$

$$T(\vec{x})=\epsilon[T^{(0)}(\vec{x})+\epsilon T^{(1)}(\vec{x})+\cdots]. \tag{1.2}$$

The Rayleigh number R, or rather $R-R_c$, is also expanded with respect to ϵ,

$$R=R_c+\epsilon R^{(1)}+\epsilon^2R^{(2)}+\cdots. \tag{1.3}$$

The hydrodynamic equations are then solved iteratively for the functions $\vec{v}^{(0)}(\vec{x})$, $T^{(0)}(\vec{x})$; $\vec{v}^{(1)}(\vec{x})$, $T^{(1)}(\vec{x})$, etc., starting with the lowest power of ϵ. Existence conditions for the solutions in each order of ϵ uniquely determine the unknowns $R^{(1)}$, $R^{(2)}$, etc., and permit ϵ to be expressed in terms of $R-R_c$ if the procedure is stopped at a given order of ϵ.

It is clear that this approach is restricted to a vicinity close to the Bénard point. However, it gives important results about the onset of the convective motion and its behavior for R slightly in excess of R_c. For $R>R_c$, the procedure outlined above, like the linear theory, allows for solutions within a finite band of wave numbers and with various cell patterns. However, it could be shown[11] that, within the Boussinesq approximation, and for R sufficiently close to R_c, only two-dimensional rolls are stable among all cell patterns.[12] Furthermore, only those rolls whose wave number $\vec{k}$ is in the range $k_c\leq|\vec{k}|<k_m$ were found to be stable. The upper bound k_m is of the order of $(k_m-k_c)\sim(R-R_c)^{1/2}$. All rolls with $|\vec{k}|<k_c$ are unstable with respect to the growth of perturbations along the roll axis; for $|\vec{k}|>k_m$, rolls are unstable, either against growth of perturbations oblique to the rolls, or against generation of new parallel rolls with wave numbers inside the region of stability.

It seems reasonable to expect that observed rolls correspond to stable rolls. The question, of course, remains: Why, among all possible wave numbers within the stability region, is just the critical wave number observed experimentally? Segel[13] considered the nonlinear coupling between parallel rolls of different wave numbers and showed that, apart from a slight dependence on the initial conditions, the roll with the largest linear growth rate is most likely to survive. In the present paper, I offer a different explanation by adding fluctuations, i.e., a probabilistic aspect, to the whole picture.[14] I will show that, since the liquid layer is subject to thermodynamic fluctuations, rolls with $|\vec{k}|=k_c$ are the most likely to occur.

There are many reasons why an inclusion of thermodynamic fluctuations into the theory of the Bénard instability is of interest. The large degeneracy of solutions for $R>R_c$ is only one of them. Another one comes from the pronounced similarity of the Bénard point to the critical point of a second-order phase transition: The transition point is experimentally well defined, and corresponds to a point where the symmetry of the system is subject to a qualitative change. One is led, therefore, to expect a pronounced enhancement and a considerable slowing down of thermodynamic fluctuations at the critical wave number near the Bénard point, similar to the occurrence of strong fluctuations near a critical point. One would like to know the typical frequency range of the enhanced fluctuations, in order to decide which method is most suitable to detect them experimentally. Theoretically, one expects that, by their enhancement, the fluctuations will determine the exact nature of the Bénard transition and its relation to second-order phase transitions.

There is an interesting and crucial difference between the fluctuations at the Bénard point and critical fluctuations. In both cases, the fluctuations originate on the microscopic level. Critical fluctuations, and all hydrodynamic fluctuations in general, are obtained by selecting only those fluctuations which have very long wavelengths and

which lie in the hydrodynamic part of the spectrum. This constitutes the macroscopic or hydrodynamic level of description. From that macroscopic level, fluctuations near the Bénard point are obtained by an additional, second selection process, which takes place if R is close to R_c and singles out fluctuations near the critical wave number k_c, which itself lies in the macroscopic range. It is this second selection process which I will analyze in detail in the present paper.

The statements made above can be summarized by saying that, at critical points of second-order phase transitions, the stability of a certain class of (equilibrium) states breaks down on a microscopic level, while, in the Bénard case, the stability of a class of (nonequilibrium steady) states breaks down on a macroscopic level. On the microscopic level, the Bénard instability is probably not felt at all, since (at least according to generally adopted assumptions, which I will not question here) the system remains in a state of stable, local, thermal equilibrium throughout the entire transition.

A further motivation for including thermodynamic fluctuations in a theory of the Bénard instability comes from the fact that a fluid layer near the Bénard point, though locally in thermal equilibrium, must be regarded as a system far from global thermal equilibrium. It has generally been shown[15,16] that, for an instability like the Bénard instability to appear, it is necessary for a system to be driven by some external force (the temperature difference between the top and the bottom of the layer, in the present case) sufficiently far away from global thermal equilibrium, so that its response (i.e., its deviation from thermal equilibrium) becomes nonlinear with respect to the external force.[17]

In the linear range, there exists a very-well-developed thermodynamic theory of nonequilibrium steady states,[18] which combines the theories of the stability and the fluctuations from the steady state in the usual fashion, well known from equilibrium thermodynamics. Much less is known about a thermodynamic theory of fluctuations from the steady state in the nonlinear domain far from equilibrium. A number of thermodynamic variational principles have been proposed for that region[19,20] which are based on variational properties of the entropy production or the energy dissipation, permit the determination of steady states, and give sufficient criteria for their stability. However, a clear-cut relation between the above-mentioned potentials and the fluctuations from the steady state does not seem to exist.[21]

The only thermodynamic theory that allows the calculation of fluctuations in the nonlinear region far from equilibrium, is the theory of Landau and Lifshitz,[22] in which thermodynamically determined fluctuating forces are added to the hydrodynamic equation.[23] In thermal equilibrium, the relevant thermodynamic potential (depending on the boundary conditions), say F, is given by solving these equations for the time-independent probability distribution W of the fluctuations, and taking $F \sim -\ln W$. The question, then, is whether a similar solution W with a similarly corresponding generalized thermodynamic potential, say Φ, can be obtained also for steady states far from thermal equilibrium.

In a recent paper[14] we have proposed such a generalized thermodynamic potential[24] in order to describe, within specified assumptions, a liquid layer near the Bénard point. That potential Φ is linked to the probability distribution W of fluctuations from the steady state by the usual Einstein formula $W \sim e^{-\Phi}$. Furthermore, it was found that, throughout the transition region, stationary states of heat transport across the layer are determined by an extremum condition for the potential, subject to given boundary conditions.

It is my aim in this paper to provide a derivation of the form and the properties of Φ through an analysis of the hydrodynamic equations with fluctuating forces in the vicinity of the Bénard point. However, we have to leave open the question of which of the formerly proposed potentials[9,20] my new potential is most closely related to. I hope to be able to return to this interesting question in future work.

The paper is organized as follows: Section II is largely a summary of my basic notation and the assumptions which establish my starting point. The latter is taken with the hydrodynamic equations in the Boussinesq approximation,[3] including the fluctuating-force terms due to Landau and Lifshitz.[22] For simplicity, I consider the case of free boundaries on top and on the bottom of the layer.

In Sec. III, the working equations are simplified by a restriction to the close vicinity of the Bénard point. To this end, we employ a modified version of the above mentioned expansion technique.[11] In this expansion,[25,26] the separation of time scales and length scales is explicitly introduced; these scales occur close to the Bénard point, due to the selection of fluctuations near the critical wave number k_c. The slowly varying mode, for the case of nearly-two-dimensional rolls and $R = R_c$, is identified in Sec. III A. The small parameter of the expansion is introduced into the equations of motion in Sec. III B. In Sec. III C, the stochastic equations of motion (Langevin equation) for the amplitude of the slow mode is determined to second

order in the small parameter. The components of the slow mode in terms of the hydrodynamical variables are determined to the same order. In Sec. III D, the stochastic force is evaluated. The Langevin equation, which is completely determined at the end of Sec. III, forms the basis of all following considerations.

In Sec. IV a number of general properties of the stochastic equation of motion are discussed. In Sec. IV A, the Fokker-Planck equation is obtained, which is stochastically equivalent to our Langevin equation. A number of useful symmetries is discussed in Sec. IV B. In Sec. IV C, it is shown that the fluctuations of the slow mode have a detailed balance symmetry. This symmetry makes it easy to obtain the time-independent probability distribution for fluctuations from the steady state in Sec. IV D. In this way, I complete the derivation of the generalized thermodynamic potential, discussed previously.[14] In Sec. IV E, I discuss extrema of my potential that correspond to stable, unstable, or metastable steady states.

In Sec. V I proceed to an evaluation of the Fokker-Planck equation in the case of strictly two-dimensional rolls. Time-independent correlation functions are obtained in Sec. V A without further approximation, by evaluating the required functional integral using the Feynman-Kac formula[27] and recent numerical solutions[28] of the resulting "Schrödinger" equation. These "exact" numerical results are analytically approximated in Secs. V B and V C for $R<R_c$ and $R>R_c$, respectively, using a quasilinear approximation. In Secs. V D and V E, the quasilinear approximation is also used to calculate time-dependent correlation functions.

In Sec. VI correlation functions are obtained for the case in which a (weak) variation of the roll amplitude along the roll axis subsists. Section VI is subdivided into two subsections according to the different cases considered. The quasilinearization is employed throughout this section, because of a lack of exact methods for evaluating the required functional integrals.

II. BASIC HYDRODYNAMIC EQUATIONS

Macroscopically, the fluid layer is described by the balance equations of hydrodynamics, which, in conventional notation,[29] take the form

$$\partial_t \rho + \partial_j(\rho v_j) = 0, \tag{2.1}$$

$$\rho(\partial_t v_i + v_j \partial_j v_i) = -\partial_i p - \rho g \delta_{i3} + \partial_j \sigma'_{ij}, \tag{2.2}$$

$$\rho(\partial_t \epsilon + v_j \partial_j \epsilon) = -p\,\partial_j v_j + \sigma'_{ij}\partial_j v_i - \partial_j q'_j, \tag{2.3}$$

with

$$\sigma'_{ij} = \eta(\partial_j v_i + \partial_i v_j) + (\zeta - \tfrac{2}{3}\eta)\delta_{ij}\partial_i v_i + s_{ij}, \tag{2.4}$$

$$q'_i = -\kappa \partial_i T + q_i. \tag{2.5}$$

The density ρ, the pressure p, and the temperature T of the liquid are connected by an equation of state, which may be taken as

$$d\rho/\rho = -\beta\, dT + \chi\, dp. \tag{2.6}$$

β is the volume-expansion coefficient, and χ is the isothermal compressibility. The internal energy density per unit mass ϵ is given by a second equation of state, which we take as

$$d\epsilon = C_v dT + (p/\rho^2)\, d\rho, \tag{2.7}$$

where C_v is the specific heat per unit mass. The forces which drive the velocity field $\vec{v}$ are gravity [which acts in the vertical direction $(i=3)$], the pressure gradient, viscous forces (which contain the coefficients of viscosity η, ζ), and a random force, which is described by the fluctuating stress tensor s_{ik}. Similarly, Eq. (2.3) contains the random heat-flux density q_i, in addition to the usual heat-conduction flux, the viscous heat source, and the work done by pressure. The averages of the random quantities have to vanish,

$$\langle s_{ij}(\vec{x}, t)\rangle = 0 = \langle q_i(\vec{x}, t)\rangle, \tag{2.8}$$

in order to recover the standard equations of fluid dynamics as equations for averages.[30]

The random flux densities can be assumed to be Gaussian, in view of the central limit theorem of statistics. A knowledge of their second-order correlation functions is then sufficient for obtaining all higher-order correlation functions. In general, the former have to be considered as phenomenological quantities for each specific experiment. Here we will make the assumption that the fluctuations have thermodynamic origin; i.e., we impose the requirement that Eqs. (2.1)–(2.5) lead to the correct probability distribution function for fluctuations from equilibrium. The latter is, in Gaussian approximation, given by the Einstein formula

$$W \sim e^{\Delta^{(2)}S/k}, \tag{2.9}$$

where k is Boltzmann's constant and $\Delta^{(2)}S$ is the second-order entropy change due to the fluctuations ΔT, $\Delta\rho$, $\vec{v}$ from equilibrium, which is given by the familiar quadratic form[22]

$$\Delta^{(2)}S = -\frac{1}{2}\int d^3x \frac{\rho}{T}\left(\frac{C_v}{T}(\Delta T)^2 + \frac{(\Delta\rho)^2}{\chi\rho^3} + \vec{v}^2\right). \tag{2.10}$$

Equations (2.9) and (2.10) are consistent with Eqs. (2.1)–(2.7) if[31]

$$\langle s_{ij}(\vec{x}, t)\, s_{lm}(\vec{x}', t')\rangle = 2kT[\eta(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) + (\zeta - \tfrac{2}{3}\eta)\delta_{ij}\delta_{lm}]\,\delta(\vec{x} - \vec{x}')\delta(t - t'), \tag{2.11}$$

$$\langle q_i(\vec{x},t)q_j(\vec{x}',t')\rangle = 2kT^2\kappa\delta_{ij}\delta(\vec{x}-\vec{x}')\delta(t-t'), \tag{2.12}$$

$$\langle q_i(\vec{x},t)s_{jl}(\vec{x}',t')\rangle = 0. \tag{2.13}$$

We will be interested in solutions of Eqs. (2.1)–(2.5) which satisfy the boundary conditions

$$\begin{aligned} &x_3 = 0:\ T = T_0;\ p = p_0;\ \rho = \rho_0; \\ &x_3 = l:\ T = T_0 - \Delta T, \end{aligned} \tag{2.14}$$

where l is the layer thickness. If we neglect all fluctuations, the simplest solution satisfying Eq. (2.14) for a very thin layer is given by

$$\begin{aligned} &\rho = \rho_c = \rho_0[1 + x_3(\beta\Delta T/l - \chi\rho_0 g)], \\ &p = p_c = p_0 - \rho_0 g x_3[1 + \tfrac{1}{2}x_3(\beta\Delta T/l - \chi\rho_0 g)], \\ &T = T_c = T_0 - \Delta T x_3/l, \\ &v = v_c = 0, \end{aligned} \tag{2.15}$$

which describes the time-independent steady state in which energy is transported across the layer by heat conduction only. We introduce

$$\tilde{T} = T - T_c,\quad \tilde{p} = p - p_c, \tag{2.16}$$

as new variables and consider their equations of motion in the Boussinesq approximation.[3] This implies that we take all fluid parameters to be constants, and make use of the facts: (i) $\beta\Delta T$ is a small parameter in most liquids; (ii) $\chi\rho_0 gl$ is negligible for thin layers, and; (iii) for convective motions, the viscous energy dissipation in Eq. (2.3) is negligible. To leading order in $\beta\Delta T$, the equations are

$$\partial_j v_j = 0, \tag{2.17}$$

$$\partial_t v_i + v_j\partial_j v_i = -\partial_i p + \partial_j\partial_j v_i + \sqrt{R}\,T\delta_{i3} + \partial_j s_{ij}, \tag{2.18}$$

$$P(\partial_t T + v_j\partial_j T) = \partial_j\partial_j T + \sqrt{R}\,v_3 - \partial_j q_j. \tag{2.19}$$

Here we have made use of dimensionless units: We scale lengths by l, times by $\rho l^2/\eta$, and temperature by $(\Delta T\,\eta^3 C_v/\rho_0^2 g\beta\kappa l^3)^{1/2}$. We also returned to the original notation for the hydrodynamical variables p, T, which now describe deviations from the purely heat-conducting steady state in dimensionless units. For all other quantities in Eqs. (2.17)–(2.19), dimensionless units are also understood. The fluid flow is then characterized by the dimensionless Rayleigh number

$$R = g\beta\Delta T\, l^3\rho_0^2 C_v/\eta\kappa$$

and the dimensionless Prandtl number

$$P = \eta C_v/\kappa.$$

The fluctuations of the fluid are characterized in the new units by the two dimensionless parameters Q_1, Q_2 defined by

$$\langle s_{ij}(\vec{x},t)s_{lm}(\vec{x}',t')\rangle = 2Q_2(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\delta(\vec{x}-\vec{x}')\delta(t-t'), \tag{2.20}$$

$$\langle q_i(\vec{x},t)q_j(\vec{x}',t')\rangle = 2Q_1\delta_{ij}\delta(\vec{x}-\vec{x}')\delta(t-t'), \tag{2.21}$$

which are obtained as

$$Q_1 = \frac{g\beta k T^2\rho_0}{C_v\Delta T\,\eta^2},\quad Q_2 = \frac{kT\rho_0}{\eta^2 l}. \tag{2.22}$$

Equations (2.17)–(2.19) are the basis of all further considerations.

The boundary conditions, which we will use for mathematical convenience, are those of free surfaces at $x_3 = 0, 1$:

$$v_3 = T = 0,\quad \partial_3 v_1 = \partial_3 v_2 = 0. \tag{2.23}$$

The results obtained for these boundary conditions give us sufficient reason to expect no qualitative change of the results if the boundary conditions are changed, e.g., from free to physically more reasonable rigid boundaries.[32] Boundary conditions in the (x_1, x_2) plane are also needed. In order to separate as much as possible the effects which are due to special boundary conditions from the effects which are due to the convection instability, we will consider, in the present paper, an infinitely extended horizontal fluid layer. It is not difficult to specialize our results for finite layers.

Before proceeding further, it will be convenient to establish some abbreviated notation. We introduce a five-component vector which contains our five hydrodynamical variables as components;

$$u \equiv (\vec{v}, T, p), \tag{2.24}$$

and define the scalar product

$$(u^{(1)}, u^{(2)}) = \int \frac{d^3x}{F}[\vec{v}^{(1)}(\vec{x})\cdot\vec{v}^{(2)}(\vec{x}) + T^{(1)}(\vec{x})\,T^{(2)}(\vec{x}) + p^{(1)}(\vec{x})p^{(2)}(\vec{x})], \tag{2.25}$$

where F is the surface area of the fluid layer. Then we define a nonlinear matrix differential operator $\underline{L}$ acting on u, such that Eqs. (2.17)–(2.19) without random flux terms may be written as

$$\underline{L}(u) = 0. \tag{2.26}$$

The random fluxes are combined to form the vector

$$I \equiv (\partial_j s_{1j}, \partial_j s_{2j}, \partial_j s_{3j}, -\partial_j q_j, 0), \tag{2.27}$$

so that Eqs. (2.17)–(2.19) take the concise form

$$\underline{L}(u) = I \tag{2.28}$$

used in Sec. III.

III. HYDRODYNAMICS NEAR THE INSTABILITY

It is now our aim to derive from the basic equations of motion a simplified equation describing the liquid near the instability. Such simplified equations have been derived before by a number of authors.[25,26] Our derivation, given here, extends earlier work by taking into account the random flux terms, which form the inhomogeneity of Eq. (2.28).[33]

A. Slow normal mode

For sufficiently small values of R, the motionless heat-conducting steady state will be stable, and we may linearize Eqs. (2.17)–(2.19) in v and T.[23] The resulting linearized inhomogeneous boundary-value problem is known to be self-adjoint.[3] The eigenfunctions of the corresponding homogeneous problem are of the form

$$u_n(t) = u_n^{(0)} e^{-\lambda_n(R)t}, \tag{3.1}$$

with real positive eigenvalues $\lambda_n(R)$. For $R \to R_c = \frac{27}{4}\pi^4$, one of the eigenvalues, say λ_0, tends to zero, $\lambda_0(R_c) = 0$. Hence, in the close vicinity of $R = R_c$, there exists one normal mode of the system which has a very slow decay rate $\lambda_0(R)$, if compared to the decay rates of all other normal modes. As a result, this slow mode will dominate the hydrodynamic behavior of the system near the instability. Let us find that normal mode for $R = R_c$.

The homogeneous linear boundary-value problem associated with Eq. (2.28), if specialized for the eigenvalue $\lambda_0(R_c) = 0$, may be written

$$\underline{L}_0(u^{(0)}) = 0, \tag{3.2}$$

with the self-adjoint matrix differential operator

$$\underline{L}_0 = \begin{bmatrix} \partial_j\partial_j & 0 & 0 & 0 & -\partial_1 \\ 0 & \partial_j\partial_j & 0 & 0 & -\partial_2 \\ 0 & 0 & \partial_j\partial_j & \sqrt{R_c} & -\partial_3 \\ 0 & 0 & \sqrt{R_c} & \partial_j\partial_j & 0 \\ \partial_1 & \partial_2 & \partial_3 & 0 & 0 \end{bmatrix}. \tag{3.3}$$

The numerical value of R_c in Eq. (3.3) is obtained as the smallest number for which $\underline{L}_0$, with boundary conditions (2.23), has the eigenvalue $\lambda_0 = 0$, and for which Eq. (3.2), therefore, has a nontrivial solution $u^{(0)}$. The possible solutions $u^{(0)}$ are highly degenerate, as was already mentioned in the introduction. In view of the stability results of Schlüter, Lortz, and Busse,[11] we will assume that $u^{(0)}$ is a plane wave $\sim e^{ik_c x_1}$ in the x_1 direction, whose amplitude does not depend on x_2. This ansatz just describes a regular lattice of two-dimensional rolls, which was found to be the only stable cell pattern for R slightly larger than, and sufficiently close to, R_c.[11] However, we thereby introduce a preferred direction in the horizontal plane, and, have to restrict our analysis to cases in which the boundary conditions in the (x_1, x_2) plane are compatible with a plane wave in x_1 direction. In order to consider more complicated situations (e.g., interaction of several plane waves in various directions), the calculations would have to be altered at this point.

Putting our ansatz for $u^{(0)}$ into Eq. (3.2) we obtain

$$R_c = \tfrac{27}{4}\pi^4; \quad k_c = \pi/\sqrt{2} \tag{3.4}$$

and

$$\begin{aligned} u^{(0)} &= w\psi_0(x_1, x_3) + \text{c.c.}, \\ \psi_0 &= e^{ik_c x_1}\,(i\sqrt{2}\cos\pi x_3, 0, \sin\pi x_3, \\ &\qquad \sqrt{3}\sin\pi x_3, -3\pi\cos\pi x_3), \end{aligned} \tag{3.5}$$

where w is an arbitrary complex amplitude whose absolute value gives the intensity, and whose phase gives the position, of the convection rolls. We may assume that after a time $t \gg |\min\lambda_n|^{-1}$ has elapsed (where $|\min\lambda_n|$ is the absolute value of the smallest nonzero eigenvalue of L_0, corresponding to a wave with wave number k_c), all normal modes on that length scale, except $u^{(0)}$, have relaxed to the steady state. On such long time scales, the five hydrodynamic variables, which make up the vector u, are determined by just one complex amplitude w. In this discussion, linearization of Eq. (2.28), neglect of fluctuations, and $R = R_c$ were assumed.

For R unequal but close to R_c, and fluctuations as well as nonlinearities switched on in Eq. (2.28), the slowest-decaying normal mode will still be very close to $u^{(0)}$, Eq. (3.5). However, its components will receive small corrections, depending on $R - R_c$, and its amplitude will be a slowly varying random function $w(x_1, x_2, t)$ of space and time. In a generalization of the work of Newell and Whitehead,[25] we will now set up a systematic approximation procedure which allows us to find both the stochastic equation of motion which governs $w(x_1, x_2, t)$, and the corrections to the components of the eigenvector $u^{(0)}$.

B. Splitting of equations of motion

It will be convenient to explicitly distinguish between the length scale of the convection cells, which is set by the layer thickness l, and the length and time scale on which the amplitude $w(x_1, x_2, t)$ changes. This goal is achieved by introducing[25]

$$x_1 = \xi/\epsilon, \quad x_2 = \eta/\sqrt{\epsilon}, \quad t = \tau/\epsilon^2, \tag{3.6}$$

and replacing

$$w(x_1, x_2, t) \to \epsilon w(\xi, \eta, \tau). \tag{3.7}$$

ϵ, in Eq. (3.6) is a small parameter. According to the choice of scales (3.6), ϵ is of the order of the deviations Δk_1 of the wave number in the x_1 direction from the critical wave number k_c. Since $R(k_1)$ must have a minimum for $k_1 = k_c$, $R(k_c) = R_c$, the quantities $|k_1 - k_c|^2 \sim |R(k_1) - R_c| \sim \epsilon^2$ are all of the same order; to take ϵ small implies a restriction to small $|R - R_c|$.

The choice of scale of x_2 is suggested by the geometric observation that, in an otherwise isotropic system, changes of the wave number in the x_2 direction, Δk_2, will influence the absolute value of the wave number only in second order; $\Delta k \sim \Delta k_2^2/2k_c$. Hence, $\Delta k_2^2 \sim \Delta k_1 \sim \epsilon$ are quantities of the same order. The choice of the time scale in Eq. (3.6) is based on the assumption that $\lambda_0 \sim |R - R_c| \sim \epsilon^2$ for R close to R_c. Finally, the choice of the prefactor ϵ in Eq. (3.7) contains the assumption that the amplitudes of $(\vec{v}, T, p)$ are proportional to $|R - R_c|^{1/2}$ for small amplitudes. Since $w(\xi, \eta, \tau)$ appears as a common factor of all five unknowns $(\vec{v}, T, p)$, we have to substitute in the equations of motion (2.17)–(2.19)

$$\partial_1 \to \partial_1 + \epsilon\, \partial_\xi, \quad \partial_2 \to \partial_2 + \sqrt{\epsilon}\, \partial_\eta, \quad \partial_t \to \epsilon^2 \partial_\tau. \tag{3.8}$$

Then, inserting the ansatz

$$u = \epsilon w(\xi, \eta, \tau)\psi(x_1, x_3) + \text{c.c.} \tag{3.9}$$

into Eqs. (2.17)–(2.19) and collecting equal powers of ϵ, we may write the operator $\underline{L}$ in Eq. (2.28) in powers of $\sqrt{\epsilon}$,

$$\underline{L} = \underline{L}_0 + \epsilon^{1/2}\underline{L}_{1/2} + \epsilon \underline{L}_1 + \epsilon^{3/2}\underline{L}_{3/2} + \epsilon^2 \underline{L}_2. \tag{3.10}$$

$\underline{L}_0$ is already given by Eq. (3.3). The other operators are

$$\underline{L}_{1/2}(u) \equiv (2\,\partial_2\partial_\eta u_1,\ 2\,\partial_2\partial_\eta u_2 - \partial_\eta u_5,\ 2\,\partial_2\partial_\eta u_3,\ 2\,\partial_2\partial_\eta u_4,\ \partial_\eta u_2), \tag{3.11}$$

$$\underline{L}_1(u) \equiv (\Omega u_1 - \partial_\xi u_5,\ \Omega u_2,\ \Omega u_3,\ [(1-P)v_j\partial_j + \Omega]u_4,\ \partial_\xi u_1), \tag{3.12}$$

with

$$\Omega \equiv 2\,\partial_1\partial_\xi + \partial_\eta^2 - v_j\partial_j, \tag{3.13}$$

$$\underline{L}_{3/2}(u) \equiv (-v_2\,\partial_\eta u_1,\ -v_2\,\partial_\eta u_2,\ -v_2\,\partial_\eta u_3,\ -Pv_2\,\partial_\eta u_4,\ 0), \tag{3.14}$$

$$\underline{L}_2(u) \equiv (\Omega' u_1,\ \Omega' u_2,\ \Omega' u_3 + (\sqrt{R} - \sqrt{R_c})u_4,\ [P\Omega' + (1-P)\partial_\xi^2]u_4 + (\sqrt{R} - \sqrt{R_c})u_3,\ 0), \tag{3.15}$$

with

$$\Omega' \equiv \partial_\xi^2 - v_1\partial_\xi - \partial_\tau. \tag{3.16}$$

Let us now turn to the inhomogeneity in Eq. (2.28), given by the random stress tensor s_{ij} and the heat flux q_i. We are interested in the fluctuations around the plane wave state $\sim e^{ik_c x_1}$. Hence, we put (cf. Sec. III D)

$$s_{ij}(x_1, x_2, x_3, t) = e^{i(\pi/\sqrt{2})x_1}\tilde{s}_{ij}(\xi, \eta, x_3, \tau) + \text{c.c.},$$
$$q_i(x_1, x_2, x_3, t) = e^{i(\pi/\sqrt{2})x_1}\tilde{q}_i(\xi, \eta, x_3, \tau) + \text{c.c.}, \tag{3.17}$$

with the convention

$$\langle \tilde{s}_{ij}\tilde{s}_{lm}\rangle = 0 = \langle \tilde{q}_i \tilde{q}_j\rangle. \tag{3.18}$$

By comparison with Eqs. (2.20) and (2.21) we find

$$\langle \tilde{s}_{ij}(\xi, \eta, x_3, \tau)\tilde{s}^*_{lm}(\xi', \eta', x_3', \tau')\rangle = \epsilon^{7/2}Q_2(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) \times \delta(\xi - \xi')\delta(\eta - \eta')\delta(x_3 - x_3')\delta(\tau - \tau'), \tag{3.19}$$

$$\langle q_i(\xi, \eta, x_3, \tau)q_j^*(\xi', \eta', x_3', \tau')\rangle = \epsilon^{7/2}Q_1\delta_{ij}\delta(\xi - \xi')\delta(\eta - \eta')\delta(x_3 - x_3')\delta(\tau - \tau'). \tag{3.20}$$

In all common liquids, Q_1 and Q_2 are very small numbers ($Q_1 \ll Q_2 \simeq 10^{-12}$ for tolulene at 20°C, $l = 0.1$ cm), and we will assume that

$$Q_1, Q_2 \lesssim \epsilon^{5/2}. \tag{3.21}$$

Equation (3.21) is satisfied as long as $|R - R_c| \gtrsim Q_1^{4/5}, Q_2^{4/5}$, a condition which will always be met in practice. In that case, $\tilde{s}_{ij}$ and $\tilde{q}_i$ are small, at least of order ϵ^3 [cf. Eqs. (3.19) and (3.20)], and contribute to Eq. (2.28) in the same order as $\underline{L}_2(u)$, Eq. (3.15). Since this will be the highest order which we shall consider, it is sufficient to evaluate the inhomogeneity I [Eq. (2.27)] in the lowest order of ϵ. Introducing the substitutions (3.8) and (3.17) into Eq. (2.26), we obtain in lowest order

$$I = [e^{i(\pi/\sqrt{2})x_1}(i\pi\tilde{s}_{11}/\sqrt{2} + \partial_3\tilde{s}_{13},\ i\pi\tilde{s}_{21}/\sqrt{2} + \partial_3\tilde{s}_{23},\ i\pi\tilde{s}_{31}/\sqrt{2} + \partial_3\tilde{s}_{33},\ -i\pi\tilde{q}_1/\sqrt{2} - \partial_3\tilde{q}_3,\ 0) + \text{c.c.}] + O(\epsilon^{7/2}). \tag{3.22}$$

With this step, we have completed our decomposition of Eq. (2.27) into parts containing different powers of $\sqrt{\epsilon}$.

C. Iteration procedure

We now look for the solution of Eq. (2.28) in the form of a power series expansion in $\sqrt{\epsilon}$; i.e., we insert

$$u = \epsilon(u^{(0)} + \epsilon^{1/2}u^{(1/2)} + \epsilon u^{(1)} + \epsilon^{3/2}u^{(3/2)} + \epsilon^2 u^{(2)} + \cdots) \tag{3.23}$$

into Eq. (2.27), make use of Eqs. (3.10) and (3.22), and collect equal powers of ϵ. In lowest order, i.e., $\sim\epsilon$, we obtain

$$\underline{L}_0(u^{(0)})=0, \tag{3.24}$$

which has already been solved above [cf. Eq. (3.5)].

1. Iteration

In order $\epsilon^{3/2}$ we obtain

$$\underline{L}_0(u^{(1/2)})+\underline{L}_{1/2}(u^{(0)})=0, \tag{3.25}$$

which has to be solved for $u^{(1/2)}$. Since $\underline{L}_0$ is self-adjoint

$$(\varphi, \underline{L}_0(\psi))=(\underline{L}_0(\varphi), \psi), \tag{3.26}$$

and $\underline{L}_0(\psi_0)=0$ [cf. Eqs. (3.2), (3.5)], we have

$$(\psi_0, \underline{L}_0(u^{(1/2)}))=(\psi_0^*, \underline{L}_0(u^{(1/2)}))=0. \tag{3.27}$$

From Eq. (3.25) we obtain the necessary condition

$$(\psi_0^*, \underline{L}_{1/2}(u^{(0)}))=0 \tag{3.28}$$

which is, in fact, satisfied by $\underline{L}_{1/2}$ [Eqs. (3.11) and (3.5)]. Solving Eq. (3.25) for $u^{(1/2)}$ yields

$$u^{(1/2)}=(0, (2/\pi)\partial_\eta w\cos\pi x_3, 0, 0, 0)e^{i(\pi/\sqrt{2})x_1}+\text{c.c.} \tag{3.29}$$

This result has a simple geometric interpretation; it simply gives the velocity component of the slow mode in x_2 direction to first order if the preferred direction, introduced in $u^{(0)}$ [Eq. (3.5)], is slightly tilted with respect to the x_1 direction.

2. Iteration

In order ϵ^2 we obtain

$$\underline{L}_0(u^{(1)})+\underline{L}_{1/2}(u^{(1/2)})+\underline{L}_1(u^{(0)})=0 \tag{3.30}$$

with the constraint

$$(\psi_0^*, \underline{L}_{1/2}(u^{(1/2)})+\underline{L}_1(u^{(0)}))=0. \tag{3.31}$$

Again, the orthogonality condition Eq. (3.31) is automatically satisfied by $\psi_0^*, \underline{L}_{1/2}, \underline{L}_1$. Solving Eq. (3.30) for $u^{(1)}$ we obtain

$$u^{(1)}=u^{(1)I}+u^{(1)nI} \tag{3.32}$$

$$u^{(1)I}=e^{i(\pi/\sqrt{2})x_1}((2\sqrt{2}\,i/\pi^2)[(\pi i/\sqrt{2})\partial_\xi+\partial_\eta^2]w\cos\pi x_3, 0, 0, (2/\sqrt{3}\,\pi^2)(\sqrt{2}\,\pi i\partial_\xi+\partial_\eta^2)w\sin\pi x_3,$$
$$(-4/\pi)(\sqrt{2}\,\pi i\partial_\xi+\partial_\eta^2)w\cos\pi x_3)+\text{c.c.}, \tag{3.33}$$

$$u^{(1)nI}=(0, 0, 0, (-\sqrt{3}\,P/4\pi)|w|^2\sin 2\pi x_3, (1+\tfrac{9}{8}P)[(\tfrac{1}{2}|w|^2)\cos 2\pi x_3+w^2e^{i\pi\sqrt{2}x_1}])+\text{c.c.}\,. \tag{3.34}$$

$u^{(1)nI}$ gives a small admixture of the spatial second-harmonic mode to the components of the slow mode, while $u^{(1)I}$ gives further corrections due to a spatial variation of the roll amplitude. The contribution of the second-harmonic mode to the slow mode is a result of the nonlinear term in Eq. (2.19).

In Eq. (3.33), we could chose one component of the solution $u^{(1)I}$ of Eq. (3.30) at will. We made use of this freedom to take its third component $v_3^{(1)I}$ equal to zero. This choice will be repeated in each further step of the iteration procedure, where the choice is possible. In physical terms, the possibility of this free choice results from the fact that we can chose with which physical variable we want to associate the amplitude w most closely. Once we have made this choice, the rest of the physical variables may be uniquely expressed by w. We emphasize, however, that the nonlinear part $u^{(1)nI}$ is uniquely determined by Eq. (3.30). Hence, the result (3.34), according to which only the temperature and pressure components contribute to $u^{(1)nI}$, is not a choice of gauge, but reflects physical reality. For $R=R_c$, this contribution turns out to be crucial for the stabilization of the slow mode since, compared to the linear theory, it provides an additional dissipation mechanism, the only one which does not vanish for $R=R_c$. Later we will make use of the proportionality to $|w|^2$ of the second-harmonic amplitude in order to discuss the mode-mode coupling near $R=R_c$.

3. Iteration

In order $\epsilon^{5/2}$ we obtain

$$\underline{L}_0(u^{(3/2)})+\underline{L}_{1/2}(u^{(1)})+\underline{L}_1(u^{(1/2)})+\underline{L}_{3/2}(u^{(0)})=0. \tag{3.35}$$

The orthogonality condition

$$(\psi_0^*, \underline{L}_{1/2}(u^{(1)})+\underline{L}_1(u^{(1/2)})+\underline{L}_{3/2}(u^{(0)}))=0 \tag{3.36}$$

is again automatically satisfied. By solving Eqs. (3.35) and (3.36) we find

$$u^{(3/2)}=(0, (4/\pi^3)(\sqrt{2}\,\pi i\partial_\xi+\partial_\eta^2)\partial_\eta w\cos\pi x_3\, e^{i(\pi/\sqrt{2})x_1}$$
$$-(3/8\pi^2)(1+\tfrac{3}{8}P)\partial_\eta|w|^2\cos 2\pi x_3, 0, 0, 0)+\text{c.c.}, \tag{3.37}$$

again with a linear and a second-harmonic part.

$u^{(3/2)}$, like $u^{(1/2)}$, is nonvanishing only if w depends explicitly on η.

4. *Iteration*

In the order ϵ^3, we must now include the right-hand side of Eq. (2.28), obtaining

$$\underline{L}_0(u^{(2)})+\underline{L}_{1/2}(u^{(3/2)})+\underline{L}_1(u^{(1)}) + \underline{L}_{3/2}(u^{(1/2)})+\underline{L}_2(u^{(0)})=I. \quad (3.38)$$

The orthogonality condition yields

$$(\psi_0^*, \underline{L}_{1/2}(u^{(3/2)})+\underline{L}_1(u^{(1)})+\underline{L}_{3/2}(u^{(1/2)})+\underline{L}_2(u^{(0)})) = (\psi_0^*, I). \quad (3.39)$$

Equation (3.39) is not automatically satisfied. Rather, it forms an equation which has to be satisfied by the yet-undetermined amplitude w. Evaluating Eq. (3.39) explicitly, we obtain the condition

$$(1+P)\partial_\tau w = \{\tfrac{3}{2}\pi^2(R-R_c)/R_c - \tfrac{1}{2}P^2|w|^2\}w + 4[\partial_\xi - (i/\sqrt{2}\,\pi)\partial_\eta^2]^2 w + \tfrac{2}{3}(\psi_0^*, I). \quad (3.40)$$

Apart from the random term (ψ_0^*, I), Eq. (3.40) has already been derived by Newell and Whitehead.[25] The random term in Eq. (3.40) acts as a driving force, thereby turning the equation of motion into a nonlinear Langevin equation which has to be analyzed by statistical methods. Before we proceed to this topic, we want to solve Eq. (3.38) for $u^{(2)}$. When doing this, we can make use of Eq. (3.40) to eliminate all time derivatives from Eq. (3.38). We wish to calculate from Eq. (3.38) only the corrections to the fundamental mode, and for this purpose we will disregard all higher-order modes, e.g., third harmonics, as well as the fluctuating terms in Eqs. (3.38) and (3.40). The following second-order corrections are found by solving Eq. (3.38) for $u^{(2)}$:

$$v_1^{(2)} = -\frac{2i\sqrt{2}}{\pi^2}\left(\partial_\xi^2 - \frac{3\sqrt{2}\,i}{\pi}\partial_\xi\partial_\eta^2 - \frac{2}{\pi^2}\partial_\eta^4\right) \times w e^{i(\pi/\sqrt{2})x_1}\cos\pi x_3 + \text{c.c.}, \quad (3.41)$$

$$v_2^{(2)} = 0 = v_3^{(2)}, \quad (3.42)$$

$$T^{(2)} = \frac{\sqrt{3}}{(1+P)}\left[\left(\frac{1-P}{2}\,\frac{R-R_c}{R_c} - \frac{P^2}{3\pi^2}|w|^2\right)w - \frac{2}{9\pi^2}\left((1+13P)\partial_\xi^2 w - \frac{4\sqrt{2}\,i(1+4P)}{\pi}\partial_\xi\partial_\eta^2 w - \frac{2(1+4P)}{\pi^2}\partial_\eta^4 w\right)\right]\exp[i(\pi/\sqrt{2})x_1]\sin\pi x_3, \quad (3.43)$$

$$p^{(2)} = \frac{-3\pi}{(1+P)}\left[\left(\frac{R-R_c}{R_c} - \frac{P^2}{3\pi^2}|w|^2\right)w - \frac{4}{3\pi^2}\left((1+3P)\partial_\xi^2 w - \frac{2\sqrt{2}\,i}{\pi}(1+2P)\partial_\xi\partial_\eta^2 w - \frac{(1+2P)}{\pi^2}\partial_\eta^4 w\right)\right]e^{i(\pi/\sqrt{2})x_1}\cos\pi x_3. \quad (3.44)$$

Once we have succeeded in solving the equation of motion (3.40), we must use Eqs. (3.5), (3.29), (3.33), (3.34), and (3.41)–(3.44) to obtain the physical variables $\vec{v}$, T, and p from the complex function $w(\xi,\eta,\tau)$. It should be noted that v_3 is related to w in a particularly simple manner, since we have

$$v_3 = (w e^{i\pi x_1/\sqrt{2}} + \text{c.c.})\sin\pi x_3 \quad (3.45)$$

to order ϵ^3. v_1 is also linearly related to w, contrary to T, which contains the second-harmonic contribution (3.34), and a cubic contribution to the fundamental mode, Eq. (3.43). The latter term makes an evaluation of the statistical properties of T from w somewhat tedious. In order to circumvent this difficulty, we will use for T the expression obtained up to the third iteration step

$$T = \{\sqrt{3}\,e^{ik_c x_1}\sin\pi x_3[w + (2/3\pi^2)(\sqrt{2}\,\pi i\partial_\xi + \partial_\eta^2)w] - (\sqrt{3}\,P/4\pi)|w|^2\sin 2\pi x_3 + \text{c.c.}\}. \quad (3.46)$$

D. Evaluation of random force

The random force of the Langevin equation (3.40) is evaluated by using the explicit form [Eq. (3.22)], which gives I in lowest order in ϵ. From Eq. (3.40) we can now appreciate that the component of I making a contribution has to be proportional to $\exp[+i(\pi/\sqrt{2})x_1]$, which justifies Eq. (3.17). Putting

$$\tfrac{2}{3}(\psi_0^*, I) = \Gamma(\xi,\eta,\tau), \quad (3.47)$$

we obtain

$$\Gamma(\xi,\eta,\tau) = \frac{2\pi}{3}\int_0^1 dx_3\left((\tilde{s}_{11}-\tilde{s}_{33}+\sqrt{3}\,\tilde{q}_3)\cos\pi x_3 - \frac{i}{\sqrt{2}}(-2\tilde{s}_{13}+\tilde{s}_{31}-\sqrt{3}\,\tilde{q}_1)\sin\pi x_3\right), \quad (3.48)$$

where we have already eliminated the derivatives

∂_3 in Eq. (3.22) by partial integration, making use of the assumption that no fluctuations occur at the boundaries.[34] We can now calculate the second-order correlation functions of Γ and Γ^* quite easily, putting Eqs. (3.18)–(3.20) to use. We obtain

$$\langle\Gamma\rangle=\langle\Gamma^*\rangle=0, \tag{3.49}$$

$$\langle\Gamma(\xi,\eta,\tau)\Gamma(\xi',\eta',\tau')\rangle=0, \tag{3.50}$$

$$\langle\Gamma^*(\xi,\eta,\tau)\Gamma(\xi',\eta',\tau')\rangle = \pi^2\epsilon^{7/2}(Q_1+Q_2)\delta(\xi-\xi')\delta(\eta-\eta')\delta(\tau-\tau'). \tag{3.51}$$

Since we want to stop the iteration procedure at this point, we need no longer explicitly distinguish several time and length scales. Formally, we simply put $\epsilon=1$ again; i.e.,

$$\xi=x_1, \quad \eta=x_2, \quad \tau=t. \tag{3.52}$$

We have thereby completed our derivation of the hydrodynamic equations (3.40) and (3.51) for the fluctuations of the slow mode near the convection instability. All following considerations will be based on these equations.

IV. FOKKER-PLANCK EQUATION OF SLOW MODE

A. Fokker-Planck equation

To our Langevin equation (3.40) there corresponds a stochastically equivalent Fokker-Planck equation,[35] which governs the probability distribution W of the fluctuations of the slow-mode amplitude w. Since this amplitude is a continuous field, i.e., a function of the spatial variables x_1 and x_2, its probability distribution must be a functional of that field and, accordingly, the Fokker-Planck equation must be a functional differential equation. We will formally treat w and w^* as independent fields and denote by $\delta_w(\vec{x})$ the functional derivative with respect to $w(\vec{x})$, where w^* is kept fixed. The horizontal vectors $\vec{x}=(x_1,x_2,0)$, $\vec{k}=(k_1,k_2,0)$, $\Delta\vec{k}=(\Delta k_1,\Delta k_2,0)$, $\vec{k}_c=(k_c,0,0)$ will be used in the following as a shorthand notation.

The Fokker-Planck equation is very simply related to the Langevin equation (3.40), and takes the form[14]

$$(1+P)\partial_t W=\int\int d^2x\left(\left\{\delta_w(\vec{x})\left[(-\tfrac{3}{2}\pi^2\nu+\tfrac{1}{2}P^2|w(\vec{x})|^2)w(\vec{x})-4\left(\partial_1-\frac{i}{\sqrt{2}\,\pi}\partial_2^2\right)^2 w(\vec{x})+Q\delta_{w^*}(\vec{x})\right]W\right\}+\text{c.c.}\right), \tag{4.1}$$

with

$$Q=\pi^2(Q_1+Q_2)/2(1+P), \quad \nu=(R-R_c)/R_c. \tag{4.2}$$

We correct here for an error of $\frac{1}{2}$ our corresponding result in Ref. 14. The spatial integral in Eq. (4.1). is taken over the entire horizontal plane. $W(\{w\},t)$ is the probability density in the function space of w, w^* for finding the complex functions $w(\vec{x})$, $w^*(\vec{x})$ at time t. Hence, W is a functional $w(\vec{x})$ and $w^*(\vec{x})$, and a function of time t.[36] The boundary conditions in function space, which have to be satisfied by acceptable solutions of Eq. (4.1), are that W and its functional derivatives vanish for $|w(\vec{x})|\to\infty$, for any (measurable) sets of points $\vec{x}$.

Two special solutions of Eq. (4.1) are particularly interesting. The first is the stationary time-independent probability density $W(\{w\})$, which is obtained as a solution of Eq. (4.1) in the limit $t\to\infty$, or by simply putting $\partial_t=0$. The second solution is the so-called conditional probability density $P(\{w\}|\{w_0\};t)$, i.e., the probability density for finding the functions $w(\vec{x}),w^*(\vec{x})$ at time t if $w_0(\vec{x})$, $w_0^*(\vec{x})$ are known to have been realized at time $t=0$. P is the solution of Eq. (4.1) which satisfies the initial condition

$$P(\{w\}|\{w_0\};0)=\delta^{(2)}(\{w\}-\{w_0\}). \tag{4.3}$$

Here, $\delta^{(2)}(\{w\})$ is the δ functional in the complex function space.[37] We will use this functional as a purely formal device, which is an obvious extension to functional integration of the common δ-function concept.

Since we are concerned with a Markoffian random process, P already contains the complete information about the fluctuations. For $t\to\infty$, it reduces to $W(\{w\})$. Once P is known, correlation functions of the random process can be evaluated from functional integrals, at least in principle. Unfortunately, only the time-independent solution of Eq. (4.1) can be obtained in closed form (see below). Furthermore, even for time-independent correlation functions, the required functional integrals can be exactly carried out only in the strictly one-dimensional case, where $w(x_1,x_2)$ reduces to a function $w(x_1)$ of x_1 alone. In all other cases, we have to resort to certain approximations. Fortunately, their quality can be checked for the exactly solvable time-independent one-dimensional case. These topics will be dealt with in detail in Secs. V and VI. In the remainder of this section, we will summarize what can be said in general about Eq. (4.1).

B. Some symmetries

As a consequence of translational invariance, Eq. (4.1) is covariant against the global gauge trans-

formation

$$w(\vec{x}) \to w(\vec{x})\, e^{i\varphi}, \tag{4.4}$$

where φ is independent of x. The time-independent solution $W(\{w\})$ of Eq. (4.1) can, therefore, be only a functional of the product $w(\vec{x})w^*(\vec{x}')$. All moments of that distribution which contain factors of w and w^* in unequal number, then have to vanish. In particular, we have $\langle w\rangle = 0$.

The only conceivable exception to this result would be the case in which the continuous translation invariance is spontaneously broken down to some discrete lattice symmetry. This could, of course, just be the case in which convection cells appear and form a regular lattice in the horizontal plane. A Goldstone mode would be connected with the spontaneously broken symmetry which, in the present case, would correspond to a collective vibration of the lattice of rolls, i.e., a phonon whose frequency approaches zero and whose lifetime becomes infinite in the long-wavelength limit.

We will analyze this question in some detail and show that, in a sense, our system comes very close to a spontaneous breakdown of symmetry, but never really does quite make it. A breakdown of translational invariance does, of course, occur for the steady state without fluctuations, which is described by a time-independent solution of the deterministic hydrodynamic equations; however, the fluctuations from the steady state are found always to restore the full continuous translational symmetry. Phrased differently, the would-be Goldstone mode of broken translational invariance in the present case retains a finite, albeit large, lifetime, even in the long-wavelength limit, and is therefore a diffusion mode, rather than a zero-frequency phonon.

Another obvious symmetry of the liquid layer, the two-dimensional rotational symmetry within the horizontal plane, is not a symmetry of our Fokker-Planck equation (4.1). Clearly, the spatial-derivative terms in Eq. (4.1) cannot be written in covariant form without introducing a vector, e.g. $\vec{k}_c$, with a preferred direction within the horizontal plane. This preferred direction has first been introduced in Eq. (3.5), where we decided to treat fluctuations of rolls in a preferred direction. By this ansatz, the continuous rotational invariance was reduced to invariance against rotations of 180° only. This latter symmetry is still present in Eq. (4.1).

In order to maintain the full rotational invariance, rolls in all directions [separated by angles of order $(\epsilon)^\circ$, however[25]] would have to be considered. From a formal point of view, our considerations up to this point would not change much. The Fokker-Planck equation (4.1) would be amended by a sum over all roll directions, in addition to the spatial integral, and it would receive additional terms, cubic in w, which describe nonlinear interactions of rolls in different directions. Thus, the rotationally invariant Fokker-Planck equation is of the form

$$(1+P)\partial_t W = \int\int d^2x \sum_l \left\{ \delta_{w_l}(\vec{x}) \left[\left(-\tfrac{3}{2}\pi^2\nu + \sum_m b_{lm} |w_m|^2 \right) w - \left(\frac{\vec{\partial}^2 + 2i\vec{k}_{cl}\cdot\vec{\partial} - (\vec{k}_{cl}\cdot\vec{\partial})^2/k_c^2}{k_c} \right)^2 w_l + Q\delta_{w_l^*}(\vec{x}) \right] W \right\} + \text{c.c.} \Big), \tag{4.5}$$

where we have introduced the horizontal gradient $\vec{\partial} = (\partial_1, \partial_2, 0)$, the index l numbers all directions of the rolls, and $\vec{k}_{cl}$ is a vector of length $\pi/\sqrt{2}$ in the l direction. The form of the mode-mode coupling terms already follows from translational invariance and local coupling. As a consequence of rotational invariance, the coupling coefficients b_{lm} can depend on l, m only via the scalar product $k_{cl}\cdot k_{cm}$. Hence, they satisfy the symmetry relation

$$b_{lm} = b_{ml}\,. \tag{4.6}$$

Since we want to restrict our analysis only to rolls in one direction, we fortunately need not calculate the coefficients $b_{l,m}$ explicitly here.[38]

C. Detailed balance

A further important symmetry of Eq. (4.1) is expressed by its transformation behavior against time reversal $t \to -t$. We will show that the conditional probability density P, as obtained from Eq. (4.1), satisfies the detailed balance symmetry

$$P(\{w\}|\{w_0\};t)W(\{w_0\}) = P(\{w_0^*\}|\{w^*\};t)W(\{w^*\}). \tag{4.7}$$

Equation (4.7) is an expression of important symmetries of correlation functions and implies, e.g., that $\langle w^*(\vec{x}t)w(\vec{x}'t')\rangle = \langle w^*(\vec{x}'t)w(\vec{x}t')\rangle$.

In order to prove Eq. (4.7), we will make use of results obtained in earlier work,[39] where the conditions which any Fokker-Planck equation has to satisfy [in order that its solution has the symmetry expressed by Eq. (4.7)], have been found. For the present case, a sufficient condition for Eq. (4.7) to meet is the requirement that Eq. (4.1) be put into the form

$$\partial_t W = \frac{Q}{(1+P)} \int\int d^2x \left(\{\delta_w(\vec{x})[\delta_{w*}(\vec{x})\Phi(\{w\}) + \delta_{w*}(\vec{x})]W\} + \text{c.c.}\right) \tag{4.8}$$

with a suitably chosen functional $\Phi(\{w\})$. It is easily seen that the sufficient criterion Eq. (4.8) is, in fact, met by Eq. (4.1). Indeed, by comparison of Eq. (4.8) with Eq. (4.1) we find

$$\Phi(\{w\}) = Q^{-1} \int\int d^2x \left(-\tfrac{3}{2}\pi^2\nu|w|^2 + \tfrac{1}{4}P^2|w|^4 + 4\left|\partial_1 w - \frac{i}{\sqrt{2}\,\pi}\partial_2^2 w\right|^2\right). \tag{4.9}$$

To be sure, the detailed balance property [Eq. (4.7)], which we have thereby established for the fluctuations of the slow mode, is a property of this slow mode alone. For example, it may be easily checked that the full hydrodynamic equations, Eq. (2.17–2.19), which still contain arbitrary hydrodynamic fluctuations, do not have a corresponding detailed balance symmetry, as long as the temperature gradient $\sim R$ is different from zero. For $R=0$, the fluid layer is, of course, in thermal equilibrium where detailed balance is present simply as a consequence of the reversibility of the microscopic motion. No such microscopic symmetry underlies the detailed balance of the slow mode, since external forces, $R \neq 0$, destroy these symmetries on the microscopic level. In the present case, we may state in retrospect that the potential, Eq. (4.9), exists, and hence the symmetry, Eq. (4.7), appears, for the combined reasons that the slow mode dominates the fluctuations, and that Eq. (4.1) is invariant against the transformation (4.4) (i.e., translational invariance), and the invariance against rotations by 180°.

It is instructive to note in this context that in the more general case, Eq. (4.5), detailed balance of the slow-mode amplitudes is still preserved. The sufficient condition, Eq. (4.8), is generalized in an obvious way by a double sum over all roll directions. The potential Φ still exists and is obtained from Eq. (4.5) in the new form

$$\Phi = Q^{-1}\int\int d^2x \sum_l \Big[-\tfrac{3}{2}\pi^2\nu|w_l(\vec{x})|^2 + \frac{1}{2}\sum_m b_{lm}|w_l(\vec{x})|^2|w_m(\vec{x})|^2 + \left|\left(\frac{\vec{\partial}^2 + 2i\vec{k}_{cl}\cdot\vec{\partial} - (\vec{k}_{cl}\cdot\vec{\partial})^2/k_c^2}{k_c}\right)w_l(\vec{x})\right|^2\Big]. \tag{4.10}$$

For the existence of the potential (4.10), the additional condition, Eq. (4.6), has to be satisfied; this was found to hold by rotational invariance. Hence, rotational invariance, in addition to translational invariance, underlies the detailed balance symmetry in this case.

D. Time-independent solution

The Fokker-Planck equation (4.8) obviously has the time-independent solution[40]

$$W(\{w\}) = Ne^{-\Phi(\{w\})}, \tag{4.11}$$

where Φ is given by Eq. (4.9) and N is a normalization constant. The same form, with Φ given by Eq. (4.10), satisfies the more general Eq. (4.5) in the time-independent case.

This result, and the results of the preceding Sec. IVC, make it clear that the potential Φ plays a central role in our theory. By Eq. (4.11), Φ is connected with the fluctuations from the steady state in much the same way as a thermodynamic potential is connected with fluctuations from an equilibrium state. The steady state itself is obtained as the most likely state by minimizing Φ with respect to $w(x)$ subject to specific boundary conditions.

Taking averages in Eq. (4.8), we obtain

$$\langle \dot{w}(\vec{x}, t)\rangle = -[Q/(1+P)]\langle \delta_{w*}(\vec{x})\Phi(\{w\})\rangle, \tag{4.12}$$

i.e., Φ acts as the potential of a generalized force $-\langle \delta_{w*}\Phi\rangle$ which tends to restore the steady state once it has been perturbed by a fluctuation. Dropping the averages in Eq. (4.12), we come back to the hydrodynamic equation (3.40) without the random term; this equation has the property

$$\dot{\Phi} = -\frac{2Q}{1+P}\int\int d^2x\,|\delta_w(\vec{x})\Phi\,|^2 \leq 0. \tag{4.13}$$

Thus Φ has to decrease monotonically in time. Since Φ itself can always be chosen positively by adding a constant, it may serve as a Lyapunoff function which, by Eq. (4.13), shows the relative stability of the states which minimize Φ.

All these properties have the potential Φ in common with thermodynamic potentials in thermal equilibrium, which is the reason why I have called it a generalized thermodynamic potential. The source of all these similarities is the detailed balance symmetry [Eq. (4.7)] which is so characteristic for systems in thermal equilibrium and which holds, as we have pointed out above, for the slow mode far from thermal equilibrium only as a consequence of other symmetries.

The explicit form of Φ, Eq. (4.9), is typical for systems in the vicinity of an instability. For $R < R_c$, the state $w \equiv 0$ minimizes Φ and corresponds to a stable steady state which realizes the

translational symmetry in a trivial way. The state $w \equiv 0$ becomes unstable if the coefficient of the second-order term in Eq. (4.9) turns negative. The fourth-order term always acts in a stabilizing way, and ensures the normalizability of the probability distribution. The spatial-derivative terms, finally, tend to suppress or make unlikely spatial fluctuations over short distances, and thereby introduce coherence lengths into the fluid flow.

Landau has made thermodynamic potentials of this form, together with the probability density, Eq. (4.11), the basis of his well-known phenomenological phase-transition theory. He also proposed hydrodynamic equations of the form (4.12) (with the averages omitted) to describe fluids near the transition to turbulence in a phenomenological way. Our results up to this point suggest that, based on the detailed balance symmetry of the slow mode, these two Landau theories can be unified to give a phenomenological description of fluctuations near hydrodynamic instabilities.

E. Extrema of generalized thermodynamic potential

Extrema of Φ, Eq. (4.9), correspond to stable, metastable, or unstable steady states. They satisfy the equation

$$Q\delta_{w^*}(\vec{x})\Phi = -\tfrac{3}{2}\pi^2\nu w + \tfrac{1}{2}P^2|w|^2 w - 4\left(\partial_1 - \frac{i}{\sqrt{2}\,\pi}\,\partial_2^2\right)^2 w = 0. \tag{4.14}$$

The only boundary condition I want to impose is that $w(\vec{x})$ remains finite as $|\vec{x}| \to \infty$. For $R < R_c$, i.e., $\nu < 0$, the only solution of Eq. (4.14) which satisfies this boundary condition is the trivial solution $w \equiv 0$, which describes the purely heat-conducting state. It is easily verified that, for $\nu < 0$, this state corresponds to the only minimum of Φ.

For $R > R_c$ ($\nu > 0$), $w = 0$ is a maximum of Φ, and nontrivial solutions of Eq. (4.14) exist, e.g., the plane waves

$$w(\vec{x}) = (3\pi^2\nu/P^2)^{1/2}(1-\kappa^2)^{1/2} e^{i(\varphi_0 + \Delta\vec{k}\cdot\vec{x})}, \tag{4.15}$$

where φ_0 is an arbitrary phase and

$$\kappa^2 = (8/3\pi^2\nu)(\Delta k_1 + \Delta k_2^2/\sqrt{2}\,\pi)^2. \tag{4.16}$$

These solutions, describing ideal lattices of rolls with wave number $\vec{k}_c + \Delta\vec{k}$ give Φ the values

$$\Phi = \Phi_\kappa = -(F/4QP^2)(3\pi^2\nu)^2(1-\kappa^2). \tag{4.17}$$

The absolute minimum of Φ is obtained for $\kappa^2 = 0$, i.e., $\Delta\vec{k} = 0$ (i.e., for rolls with the critical wave number $\vec{k}_c$). This minimum is very sharp, since F/Q in Eq. (4.17) is a very large number.

The stability of the states (4.15) is determined by the second-order derivatives of Φ. Expanding Φ up to second order in the deviations from Eq. (4.15), it is easily shown that the states within the range

$$0 \leq \Delta k_1 < (\tfrac{1}{8}\pi^2\nu)^{1/2}, \quad 0 \leq \kappa \leq 1/\sqrt{3} \tag{4.18}$$

are stable, that states with $\Delta k_1 < 0$ are unstable with respect to the growth of oblique modes, and that sideband instabilities occur for $\Delta k_1 > (\tfrac{1}{8}\pi^2\nu)^{1/2}$.

There are further solutions of Eq. (4.14) which describe slightly disturbed lattices. In particular, we look for perturbations which are localized in the x_1 direction; i.e., we require $|w(\vec{x})|^2 \to 3\pi^2\nu(1-\kappa^2)/P^2$ for $|x_1| \to \infty$. Looking for solutions which do not depend on x_2, we obtain[25,41]

$$w(x_1) = (3\pi^2\nu/P^2)^{1/2}[(1-3\kappa^2)^{1/2}\tanh(x_1/l_\kappa) - i\sqrt{2}\,\kappa] \times e^{i(\varphi_0 + \Delta k_1 x_1)}, \tag{4.19}$$

with

$$\kappa = (8/3\pi^2\nu)^{1/2}\Delta k_1, \tag{4.20}$$

$$l_\kappa = 4/[3\pi^2\nu(1-\kappa^2)]^{1/2}. \tag{4.21}$$

Equation (4.19) describes two different ideal lattices for $x_1 \to \infty, x_1 \to -\infty$, respectively, which differ only by their phase, and which are joined by distorting them smoothly in a region of size l_κ around $x_1 = 0$. As one might expect, an increase $\Delta\Phi_\kappa$ of Φ is associated with the lattice distortion. Its value is obtained as

$$\Delta\Phi_\kappa = 8\sqrt{3}\,\pi^3 L_2 \nu^{3/2}(1-3\kappa^2)^{1/2}/QP^2, \tag{4.22}$$

where L_2 is the diameter of the horizontal plane in the 2 direction. The vanishing of $\Delta\Phi_\kappa$ for $\Delta k_1^2 = \tfrac{1}{8}\pi^2\nu$, i.e., $\kappa^2 = \tfrac{1}{3}$, signals the disappearance of the probability barrier for a lattice distortion at this point and confirms the instability which sets in for $\Delta k_1 > \pi\sqrt{\nu}/2\sqrt{2}$.

From discussions of free energies similar to Eq. (4.9),[41] it is known that the states (4.19) correspond to saddle points of Φ, which are located in function space between the minima of Φ given by Eq. (4.15). Depending on the topology of the underlying function space, these saddle points may or may not become instrumental in determining the fluctuation processes which connect the plane-wave states, Eq. (4.15). Thus, systems which are unbounded in the x_1 direction can pass between two minima of Φ described by Eq. (4.15), by a continuous diffusion process, bypassing the saddle points (4.19) and changing the phase factor of $w(\vec{x})$ in Eq. (4.15) in a continuous way. For systems which are bounded in the x_1 direction, the function space is more restricted than before, since the functions w, w^* have to satisfy an additional boundary condition. Typically, the total phase change of $w(\vec{x})$ along the x_1 axis is fixed and quantized by the boundary condition. In this case, two

neighboring minima of Φ can only be connected by a macroscopic fluctuation (which realizes a discrete phase jump by taking w through zero somewhere) which is most likely to pass over a saddle point of Φ given by Eq. (4.19).

The rate Γ of such fluctuation processes for $\Delta\Phi \gg 1$ can be calculated from the Fokker-Planck equation,[42] and may be written in the form

$$\Gamma = \Omega e^{-\Delta\Phi}, \tag{4.23}$$

where Ω is a frequency factor and $\Delta\Phi$ is the difference of Φ evaluated at the saddle point and at the initial minimum of Φ.[43] The fluctuation rate (4.23) has been evaluated in detail[41] both for Δk increasing and decreasing fluctuations. In the limit $\Delta k \to 0$, these results take the form

$$\Gamma = \frac{(3\nu)^{9/4}}{4P(1+P)} \frac{F}{(QL_2)^{1/2}} \pi^3 \exp\left(-\frac{8\sqrt{3}\,\pi^3 L_2 \nu^{3/2}}{QP^2}\right). \tag{4.24}$$

For systems which are infinitely extended in the x_1 direction, the result (4.24) becomes meaningless, since it then describes a diverging jump rate between states which lie infinitely close together. A different approach to the fluctuation rates then has to be used; it is described in Secs. V and VI.

V. FLUCTUATIONS IN ONE DIMENSION

For the sake of simplicity, we shall first restrict our attention to liquid layers which are infinitely extended only in the preferred x_1 direction, i.e., in the direction of the critical wave number $\vec{k}_c$. It is assumed that the boundary conditions in the orthogonal x_2 direction are such as to strongly favor all modes with $k_2 = 0$, compared to modes with $k_2 \neq 0$. This condition requires that

$$L_2^2 \ll L_1 l. \tag{5.1}$$

We may then simplify the potential Φ, Eq. (4.9), by putting $\partial_2 w = 0$, and obtain

$$\Phi = \frac{L_2}{Q} \int dx_1 \left(-\tfrac{3}{2}\pi^2\nu|w|^2 + \tfrac{1}{4}P^2|w|^4 + 4|\partial_1 w|^2\right). \tag{5.2}$$

We drop the unnecessary index 1 and take $x_1 = x$ in the following calculations.

A. Time-independent correlation functions

We can calculate all time-independent two-point correlation functions, once we succeed in evaluating the average

$$W_2(wx, \tilde{w}\tilde{x}) \equiv \langle \delta^{(2)}(w - w(x))\, \delta^{(2)}(\tilde{w} - w(\tilde{x})) \rangle. \tag{5.3}$$

Equation (5.3) defines the time-independent joint probability density for $w(x)$ to have the value w at point x, and the value $\tilde{w}$ at point $\tilde{x}$. The angular brackets in Eq. (5.3) define the averaging of the enclosed expression with the probability density $W \sim e^{-\Phi}$. Since W is a functional, the averaging requires the evaluation of a functional integral.

In the case in which Φ is given by a one-dimensional integral of the form (5.2), functional integrals of the form (5.3) may be evaluated by solving a partial differential equation according to the well-known Feynman-Kac theorem.[27] In the present case, that partial differential equation takes the form

$$\partial_x W_2 = -HW_2, \tag{5.4}$$

with the Hamiltonian

$$H = -(Q/4L_2)\,\partial_w \partial_{w^*} + (L_2/Q)\left(-\tfrac{3}{2}\pi^2\nu|w|^2 + \tfrac{1}{4}P^2|w|^4 + c_0\right), \tag{5.5}$$

where c_0 is a constant which has yet to be determined. By physical arguments, we have to require that the solution of Eq. (5.5) factorizes into

$$W_2(wx, \tilde{w}\tilde{x}) \to W_1(w)\, W_2(\tilde{w}) \quad \text{for } |x - \tilde{x}| \to \infty.$$

Here $W_1(w)$ is the probability density that $w(x)$ takes on the value w at point x. By translational invariance, $W_1(w)$ is independent of x. Hence, Eq. (5.4) has to have an x-independent solution, which requires that c_0 is given by

$$c_0 = \tfrac{3}{2}\pi^2\nu\langle|w|^2\rangle - \tfrac{1}{4}P^2\langle|w|^4\rangle. \tag{5.6}$$

The "initial condition" to be satisfied by W_2 is

$$W_2(wx, \tilde{w}x) = \delta^{(2)}(w - \tilde{w})\, W_1(\tilde{w}). \tag{5.7}$$

A numerical solution of Eq. (5.4) has been given by several authors.[28] Figures 1 and 2 give the results for

$$\langle|w|^2\rangle = \int d^2w\, |w|^2 W_1(w), \tag{5.8}$$

$$\langle|w|^4\rangle - \langle|w|^2\rangle^2 = \int d^2w\, |w|^4 W_1(w) - \langle|w|^2\rangle^2, \tag{5.9}$$

as a function of ν in normalized units. In Fig. 3 we plot as a function of ν the two correlation lengths ξ_1, ξ_2 which are obtained by fitting to exponentials the correlation functions

$$\langle w^*(x) w(\tilde{x})\rangle = \langle|w|^2\rangle\, e^{-|x-\tilde{x}|/\xi_1}, \tag{5.10}$$

$$\langle[|w(x)|^2 - \langle|w(x)|^2\rangle][|w(\tilde{x})|^2 - \langle|w(\tilde{x})|^2\rangle]\rangle = \langle(|w|^2 - \langle|w|^2\rangle)^2\rangle\, e^{-|x-\tilde{x}|/\xi_2}. \tag{5.11}$$

We find that ξ_1 and ξ_2 are both small for $\nu < 0$ [$\xi_1 \simeq 2\xi_2 \simeq (8/3\pi^2|\nu|)^{1/2}$ in that region], but increase

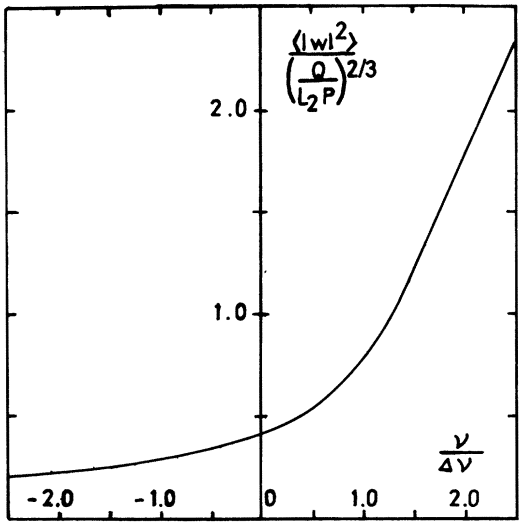

FIG. 1. Intensity of the slow mode as a function of the Rayleigh number $\nu = (R - R_c)/R_c$ in convenient units $[\Delta\nu = (P^2Q/L_2)^{2/3}/3\pi^2]$; (after Scalapino *et al.*, Ref. 28).

as ν approaches zero, to within a transition region of width $\Delta\nu = (P^2Q/L_2)^{2/3}/3\pi^2$. It should be noted that a transition region of the same width follows from the exponent in Eq. (4.24). ξ_2 has a maximum of $\xi_{2\max} \simeq 2(L_2/P^2Q)^{1/3}$ at $\nu_{\max} \simeq (P^2Q/L_2)^{2/3}/3\pi^2$, and decreases again for $\nu > \nu_{\max}$ $[\xi_2 \simeq (\frac{4}{3}\pi^2|\nu|)^{1/2}$ in that region]. ξ_1 increases monotonically as ν grows, and $\xi_1 \simeq 24\pi^2\nu L_2/P^2Q$ for $\nu > \nu_{\max}$.

Since, as will be seen in more detail below, ξ_1 is associated with fluctuations of the phase of w, its increase describes a change from a spatially disordered roll pattern to one of increasing regularity. In order to interpret our result for ξ_2, we recall that the amplitude of the second-harmonic temperature mode, according to Eq. (3.34), is proportional to $|w|^2$. On both sides of the transition region, the second-harmonic mode is dominated by short-ranging fluctuations. Within the transition region, the range of these fluctuations increases drastically; i.e., the coherence of the fundamental mode is partially transferred to the second-harmonic mode in that region.

In order to express correlation functions of the physical variables in terms of our results [Eqs.

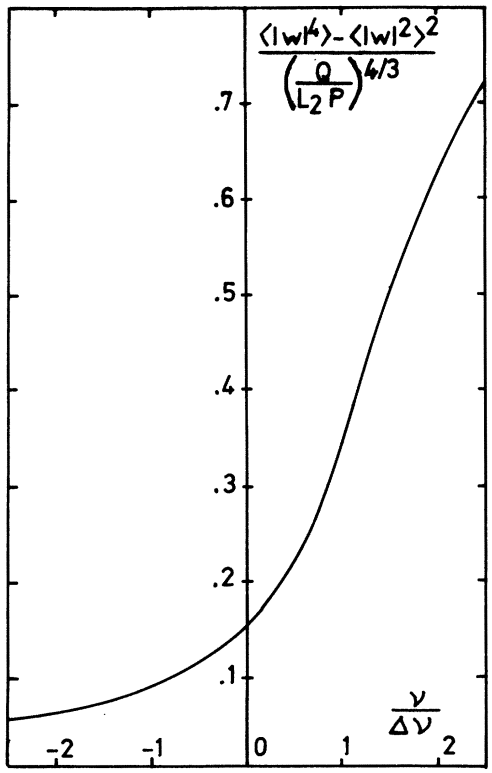

FIG. 2. Second-order cumulant of the slow-mode amplitude as a function of the Rayleigh number (same units as Fig. 1); (after Scalapino *et al.*, Ref. 28).

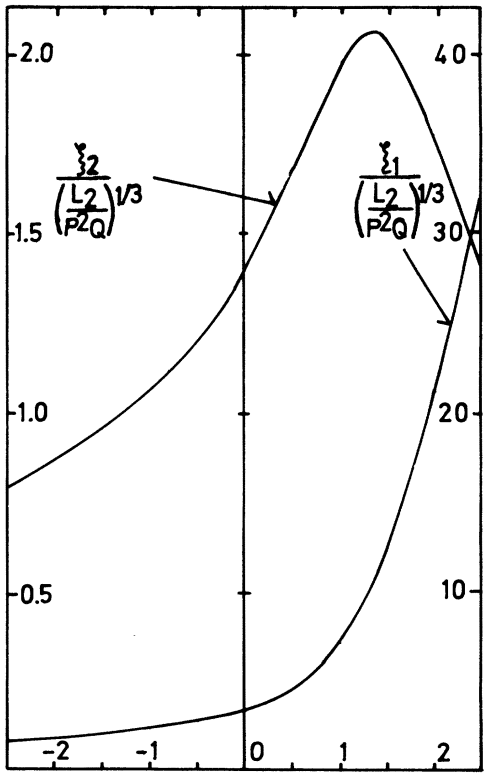

FIG. 3. Coherence lengths ξ_1 (right-hand scale) and ξ_2 (left-hand scale) as functions of the Rayleigh number (same units as Fig. 1); (after Scalapino *et al.*, Ref. 28).

(5.10), (5.11)], we have to make use of the results of Sec. III. For the correlation function of v_3, we directly obtain

$$\langle v_3(x_1x_3)v_3(x_1'x_3')\rangle = 2\langle|w|^2\rangle \sin\pi x_3 \sin\pi x_3' \times \cos[(\pi/\sqrt{2})(x_1-x_1')]e^{-|x_1-x_1'|/\xi_1}. \tag{5.12}$$

According to Eq. (3.46), T is obtained from w in the one-dimensional case by

$$T = \{\sqrt{3}\, e^{i\pi x_1/\sqrt{2}} \sin\pi x_3 [w(x_1) + (2\sqrt{2}\, i/3\pi)\partial_1 w(x_1)] - (\sqrt{3}\, P/4\pi)|w|^2 \sin 2\pi x_3\} + \text{c.c.}. \tag{5.13}$$

In the evaluation of the correlation function of T, we make use of the relation

$$\langle \partial_1 w^*(x_1) w(x_1')\rangle = -\frac{(x_1-x_1')}{|x_1-\tilde{x}_1|}\frac{\langle|w|^2\rangle}{\xi_1} e^{-|x_1-x_1'|/\xi_1}. \tag{5.14}$$

Consistently neglecting the $\langle \partial_1 w \partial_1 w^*\rangle$ term, we obtain

$$\langle T(x_1x_3)\, T(x_1'x_3')\rangle = 6\sin\pi x_3 \sin\pi x_3' \langle|w|^2\rangle \times e^{-|x_1-x_1'|/\xi_1}\left[\cos\left(\frac{\pi}{\sqrt{2}}(x_1-x_1')\right) + \frac{(x_1-x_1')}{|x_1-x_1'|}\frac{4\sqrt{2}}{3\pi\xi_1}\sin\left(\frac{\pi}{\sqrt{2}}(x_1-x_1')\right)\right] + (3P^2/4\pi^2)\sin 2\pi x_3 \sin 2\pi x_3'[(\langle|w|^4\rangle - \langle|w|^2\rangle^2)\exp(-|x_1-x_1'|/\xi_2) + \langle|w|^2\rangle^2]. \tag{5.15}$$

Experimentally, one should be able to distinguish the second-harmonic fluctuations in Eq. (5.15) by their typical x_3 dependence. Hence, ξ_1 should be determined by examining either the velocity fluctuations or the temperature fluctuations at the vertical wave number π, while ξ_2 has to be determined by examining the temperature fluctuations at the vertical wave number 2π.

B. Quasilinear approximation in heat-conduction region

Since exact methods of solution are not available for time-dependent correlation functions or, generally, for fluctuations in systems with more than one spatial dimension, we will evaluate correlation functions by introducing a quasilinear approximation. First, some of the results of the preceding sections will be rederived in approximate but analytical form. Then, the same method will be applied to more general cases.[44]

For $R<R_c$, i.e., $\nu<0$, the term of fourth order in the potential (5.2) is hardly crucial, since the second-order term already limits the size of the fluctuations. Its main effect will be to change the size of the second-order term slightly. This effect is taken into account in an averaged fashion by replacing $|w|^4$ by the partly averaged term $4\langle|w|^2\rangle|w|^2$. Hence, we replace Eq. (5.2) by

$$\Phi = \frac{L_2}{Q}\int dx\,[(-\tfrac{3}{2}\pi^2\nu + P^2\langle|w|^2\rangle)|w|^2 + 4|\partial_1 w|^2], \tag{5.16}$$

where $\langle|w|^2\rangle$ is still to be determined self-consistently.

Introducing spatial Fourier transforms

$$w(q) = \int dx\, e^{-iqx} w(x), \quad w^*(q) = (w(q))^*, \tag{5.17}$$

we may write the Fourier transform of the correlation function

$$K_w(q) = \int dx\, e^{iqx}\langle w^*(x)w(x')\rangle \tag{5.18}$$

in the form

$$\langle w^*(q)w(q')\rangle = 2\pi\delta(q-q')K_w(q). \tag{5.19}$$

The potential (5.16) is diagonalized by Eq. (5.17), and the average (5.19) is readily evaluated. We obtain

$$K_w(q) = \frac{Q/L_2}{-\frac{3}{2}\pi^2\nu + P^2\langle|w|^2\rangle + 4q^2}. \tag{5.20}$$

Equation (5.19) implies the sum rule

$$\int \frac{dq}{2\pi} K_w(q) = \langle|w|^2\rangle \tag{5.21}$$

which, together with Eq. (5.21), gives a closed expression for $\langle|w|^2\rangle$ which may be cast into the form

$$\langle|w|^2\rangle^3 - \tfrac{3}{2}(\pi^2/P^2)\nu\langle|w|^2\rangle^2 - Q^2/16P^2L_2^2 = 0. \tag{5.22}$$

We obtain the asymptotic results

$$\langle|w|^2\rangle \to (Q/4PL_2)^{2/3} \quad \text{for } \nu \to 0 \tag{5.23}$$

and

$$\langle|w|^2\rangle \to Q/2\pi L_2(6|\nu|)^{1/2} \quad \text{for } \nu<0, \quad |\nu|\gg 1. \tag{5.24}$$

These results are in reasonable agreement with the numerical result of Sec. V A.

Equation (5.20) shows the advantage which the quasilinearization procedure has over a simpler straightforward linearization.[23] The latter procedure would give Eq. (5.20) without $P^2\langle|w|^2\rangle$ in the denominator. It would thus describe a diverging fluctuation spectrum for $\nu \to 0$. In the present approximation, the point of this divergence is shifted by $2P^2\langle|w|^2\rangle/3\pi^2$ in Eq. (5.20). Since $\langle|w|^2\rangle$ diverges itself at this point, the critical point $\nu=0$ is shifted to $\nu\to\infty$; i.e., it is completely re-

moved. Of course, for $\nu > 0$, our quasilinearization procedure lacks any justification, and the results of Eq. (5.22) cannot be simply extended into that region. However, we may expect that Eq. (5.20) retains its validity for small positive

$$\nu < \left(\frac{2P^2\langle|w|^2\rangle}{3\pi^2}\right)_{\nu=0} = \frac{2(QP^2/4L_2)^{2/3}}{3\pi^2}. \tag{5.25}$$

Equation (5.25) gives a measure of the width of the region in which the continuous transition from the heat-conducting state to the heat-convecting state takes place. It agrees well with the previous numerical result.

Equation (5.20) may be written in the form

$$K_w(q) = \frac{2\langle|w|^2\rangle\,\xi_1}{1+(\xi_1 q)^2}, \tag{5.26}$$

where the correlation length

$$\xi_1 = (-\tfrac{3}{8}\pi^2\nu + \tfrac{1}{4}P^2\langle|w|^2\rangle)^{-1/2} = 8L_2\langle|w|^2\rangle/Q \tag{5.27}$$

takes on the limiting values

$$\xi_1 \to 2(2L_2/P^2Q)^{1/3} \quad \text{for } \nu \to 0^-,$$

$$\xi_1 \to \left(\frac{3\pi^2|\nu|}{8} + \frac{P^2Q}{8L_2(6|\nu|)^{1/2}}\right)^{-1/2} \quad \text{for } \nu < 0,\ |\nu| \gg 1. \tag{5.28}$$

Again, these results are reasonable approximations of the previous numerical results.

Within our present Gaussian approximation, higher-order correlation functions may be expressed in terms of $\langle w^*(x)w(x')\rangle$; i.e., we have

$$\langle|w(x)|^2|w(x')|^2\rangle = \langle|w|^2\rangle^2 + |\langle w^*(x)w(x')\rangle|^2.$$

It is clear, therefore, that the correlation length ξ_2 defined by Eq. (5.11) is given by $\xi_2 = \frac{1}{2}\xi_1$ within the present approximation.

C. Quasilinear approximation in heat-convection region

We will restrict our attention to the region where R is well above R_c so that the convection rolls are well established. We will not try to give an approximation which remains valid near the convection threshold from above. Within these restrictions we may write the complex amplitude w in the form[44]

$$w(x) = (3\pi^2\nu/P^2)^{1/2}[1+\rho(x)]\,e^{i\varphi(x)}, \tag{5.29}$$

where $\rho(x)$ represents a small amplitude fluctuation, which we will treat as a Gaussian random field, and $\varphi(x)$ represents a slowly varying phase fluctuation. Inserting Eq. (5.29) into Eq. (5.2), keeping only terms up to second order in ρ and approximating functions of ρ which are multiplied with $(\partial_x\varphi)^2$ by their averages, we obtain

$$\Phi = \frac{L_2}{Q}\,\frac{(3\pi^2\nu)^2}{P^2} \times \int dx \left(\rho^2(x) + \frac{4}{3\pi^2\nu}\left[(\partial_x\rho)^2 + \langle 1+\rho^2\rangle(\partial_x\varphi)^2\right]\right). \tag{5.30}$$

The volume element in function space will be changed by the nonlinear transformation (5.29), in general. However, in the region where R is well above R_c, i.e., $|\rho(x)| \ll 1$, that change of the volume element may be neglected. Averages of functionals of $\rho(x)$ and $\varphi(x)$ are then easily calculated from the Gaussian form (5.30).

Introducing spatial Fourier transforms of $\rho(x)$ and $\varphi(x)$, as in Eq. (5.17), and defining $K_\rho(q), K_\varphi(q)$ as in Eq. (5.18), we obtain

$$K_\rho(q) = 2\langle\rho^2\rangle\,\xi_2/[1+(\xi_2 q)^2], \tag{5.31}$$

with

$$\langle\rho^2\rangle = P^2Q/8L_2(3\pi^2\nu)^{3/2}, \tag{5.32}$$

$$\xi_2 = 2/(3\pi^2|\nu|)^{1/2}, \tag{5.33}$$

and

$$K_\varphi(q) = 2\xi_1/(\xi_1 q)^2, \tag{5.34}$$

with

$$\xi_1 = 16L_2\langle|w|^2\rangle/Q. \tag{5.35}$$

Clearly, the correlation function $K_\varphi(q)$ diverges in the limit $q \to 0$ and therefore dominates the long-range part of the correlation function $\langle w^*(x)w(x')\rangle$. From Eq. (5.29) and the Gaussian property of $\varphi(x)$, we have in the long-wavelength limit

$$\langle w^*(x)w(x')\rangle \to \langle|w|^2\rangle \exp\{-\tfrac{1}{2}\langle[\varphi(x)-\varphi(x')]^2\rangle\}. \tag{5.36}$$

The average in the exponent is evaluated from Eq. (5.34) by the relation

$$\langle[\varphi(x)-\varphi(x')]^2\rangle = 2\int \frac{dq}{2\pi} K_\varphi(q)(1-e^{-iq(x-x')}), \tag{5.37}$$

where the principal part of the (otherwise divergent) integral has to be taken. We obtain

$$\langle[\varphi(x)-\varphi(x')]^2\rangle = 2|x-x'|/\xi_1. \tag{5.38}$$

This result, together with Eqs. (5.35) and (5.36), correctly reproduces the asymptotic behavior of the correlation function (5.10). A comparison of Eq. (5.35) with the result (5.26) for $\nu < 0$ shows the latter result to be incorrect by a factor $\frac{1}{2}$ for $\nu > 0$. Finally, from Eqs. (5.29) and (5.31) we obtain the correlation function

$$\langle[|w(x)|^2-\langle|w(x)|^2\rangle][|w(x')|^2-\langle|w(x')|^2\rangle]\rangle = [Q(3\pi^2|\nu|)^{1/2}/2L_2P^2]\,e^{-|x-x'|/\xi_2}, \quad (5.39)$$

again in agreement with the asymptotic behavior of the previous result (5.11).

D. Time-dependent correlation functions; $\nu<0$

Correlation functions which depend on time have to be evaluated from the Fokker-Planck equation (4.8). While an exact solution of Eq. (4.8) for time-dependent distribution functions is impossible, it is very simple to solve that equation within the quasilinear approximations introduced in Secs. V B and C. An extension of these approximations to the time-dependent case is simply achieved by inserting the approximate forms of Φ [Eqs. (5.16) and (5.30), respectively], into Eq. (4.8).

First, we will treat the case $\nu<0$. Introducing the Laplace Fourier transform

$$K_w^{(L)}(qz)=\int_0^\infty dt'\int dx'\,e^{iqx'-izt'} \times\langle w^*(x+x',t+t')w(xt)\rangle, \quad (5.40)$$

we obtain from Eqs. (4.8) and (5.16)

$$[(1+P)iz-\tfrac{3}{2}\pi^2\nu+\tfrac{1}{2}P^2\langle|w|^2\rangle+4q^2]K_w^{(L)}(qz) = (1+P)K_w(q), \quad (5.41)$$

which is readily solved for $K_w^{(L)}(qz)$

$$K_w^{(L)}(qz)=\frac{K_w(q)(1+P)}{(1+P)iz+\frac{3}{2}\pi^2|\nu|+P^2\langle|w|^2\rangle+4q^2}. \quad (5.42)$$

From the detailed balance symmetry, Eq. (4.7), the following connection between the Laplace transform $K_w^{(L)}$ and the Fourier transform

$$K_w^{(F)}(q\omega)=\int dt'\int dx\,e^{iqx'-i\omega t'} \times\langle w^*(x+x',t+t')w(xt)\rangle, \quad (5.43)$$

is easily established:

$$K_w^{(F)}(q\omega)=K_w^{(L)}(q\omega)+[K_w^{(L)}(-q\omega)]^*. \quad (5.44)$$

Making use of Eq. (5.20), we obtain

$$K_w^{(F)}(q\omega)=\frac{4\langle|w|^2\rangle\,\xi_1\tau_1}{(\tau_1\omega)^2+[1+(\xi_1q)^2]^2}, \quad (5.45)$$

or, in the space-time domain,

$$\langle w^*(xt)w(x't')\rangle=\langle|w|^2\rangle\exp\left(-\frac{|x-x'|}{\xi_1}-\frac{|t-t'|}{\tau_1}\right), \quad (5.46)$$

where ξ_1 and $\langle|w|^2\rangle$ are given by Eqs. (5.26) and (5.22) respectively, and the correlation time τ_1 is given by

$$\tau_1=\frac{2(1+P)}{-3\pi^2\nu+2P^2\langle|w|^2\rangle}=\frac{1+P}{4}\,\xi_1^2. \quad (5.47)$$

In particular, we obtain the asymptotic results

$$\tau_1\to(1+P)(4L_2/P^2Q)^{2/3} \text{ for } \nu\to0^-,$$
$$\tau_1\to\frac{2(1+P)}{3\pi^2|\nu|+P^2Q/L_2(6|\nu|)^{1/2}} \text{ for } \nu<0,\ |\nu|\gg1. \quad (5.48)$$

It is plain from Eq. (5.47) that the increasing order of the roll lattice, which is established when the Bénard point is approached from below, is accompanied by a slowing down of the fluctuations. Invoking once again the assumed Gaussian property $w(xt)$, we have

$$\langle|w(xt)|^2|w(x't')|^2\rangle-\langle|w|^2\rangle^2 = \langle|w|^2\rangle^2\exp\left(-\frac{|x-x'|}{\xi_2}-\frac{|t-t'|}{\tau_2}\right), \quad (5.49)$$

where

$$\xi_2=\tfrac{1}{2}\xi_1,\quad \tau_2=\tfrac{1}{2}(1+P)\,\xi_2^2=\tfrac{1}{2}\tau_1. \quad (5.50)$$

Hence, the slowing down expressed by Eq. (5.48) also occurs in a somewhat milder form in correlation functions of higher than the second order, e.g., Eq. (5.49). Physically, such higher-order correlation functions describe the statistical properties of admixtures to modes which are spatial harmonics of the hydrodynamic modes at the critical wave number.

E. Time-dependent correlation functions; $\nu>0$

We proceed as in Sec. V D. Inserting Eq. (5.30) into Eq. (4.8) we obtain

$$K_\rho^{(L)}(qz)=\frac{(1+P)K_\rho(q)}{(1+P)iz+3\pi^2\nu+4q^2}, \quad (5.51)$$

$$K_\varphi^{(L)}(qz)=\frac{(1+P)K_\varphi(q)}{(1+P)iz+4q^2} \quad (5.52)$$

in a notation analogous to Eq. (5.40). Making use of detailed balance, we express the Laplace transforms in terms of Fourier transforms as in Eq. (5.44), we obtain

$$K_\rho^{(F)}(q\omega)=\frac{4\langle\rho^2\rangle\,\xi_2\tau_2}{(\tau_2\omega)^2+[1+(q\xi_2)^2]^2}, \quad (5.53)$$

$$K_\varphi^{(F)}(q\omega)=\frac{2\xi_1\tau_1}{(\tau_1\omega)^2+(\xi_1q)^4}, \quad (5.54)$$

where $\langle\rho^2\rangle$, ξ_1, and ξ_2 are given by Eqs. (5.32), (5.35), and (5.33) respectively, and

$$\tau_1=\tfrac{1}{4}(1+P)\xi_1^2,\quad \tau_2=\tfrac{1}{4}(1+P)\xi_2^2. \quad (5.55)$$

A comparison of Eq. (5.55) with the results (5.47) and (5.50), which hold for $\nu<0$, shows that the quantities τ_1 and ξ_1 are related to each other in the same way, regardless of the size of ν. It appears, therefore, that $\tau_1=\frac{1}{4}(1+P)\xi_1^2$ is also a good approximation within the transition region. The same is not true for τ_2, since the relations (5.55) and (5.50) differ by a factor of 2. By interpolation, $\tau_2\simeq\frac{1}{3}(1+P)\xi_2^2$ is likely to be a good guess of the correct size of τ_2 within the transition region.

VI. FLUCTUATIONS IN TWO DIMENSIONS

The results of Sec. V are subject to the restriction (5.1). We will now lift this limitation and treat the case of fluctuations propagating in all directions within the horizontal plane. The x_1 direction is, of course, still preferred, since it is the direction of the wave number $\vec{k}_c$ of the rolls which are finally established. Since exact methods of solution do not exist for this problem, we will employ the quasilinear approximation which was introduced in Secs. V B–V E, and which was found to be in reasonable agreement with exact results.

A. Heat-conduction region: $\nu<0$

A quasi-Gaussian approximation of the potential Φ, Eq. (4.9), is obtained as in Eq. (5.16),

$$\Phi=Q^{-1}\int\int d^2x\left(\left(-\tfrac{3}{2}\pi^2\nu+P^2\langle|w|^2\rangle\right)|w|^2+4\left|\partial_1 w-\frac{i}{\sqrt{2}\,\pi}\,\partial_2^2 w\right|^2\right). \tag{6.1}$$

Instead of Eq. (5.20), we obtain

$$K_w(\vec{q})=Q\left[-\tfrac{3}{2}\pi^2\nu+P^2\langle|w|^2\rangle+4(q_1+q_2^2/\sqrt{2}\,\pi)^2\right]^{-1}. \tag{6.2}$$

Defining the quantities

$$\vec{k}\equiv\vec{k}_c+\vec{q},\quad \hat{K}(\vec{k})=K(\vec{q}), \tag{6.3}$$

and noting that to within the accuracy used in Eq. (6.1) the relation

$$(q_1+q_2^2/2k_c)^2=\left[(\vec{k}^2-\vec{k}_c^2)/2k_c\right]^2 \tag{6.4}$$

is applicable, we may write our result (6.2) in the rotation invariant form

$$\hat{K}(\vec{k})=\frac{\frac{1}{4}Q\,\xi_1^2}{1+\left[\xi_1(\vec{k}^2-\vec{k}_c^2)/2k_c\right]^2}, \tag{6.5}$$

with

$$\xi_1=\left(-\tfrac{3}{8}\pi^2\nu+\tfrac{1}{4}P^2\langle|w|^2\rangle\right)^{-1/2}. \tag{6.6}$$

The result (6.5) is formally identical to Eq. (5.26). However, $\langle|w|^2\rangle$ has to be determined from the new sum rule

$$\int\frac{d^2k}{(2\pi)^2}\,\hat{K}(\vec{k})=\langle|w|^2\rangle, \tag{6.7}$$

which yields the self-consistency condition

$$\langle|w|^2\rangle=\frac{Q}{8\left[-3\nu+(2P^2/\pi^2)\langle|w|^2\rangle\right]^{1/2}}\times\left\{1+\frac{2}{\pi}\arctan\left[\left(-3\nu+\frac{2P^2}{\pi^2}\langle|w|^2\rangle\right)^{-1/2}\right]\right\}. \tag{6.8}$$

For

$$3|\nu|\gg 2P^2\langle|w|^2\rangle/\pi^2,\quad (3|\nu|)^{1/2}\ll 1 \tag{6.9}$$

Eq. (6.8) reduces to

$$\langle|w|^2\rangle=Q/4(3|\nu|)^{1/2} \tag{6.10}$$

which, according to Eq. (6.9), is correct for

$$\nu<0,\quad \tfrac{1}{3}\gg|\nu|\gg\tfrac{1}{3}(P^2Q/2\pi^2)^{2/3}. \tag{6.11}$$

The right-hand side of Eq. (6.11) is a measure of the width of the transition region. A comparison with the result (5.25) for the one-dimensional case shows that, apart from numerical factors, the length L_2 of the layer in the x_2 direction is replaced by the basic interaction and coherence length $l=1$ in the result for the two-dimensional case. The same rule is found to be valid for all other results as well. In the case

$$\nu=0,\quad 2P^2\langle|w|^2\rangle/\pi^2\ll 1, \tag{6.12}$$

Eq. (6.8) is reduced to

$$\langle|w|^2\rangle=\tfrac{1}{4}(\pi Q/\sqrt{2}\,P)^{2/3}. \tag{6.13}$$

The transition point, according to Eq. (6.2), is effectively shifted from $\nu=0$ to small positive ν, which may be estimated from Eq. (6.9)

$$\nu_{\max}\simeq(2P^2/3\pi^2)(\langle|w|^2\rangle)_{\nu=0}=\tfrac{1}{6}(QP^2/\sqrt{2}\,\pi^2)^{2/3}, \tag{6.14}$$

and is of the same order as the right-hand side of Eq. (6.7). The correlation length ξ_1 as obtained from Eq. (6.6), has its old value

$$\xi_1=2\sqrt{2}/\pi(3|\nu|)^{1/2} \tag{6.15}$$

outside the transition region while, for $\nu\to 0$ it is given by[45]

$$\xi_1=4(\sqrt{2}/\pi QP^2)^{1/3}. \tag{6.16}$$

Time-dependent correlation functions are obtained as in Sec. V. For the Laplace Fourier transform $K_w^{(L)}(\vec{q}z)$ we obtain

$$K_w^{(L)}(\vec{q}z)=\frac{K_w(\vec{q})(1+P)}{(1+P)iz-\frac{3}{2}\pi^2\nu+P^2\langle|w|^2\rangle+4(q_1+q_2^2/\sqrt{2}\,\pi)^2}. \tag{6.17}$$

The transition to the Fourier transform with respect to time can be made as in Sec. VD invoking detailed balance. Also making use of Eqs. (6.3) and (6.4), we finally obtain

$$K_w^{(F)}(\vec{q}\omega) \equiv \hat{K}^{(F)}(\vec{k}\omega) = \frac{8Q\tau_1\xi_1^2}{\tau_1^2\omega^2 + [1+\xi_1^2(\vec{k}^2-\vec{k}_c^2)^2/4k_c^2]^2}, \quad (6.18)$$

with

$$\tau_1 = \tfrac{1}{4}(1+P)\xi_1^2. \quad (6.19)$$

The same simple relation between correlation time and correlation length, which was found for the one dimensional case subsists, therefore, in this more general case also.

The Gaussian assumption upon which the present results are based, allows in principle the calculation of all higher-order correlation functions from the second-order correlation functions given above. However, since the results are algebraically much more complicated than in the one-dimensional case, we will not proceed in that direction.

We note that for $\nu<0$ outside the transition region, our results (6.15) and (6.19) are in agreement with the earlier results of Zaitsev and Shliomis,[23] if the latter are corrected for a misprint. Our Eqs. (6.6) and (6.8) give an extension of these earlier results and cover the transition region also, where Eqs. (6.14), (6.16), and (6.19) give at least a qualitatively correct description.[45]

Let us finally express correlation functions of physical variables in terms of our result (6.18). Defining the Fourier transform

$$K_3(\vec{k}\omega x_3 x_3') = \int_{-\infty}^{+\infty} dt' \int d^2x' \, e^{i\vec{k}\cdot\vec{x}' - i\omega t'} \langle v_3(\vec{x}'x_3't')v_3(\vec{x}x_3t)\rangle, \quad (6.20)$$

and, correspondingly, $K_T(\vec{k}\omega x_3 x_3')$ we have

$$K_3(\vec{k}\omega x_3 x_3') = 2\sin\pi x_3 \sin\pi x_3' \hat{K}_w^{(F)}(\vec{k}\omega), \quad (6.21)$$

$$K_T(\vec{k}\omega x_3 x_3') = 6\sin\pi x_3 \sin\pi x_3' \times (1-4(\vec{k}^2-\vec{k}_c^2)/3\pi^2)\hat{K}_w^{(F)}(\vec{k}\omega). \quad (6.22)$$

In Eq. (6.22), use has been made of Eq. (6.4), in order to cast the result into a rotation invariant form. Furthermore, nonlinear terms in the relation between T and w have been discarded, as is consistent with our earlier approximations.

B. Heat-convection region: $\nu>0$

As in the one-dimensional case, we will restrict our analysis to the domain $\nu \gg \nu_{\max}$. Inserting Eq. (5.29) into Eq. (4.9), and keeping only functionals to second order in ρ and φ, we obtain

$$\Phi = \frac{(3\pi^2\nu)^2}{P^2Q}\int\int d^2x\left[\rho^2 + \frac{4}{3\pi^2\nu}\left(\partial_1\rho + \frac{1}{\sqrt{2}\,\pi}\partial_2^2\varphi\right)^2 + \frac{4}{3\pi^2\nu}\left(\partial_1\varphi - \frac{1}{\sqrt{2}\,\pi}\partial_2^2\rho\right)^2\right]. \quad (6.23)$$

This expression should be compared with the result (5.30) for the one-dimensional case. Algebraically, Eq. (6.23) is more complicated, since fluctuations of ρ and φ are no longer independent of each other. Nevertheless, the Gaussian form (6.23) may be diagonalized with respect to the integration variable by introducing Fourier transforms. In addition to the two-dimensional analogs of K_ρ Eq. (5.31), and K_φ, Eq. (5.34), we have to consider the cross-correlation function

$$K_{\rho\varphi}(\vec{q}) = \int d^2x' \, e^{i\vec{q}\cdot\vec{x}'}\langle\rho(\vec{x}'t)\varphi(\vec{x}t)\rangle, \quad (6.24)$$

which is connected to the correlation function of Fourier transforms by

$$\langle\rho(\vec{q}t)\varphi^*(\vec{q}'t)\rangle = (2\pi)^2 K_{\rho\varphi}(\vec{q})\delta^{(2)}(\vec{q}-\vec{q}'). \quad (6.25)$$

From Eq. (6.23) we obtain, by averaging,

$$K_\rho(\vec{q}) = \frac{P^2Q/(3\pi^2\nu)^2}{1+4(q_1^2-q_2^4/2\pi^2)^2/3\pi^2\nu(q_1^2+q_2^4/2\pi^2)}, \quad (6.26)$$

$$K_\varphi(\vec{q}) = K_\rho(\vec{q})\left(1+\frac{3\pi^2\nu}{4(q_1^2+q_2^4/2\pi^2)}\right), \quad (6.27)$$

$$K_{\rho\varphi}(\vec{q}) = K_\rho(\vec{q})\left(\frac{-i\sqrt{2}\,q_1q_2^2}{\pi(q_1^2+q_2^4/2\pi^2)}\right). \quad (6.28)$$

The function $K_\varphi(\vec{q})$ diverges for $|\vec{q}|\to 0$ and dominates the long-range part of the correlation function

$$\langle w^*(\vec{x}t)w(\vec{x}'t)\rangle \to (3\pi^2\nu/P^2) \times \exp\{-\tfrac{1}{2}\langle[\varphi(\vec{x}t)-\varphi(\vec{x}'t)]^2\rangle\} \quad (6.29)$$

similar to the one-dimensional case. From

$$\langle[\varphi(\vec{x}t)-\varphi(\vec{x}'t)]^2\rangle = 2\int\frac{d^2q}{(2\pi)^2}K_\varphi(\vec{q})(1-e^{-i\vec{q}\cdot(\vec{x}-\vec{x}')}), \quad (6.30)$$

we obtain, retaining only the infrared divergent part of Eq. (6.27),

$$\langle[\varphi(\vec{x}t)-\varphi(\vec{x}'t)]^2\rangle = \frac{P^2Q}{12\pi^2\nu}\left(\frac{\pi}{\sqrt{2}}|x_2-x_2'|+(\sqrt{2}\,|x_1-x_1'|)^{1/2}\right)\exp\left(-\frac{\pi(x_2-x_2')^2}{4\sqrt{2}\,|x_1-x_1'|}\right). \tag{6.31}$$

In the limit

$$|x_2-x_2'| \ll |x_1-x_1'|/k_c \tag{6.32}$$

we obtain

$$\langle w^*(\vec{x}t)w(\vec{x}'t)\rangle \to \frac{3\pi^2\nu}{P^2}\exp\left[-\left(\frac{\sqrt{2}\,|x_1-x_1'|}{\xi_{\varphi 1}}\right)^{1/2}\right], \tag{6.33}$$

with the correlation length

$$\xi_{\varphi 1} = (24\pi^2\nu/P^2Q)^2. \tag{6.34}$$

In the opposite limit

$$|x_2-x_2'|^2 \gg |x_1-x_1'|/k_c, \tag{6.35}$$

our result (6.31) takes the form

$$\langle w^*(\vec{x}t)w(\vec{x}'t)\rangle \to \frac{3\pi^2\nu}{P^2}\exp\left(-\frac{|x_2-x_2'|}{\xi_{\varphi 2}}\right), \tag{6.36}$$

with

$$\xi_{\varphi 2} = 24\sqrt{2}\,\pi\nu/P^2Q. \tag{6.37}$$

For large ν the correlation length in the x_1 direction is much larger than the correlation length in the x_2 direction. Hence, we expect the periodicity of the roll lattice in the x_1 direction to be rather pronounced, while there should be less stability of the lattice against fluctuations along the axis of the rolls in the x_2 direction. For example, fluctuations which involve a simultaneous bending of all rolls would leave the periodicity transverse to the axis of the rolls unchanged, but would reduce the correlation length in the x_2 direction.

The result (6.26) may be used to calculate the correlation function

$$K_{|w|^2}(\vec{q}) = (2\pi)^2\langle|w|^2\rangle^2\delta^{(2)}(\vec{q}) + (6\pi^2\nu/P^2)^2K_\rho(\vec{q}), \tag{6.38}$$

where the notation is analogous to Eq. (5.18). In order to calculate also the short-range part of the correlation function $\langle w^*(\vec{x}t)w(\vec{x}'t)\rangle$ we expand Eq. (5.29) to second order in ρ and φ

$$w(\vec{x}t) = (3\pi^2\nu/P^2)^{1/2}[1+\rho(\vec{x}t)+i\varphi(\vec{x}t) + i\rho(\vec{x}t)\varphi(\vec{x}t) - \tfrac{1}{2}\varphi(\vec{x}t)^2] \tag{6.39}$$

and obtain

$$K_w(\vec{q}) = \frac{3\pi^2\nu}{P^2}\left[(2\pi)^2\left(1-\int\frac{d^2q'}{(2\pi)^2}K_\varphi(\vec{q}')\right)\delta^{(2)}(\vec{q}) + K_\rho(\vec{q})+K_\varphi(\vec{q})+2\,\mathrm{Im}K_{\rho\varphi}(\vec{q})\right]. \tag{6.40}$$

This expression, indeed, satisfies the sum rule

$$\int\frac{d^2q}{(2\pi)^2}K_w(\vec{q}) = \frac{3\pi^2\nu}{P^2}\left(1+\int\frac{d^2q}{(2\pi)^2}K_\rho(\vec{q})\right) = \langle|w|^2\rangle. \tag{6.41}$$

However, since the expansion (6.39) can be justified only for $|\vec{q}|\neq 0$ [since $\varphi(\vec{q})$ becomes soft in the limit $|\vec{q}|\to 0$ and need not be small], the result (6.40) is only valid for "large" $|\vec{q}|$. In fact, our previous results (6.29) and (6.31) show that the singular $\delta^{(2)}(\vec{q})$ in Eq. (6.40) has to be replaced by a function of $\vec{q}$ which, though sharply peaked around $|\vec{q}|=0$, must have a finite width of the order $\xi_{\varphi 1}^{-1}$, $\xi_{\varphi 2}^{-1}$ with respect to q_1 and q_2, respectively. For large $|\vec{q}|$, this sharply peaked term may be omitted altogether. We obtain then

$$K_w(\vec{q}) = \frac{3\pi^2\nu}{P^2}[K_\rho(\vec{q})+K_\varphi(\vec{q})+2\,\mathrm{Im}K_{\rho\varphi}(\vec{q})] = \frac{Q}{4}\,\frac{3\pi^2\nu+8(q_1-q_2^2/\sqrt{2}\,\pi)^2}{3\pi^2\nu(q_1^2+q_2^4/2\pi^4)+4(q_1^2-q_2^4/2\pi^2)^2}. \tag{6.42}$$

Hence, the fluctuation spectrum behaves like $|\vec{q}|^{-2}$ for large wave numbers, and has a resonance for $q_1 = \pm q_2^2/2k_c$; i.e., for fluctuations whose wave vector $\vec{q}$ satisfies the condition $|\vec{q}\pm\vec{k}_c| = |\vec{k}_c|$.

For the sake of completeness, let us finally analyze the time-dependent correlation functions of ρ and φ. Introducing Laplace transforms, we obtain the set of equations

$$\begin{aligned}
&[(1+P)iz+3\pi^2\nu+4q_1^2+2q_2^4/\pi^2]K_\rho^{(L)}(\vec{q}z) - (4\sqrt{2}\,iq_1q_2^2/\pi)K_{\varphi\rho}^{(L)}(\vec{q}z) = (1+P)K_\rho(\vec{q}),\\
&[(1+P)iz+4q_1^2+2q_2^4/\pi^2]K_{\varphi\rho}^{(L)}(\vec{q}z) + (4\sqrt{2}\,iq_1q_2^2/\pi)K_\rho^{(L)}(\vec{q}z) = (1+P)K_{\varphi\rho}(\vec{q}),\\
&[(1+P)iz+4q_1^2+2q_2^4/\pi^2]K_\varphi^{(L)}(\vec{q}z) + (4\sqrt{2}\,iq_1q_2^2/\pi)K_{\rho\varphi}^{(L)}(\vec{q}z) = (1+P)K_\varphi(\vec{q}),\\
&[(1+P)iz+3\pi^2\nu+4q_1^2+2q_2^4/\pi^2]K_{\rho\varphi}^{(L)}(\vec{q}z) - (4\sqrt{2}\,iq_1q_2^2/\pi)K_\varphi^{(L)}(\vec{q}z) = (1+P)K_{\rho\varphi}(\vec{q}).
\end{aligned} \tag{6.43}$$

The explicit solutions of Eqs. (6.43) become rather opaque. The Hermiticity of the matrices in Eq. (6.43) implies that the strength of the fluctuations is always centered around zero frequency. Both $K_\rho^{(L)}$ and $K_\varphi^{(L)}$ contain two different correlation times. In the limit $|\vec{q}|\to 0$, these are the time constants

$$\tau_\varphi \sim (1+P)\xi_{\varphi 1}^2 \sim (1+P)\xi_{\varphi 2}^4, \quad \tau_2 = (1+P)/3\pi^2\nu \tag{6.44}$$

associated with phase and amplitude fluctuations, respectively. Since the coupling between ρ and φ in Eq. (6.43) is proportional to $q_1 q_2^2$, the amplitude and phase fluctuations are completely decoupled in the x_1 and x_2 directions, respectively, but are mixed in all other directions. As in the time-independent case, the main effect of this coupling is to increase the correlations at wave numbers q_1, q_2 satisfying $|q_1| = q_2^2/2k_c$; i.e., $|\vec{q} \pm \vec{k}_c| = |\vec{k}_c|$.

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[30]The factorization of averaged products has to be assumed in addition.

[31]The δ functions in space are required by the locality of the hydrodynamic equations. If problems of convergence arise, it is useful to recall that locality is only required in the hydrodynamic limit, and hence the δ functions may always be assumed to be a bit broad. The δ functions in time express a Markoff assumption.

[32]Rigid boundaries as well as mixed rigid free boundaries lead mainly to a shift of the critical Rayleigh number R_c (cf. Ref. 3). If surface effects are included, the results change qualitatively (cf. Refs. 5 and 12) and hexagons instead of rolls are stabilized near R_c.

[33]A problem of this type has appeared earlier in laser theory. Its solution there became known as the "adiabatic elimination procedure"; see H. Haken in *Encyclopedia of Physics* (Springer, New York, 1970), Vol. 25, and references given there.

[34]If there are fluctuations at the surfaces, these couple in lowest order only to v_1, since they are otherwise orthogonal to ψ_0, Eq. (3.5). Γ, Eq. (3.48), is then supplemented by the surface term $(\frac{2}{3}\sqrt{2}i)\,[\tilde{S}_{13}(\xi, \eta, \tau)|_{x_3=1} + \tilde{S}_{13}(\xi, \eta, \tau)|_{x_3=0}]$. Large surface fluctuations invalidate our procedure.

[35]R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963).

[36]For conciseness of notation, we display only $\{w\}$ in the argument of W.

[37]We use the notation $\delta^{(2)}(\{w\}) = \delta(\{\mathrm{Re}\, w\})\delta(\{\mathrm{Im}\, w\})$.

[38]In the limit $P \to \infty$, these coefficients are given in Ref. 25. Surface tension effects and non-Boussinesq terms of the original hydrodynamic equations lead to terms of second order in the amplitudes w in Eq. (4.5) (stabilizing hexagons), which still preserve detailed balance [cf. Haken's paper (Ref. 14) in this context]. Their coefficients were calculated in the papers of Ref. 12.

[39]R. Graham and H. Haken, Z. Phys. 243, 289 (1971); 245, 141 (1971); R. Graham, *Springer Tracts in Modern Physics* (Springer, New York, 1973), Vol. 66. H. Risken, Z. Phys. 251, 231 (1972).

[40]This solution is possible because detailed balance holds (Ref. 39) or, more mathematically, because a set of integrability conditions are satisfied. The latter are given in Ref. 35 and, more generally, in H. Haken, Z. Phys. 219, 246 (1969), and imply detailed balance.

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[43]Note that $\Delta\phi$ is not given precisely by Eq. (4.22), since the boundary conditions require slightly different Δk values in Eqs. (4.15), (4.16), and (4.18)–(4.21), respectively. For details, see Ref. 41. In the $\Delta k \to 0$ limit, this difference vanishes.

[44]Quasilinearization procedures of this kind have been applied to lasers [see H. Haken, Z. Phys. 181, 96 (1964); and Ref. 33], and superconductors (see Ref. 41).

[45]Since ξ_1 and τ_1 remain finite throughout (although they become extremely large for $\nu \to 0$ under typical conditions), the point $R = R_c$ is a critical point (with classical critical indices), not in principle, but in all practice. A strict symmetry breaking would, of course, require $\xi_1, \tau_1 \to \infty$ for $R \to R_c$.