PHYSICAL REVIEW A VOLUME 10, NUMBER 5 NOVEMBER 1974

Spin waves in superfluid ^{3}He: Collisionless regime*

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(Received 4 June 1974)

Spin waves in superfluid ^{3}He are studied in the collisionless regime by introducing a canonical transformation in the kinetic equation for the distribution-function matrix. This transformation is a space-time–dependent spin rotation, which takes into account the fluctuations of the direction of the order parameter which are coupled to the spin-density fluctuations. After the transformation, the order parameter is invariant. The kinetic equation is then expanded to lowest order in ω and $\vec{q}$ and diagonalized. Fermi-liquid effects are taken into account. The result is a set of equations describing the coupled motion of the superfluid parameters and the quasiparticle distribution. In the axial state, only two degenerate spin-wave modes exist. These have their spin polarization perpendicular to the direction of the order parameter in spin space. In the isotropic state, one longitudinal and two transverse modes are obtained. Equations giving the velocity of these modes are written.

I. INTRODUCTION

The recent discovery of superfluid phases for ^{3}He at very low temperature[1,2] has given rise to an increasing interest in the theoretical understanding of these phases. The early hypothesis[3–5] of a BCS-type superfluid has been confirmed, but it has always been recognized that the pairing in the superfluid phases cannot be of s-wave type. A problem of interest, therefore, is to find which nonzero-momentum pairing is responsible for the various superfluid phases. There is some evidence[6] that only odd-momentum pairing is present. Accordingly, theoretical work has been mainly focused on the simplest case, p-wave pairing. However, a general analysis[7] of p-wave pairing in the framework of the Landau-Ginzburg theory shows that many possible states must be considered, depending on the value of the parameters appearing in this theory. So experiments are needed to rule out some of these states. Among many possible experiments, those that excite various collective modes of the system appear to be a powerful probe of the microscopic nature of the involved states. In fact, possibly the strongest support that the A phase of ^{3}He is the so-called axial state is from NMR experiments,[2,8,9] interpreted by Leggett.[10] In the same way, zero-sound[11,12] and fourth-sound[13,14] experiments are useful tools in testing existing theories.

One way of investigating these collective modes is to calculate by the standard Green's-function technique the appropriate correlation function and to find its poles. This method is very powerful and is especially convenient if the collective modes are ill defined. This approach has been used recently within the random-phase approximation (RPA) by Maki and Ebisawa to study zero-sound propagation and attenuation,[15] nuclear magnetic resonance,[16] and spin-wave propagation[17] in the A phase of ^{3}He; however, this method provides little physical insight. Another approach is to perform a canonical transformation to introduce physical quantities which describe explicitly the collective modes. One can then write transport equations governing the evolution of the collective mode parameters. This method is more tractable than the first one and yields a better physical understanding of the problem. This approach also allows the introduction of a two-fluid description of the superfluid. Such a description is known to be extremely useful. The two-fluid approach has been used by Betbeder-Matibet and Nozières[18] for a superfluid Fermi liquid with s-wave pairing. More recently, Wölfle[19] has applied this method to study zero-sound propagation in superfluid ^{3}He. Although his paper deals more with an anisotropic s-wave superfluid than with p-wave pairing, his results are valid for zero sound in ^{3}He.

In this paper, we extend the method of canonical transformation to the study of spin-wave propagation in superfluid ^{3}He, assuming p-wave pairing to be responsible for the superfluidity. The complexity of the order parameter for p-wave pairing leads to a large number of possible modes. For example, if the dipole-dipole interaction is neglected, there are separate degeneracies with respect to rotations in the spin space and in the k space. This permits modes associated with the k-space degeneracy. We ignore this possibility, leaving it to further study. Although the weak dipole-dipole interaction is very important in NMR, we neglect it. Throughout the paper, quasiparticle relaxation is also ignored, and only the collisionless regime is considered. Since the low relaxation time of the quasiparticles is of order of 1 MHz in the millikelvin temperature range, this regime can easily be reached experimentally.

In Sec. II, the theory is developed without consideration of Fermi-liquid effects. These are introduced in Sec. III.

II. FORMALISM

When zero sound is propagating in superfluid ^{3}He, the density fluctuations are coupled to the phase fluctuations of the order parameter. In the spirit of the method of canonical transformation, these phase fluctuations are described by a superfluid velocity, and one gets equations coupling the superfluid parameters and the quasiparticle distribution. In the same way, spin waves are coupled to order parameter rotations in the spin space. For p-wave pairing, the order parameter is a 2×2 matrix and may be written

$$\Delta(\vec{k}) = i\vec{\sigma}\cdot\vec{d}(\vec{k})\sigma_y,$$

where σ_x, σ_y, and σ_z are the Pauli matrices and $\vec{d}(\vec{k})$ is a spin vector which depends linearly on $\vec{k}$. The spin waves will be coupled to rotations of $\vec{d}(\vec{k})$. Naturally, it is, in general, invalid to use a space-dependent $\vec{d}(\vec{k})$ unless only processes on the scale of the coherence length are of interest. In fact, we are interested in fluctuations on an even larger scale. If we consider spin waves with wave vector $q \sim 1/\xi_0$ and frequency $\omega \sim \Delta$, they merge into the continuum of the excitation spectrum and are no longer well-defined collective modes. Accordingly, we will be interested in the long-wavelength, $qv_F \ll \Delta$, and low-frequency, $\omega \ll \Delta$, range. In this region, spin waves are well defined, because of the gap in the excitation spectrum. Naturally, at nonzero temperatures, there is some attenuation due to thermal excitation. This attenuation increases with increasing ω and q and with increasing temperature. When the temperature is such that $\Delta(T) \sim \hbar/\tau$, where τ is the quasiparticle relaxation time, our calculation is no longer valid, since it is impossible to satisfy $\hbar/\tau \ll \omega \ll \Delta(T)$. A simple calculation, however, indicates this situation occurs only very near T_c ($\sim 10^{-5}$ °K from T_c) so that one can ignore this region. Actually we are interested in states where the gap may vanish at some point on the Fermi surface, allowing Landau damping at any temperature, but by a convenient choice of the frequency, the phase space corresponding to this process may be made small, except near T_c, just as before. So this effect may also be neglected.

In order to describe the spin waves, we perform a canonical transformation to a new representation where the $\vec{d}(\vec{k})$ vector is space and time independent. In other words, we follow the motion of the $\vec{d}$ vector by a local spin rotation. Naturally, the canonical transformation is space and time dependent. Since it is more general and not more complicated, we also include phase variations which lead to gauge transformation.

We start with the BCS Hamiltonian. In the space representation ($\hbar = 1$),

$$\mathcal{H}_0 = \frac{1}{2m}\int d^3r\,[\vec{\nabla}\psi^\dagger_\alpha(\vec{r})][\vec{\nabla}\psi_\alpha(\vec{r})] + \frac{1}{2}\int d^3r\,d^3r'\,g(\vec{r}-\vec{r}\,')\psi^\dagger_\alpha(\vec{r})\psi^\dagger_\beta(\vec{r}\,')\psi_\beta(\vec{r}\,')\psi_\alpha(\vec{r}), \tag{1}$$

where m is the (bare) ^{3}He mass and the second term is the short-range BCS interaction where only the p-wave anomalous part is retained. We perform the following canonical transformation:

$$\psi_\alpha(\vec{r},t) = \{\exp[i\sigma_\lambda\theta_\lambda(\vec{r},t)]\}_{\alpha\beta}\varphi_\beta(\vec{r},t), \tag{2}$$

where $\varphi_\beta(\vec{r},t)$ is the new field variable, α and β are spin indices, and σ_λ ($\lambda = 0, 1, 2, 3$) are the Pauli matrices; $\theta_\lambda(\vec{r},t)$ are the parameters describing the spin rotation and phase change of the order parameter. We will assume that the departure from equilibrium is small and work in the linear approximation. Accordingly, $\theta_\lambda(\vec{r},t)$ will be small compared to unity, and we may expand the exponential in Eq. (2) to first order in $\theta_\lambda(\vec{r},t)$. As long as the BCS interaction is a density-density (or spin-density–spin-density) interaction, it is invariant under the transformation. If one considers an interaction which is not a density-density one, the variation of θ_λ over the range of the interaction, namely the coherence length, enters the problem, but this leads to a correction which is only of order of $q\xi_0$ times the BCS interaction (q is the perturbation wave-vector) and may safely be neglected. The transformed Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\int d^3r[\vec{\nabla}\varphi^\dagger_\alpha(\vec{r})][\vec{\nabla}\varphi_\alpha(\vec{r})] + \tfrac{1}{2}\int d^3r\,d^3r'\,g(\vec{r}-\vec{r}\,')\varphi^\dagger_\alpha(\vec{r})\varphi^\dagger_\beta(\vec{r}\,')\varphi_\beta(\vec{r}\,')\varphi_\alpha(\vec{r}) + i\int d^3r\{\vec{\nabla}\varphi^\dagger_\alpha(\vec{r})\cdot\vec{A}_{\alpha\beta}\varphi_\beta(\vec{r}) - \varphi^\dagger_\alpha(\vec{r})\cdot\vec{A}_{\alpha\beta}\vec{\nabla}\varphi_\beta(\vec{r})] + \int d^3r\,\varphi^\dagger_\alpha(\vec{r})V_{\alpha\beta}\varphi_\beta(\vec{r}), \tag{3}$$

where

$$\vec{A}_{\alpha\beta} = \frac{1}{m}\vec{\nabla}\theta_\lambda(\sigma_\lambda)_{\alpha\beta}, \quad V_{\alpha\beta} = \frac{\partial\theta_\lambda}{\partial t}(\sigma_\lambda)_{\alpha\beta}. \tag{4}$$

The last term in the Hamiltonian comes from the time dependence of our canonical transformation. This transformation looks, in fact, like a generalized gauge transformation, and $\vec{A}$ and V are (except for the sign) natural generalizations of the usual vector and scalar potential.

We now consider a perturbation that is frequency ω and wave-vector $\vec{q}$ dependent; $\vec{A}$ and V depend on space and time only through the factor $e^{i(\vec{q}\cdot\vec{r}-\omega t)}$. After a Fourier transform, the Hamiltonian becomes

$$\mathcal{H}=\sum_k \xi_k c_{k,\alpha}^\dagger c_{k,\alpha}+\sum_k c_{k+q/2,\alpha}^\dagger M_{k\alpha\beta}c_{k-q/2,\beta}+\tfrac{1}{2}\sum_{k,k',k_0} V_{kk'}c_{k+k_0,\alpha}^\dagger c_{-k,\beta}^\dagger c_{-k',\beta}c_{k'+k_0,\alpha}, \tag{5}$$

where

$$M_{k\alpha\beta}=\vec{k}\vec{A}_{\alpha\beta}+V_{\alpha\beta} \tag{6}$$

and $\vec{A}$ and V must now be understood to be without space or time dependence, that is, the exponential factor has been removed. The zero of energy is now taken at the Fermi level,

$$\xi_k=k^2/2m-\mu\,.$$

To find the propagation of the spin waves, we write the equations of motion for the distribution function. The 4×4 distribution matrix $n_{kk'}$ is defined as

$$n_{kk'}=\begin{pmatrix}\langle c_{k\alpha}^\dagger c_{k'\beta}\rangle & \langle c_{k\alpha}^\dagger c_{-k'\beta}^\dagger\rangle\\ \langle c_{-k\alpha}c_{k'\beta}\rangle & \langle c_{-k\alpha}c_{-k'\beta}^\dagger\rangle\end{pmatrix}. \tag{7}$$

At equilibrium, $n_{kk'}=n_k^0\delta_{k,k'}$. We make the usual assumption that the order parameter is a unitary matrix. In this case,

$$n_k^0=\tfrac{1}{2}+\epsilon_k^0\varphi_k/E_k, \tag{8}$$

where

$$E_k^2=\xi_k^2+(\Delta_k\Delta_k^\dagger),\quad \varphi_k=-\tfrac{1}{2}\tanh(\tfrac{1}{2}\beta E_k)$$

and

$$\epsilon_k^0=\begin{pmatrix}\xi_k\delta_{\alpha\beta} & \Delta_{k\,\alpha\beta}^\dagger\\ \Delta_{k\alpha\beta} & -\xi_k\delta_{\alpha\beta}\end{pmatrix}. \tag{9}$$

In the presence of the perturbation, $n_{kk'}$ also contains terms with $k'=k+q$; we write these terms as

$$\delta n_k\equiv n_{k-q/2,k+q/2}\equiv\begin{pmatrix}\delta n_e & \delta n_+\\ \delta n_- & \delta n_t\end{pmatrix}. \tag{10}$$

Higher harmonics may be neglected as long as we are working in first-order perturbation theory.

Now we can write the equation of motion for δn_k. We will treat this equation in the usual Hartree-Fock self-consistent scheme. We note that this approximation is completely equivalent, in principle, to the RPA used by Maki and Ebisawa. Doing so, we get the following kinetic equation[18]:

$$\omega\delta n_k=\delta n_k\epsilon_{k+q/2}^0-\epsilon_{k-q/2}^0\delta n_k+n_{k-q/2}^0\delta\epsilon_k-\delta\epsilon_k n_{k+q/2}^0, \tag{11}$$

where

$$\delta\epsilon_k=\begin{pmatrix}M_{k,\beta\alpha} & \delta\Delta_{k,\alpha\beta}^\dagger\\ \delta\Delta_{k,\alpha\beta} & -M_{-k,\alpha\beta}\end{pmatrix} \tag{12}$$

and $\delta\Delta$ is self-consistently related to δn by

$$\delta\Delta_{k,\alpha\beta}=\sum_{k'}V_{kk'}\delta n_{-k',\alpha\beta},\qquad \delta\Delta_{k,\alpha\beta}^\dagger=\sum_{k'}V_{kk'}^*\delta n_{+k',\alpha\beta}. \tag{13}$$

Since we are interested in the large wavelength limit, the kinetic equation (11) may be expanded to first order in $\vec{q}$,

$$\omega\delta n_k=[\delta n_k,\epsilon_k^0]+[n_k^0,\delta\epsilon_k]+\frac{\vec{q}\cdot\vec{k}}{2m}\left\{\left[\delta n_k,\frac{\partial\epsilon_k^0}{\partial\xi_k}\right]_+-\left[\delta\epsilon_k,\frac{\partial n_k^0}{\partial\xi_k}\right]_+\right\}. \tag{14}$$

In calculating $\partial\epsilon_k^0/\partial\xi_k$ and $\partial n_k^0/\partial\xi_k$, the derivative $\partial\Delta_k/\partial\xi_k$ may be neglected since it leads to terms of order q/k_F smaller than other terms. This equation may be greatly simplified by going to a representation which diagonalizes the BCS Hamiltonian, that is, performing the Bogoliubov-Valatin transformation. The unitary transformation matrix may be chosen as

$$U_k=\begin{pmatrix}u_k & v_k\\ -v_k^\dagger & u_k\end{pmatrix}, \tag{15}$$

where u_k and v_k are the following 2×2 matrices:

$$(u_k)_{\alpha\beta}=u_k\delta_{\alpha\beta}=[(E+\xi)/2E]^{1/2}\delta_{\alpha\beta},\qquad (v_k)_{\alpha\beta}=[u_k/(E+\xi)]\Delta_{k,\alpha\beta}^\dagger. \tag{16}$$

With this transformation, ϵ_k^0 and n_k^0 are diagonalized,

$$U_k\epsilon_k^0U_k^\dagger=\begin{pmatrix}E_k\delta_{\alpha\beta} & 0\\ 0 & -E_k\delta_{\alpha\beta}\end{pmatrix},$$

$$U_kn_k^0U_k^\dagger=\begin{pmatrix}(\tfrac{1}{2}+\varphi_k)\delta_{\alpha\beta} & 0\\ 0 & (\tfrac{1}{2}-\varphi_k)\delta_{\alpha\beta}\end{pmatrix}.$$

Defining

$$\delta\nu_k=\begin{pmatrix}\delta\nu_e & \delta\nu_+\\ \delta\nu_- & \delta\nu_t\end{pmatrix}$$

and

$$\delta E_k=\begin{pmatrix}\delta E_e & \delta E_+\\ \delta E_- & \delta E_t\end{pmatrix},$$

as, respectively, the transforms of δn_k and $\delta\epsilon_k$, we note that the kinetic equation breaks into four independent equations, if we retain only the lowest order terms in ω and $\vec{q}$ in each equation. The resulting kinetic equations are

$$(\omega - \vec{q}\cdot\vec{\nabla}_k E_k)\delta\nu_e = -\vec{q}\cdot\vec{\nabla}_k E_k \varphi_k' \delta E_e , \quad (17a)$$

$$(\omega + \vec{q}\cdot\vec{\nabla}_k E_k)\delta\nu_t = +\vec{q}\cdot\vec{\nabla}_k E_k \varphi_k' \delta E_t , \quad (17b)$$

$$\delta\nu_\pm = (\varphi_k/E_k)\delta E_\pm \quad (17c)$$

where $\varphi_k' = \partial\varphi_k/\partial E_k$.

In order to use these equations, we must express δE_e in terms of $\delta\nu_e$ and of the collective modes parameters. We need to know explicitly δE_k in terms of $\delta\epsilon_k$. Omitting the $\vec{k}$ index for simplicity, we find

$$\begin{aligned} \delta E_e &= u\,\delta\epsilon_e u + v\delta\epsilon_t v^\dagger + u\,\delta\epsilon_+ v + v\delta\epsilon_- u , \\ \delta E_t &= v^\dagger \delta\epsilon_e v + u\delta\epsilon_t u - v^\dagger \delta\epsilon_+ u - u\delta\epsilon_- v , \\ \delta E_+ &= -u\delta\epsilon_e v + v\delta\epsilon_t u + u\delta\epsilon_+ u - v\delta\epsilon_- v , \\ \delta E_- &= -v^\dagger \delta\epsilon_e u + u\delta\epsilon_t v^\dagger - v^\dagger \delta\epsilon_+ v^\dagger + u\delta\epsilon_- u . \end{aligned} \quad (18)$$

Similar relations hold between $\delta\nu_e$ and δn_e; the inverse transformation is simply obtained by changing v to $-v$.

We point out some useful relations. From the definition of $\delta\epsilon_k$, it may be verified that

$$^{tr}\delta\epsilon_{e,-k} = -\delta\epsilon_{t,k}, \quad ^{tr}\delta\epsilon_{\pm,-k} = -\delta\epsilon_{\pm,k} ,$$

where $^{tr}\delta\epsilon_e$ is, for example, the transpose of $\delta\epsilon_e$ in the spin space. Similar relations hold for δn_k. From the transformation relations (18), it may be seen that the above relations hold also for δE_k and $\delta\nu_k$. In the same way, $\delta\epsilon_e$ and $\delta\epsilon_t$ are Hermitian by definition, and $\delta\epsilon_+$ and $\delta\epsilon_-$ are Hermitian conjugates. From the transformation relations (18) and the kinetic equation (17), it may also be seen that these properties hold for δE_k, $\delta\nu_k$, and δn_k.

We recall that our initial canonical transformation is such that $\vec{d}(\vec{k})$ is stationary in the spin space. So the only remaining possibility for Δ_k is that its amplitude may vary, but not its structure. Accordingly,

$$\delta\epsilon_{-,k} = \delta\Delta_k = \eta\Delta_k , \quad (19)$$

where η is some small real parameter of first order in the perturbation. Once η is known, δE_e is given explicitly by the first of Eqs. (18).

We define

$$\delta\epsilon_{e,k} = \delta\epsilon^0_{e,k}\sigma_0 + \delta\vec{\epsilon}_{e,k}\cdot{}^{tr}\vec{\sigma} , \quad (20)$$

and similarly for $\delta E_{e,k}$, $\delta\nu_{e,k}$, and $\delta n_{e,k}$. Using this relation with Δ_k expressed in terms of $\vec{d}_k$ and making use of the unitary assumption which implies $\vec{d}\times\vec{d}^* = 0$ and of the relation, $\delta\epsilon_{t,k} = -{}^{tr}\delta\epsilon_{e,-k}$, we finally obtain (omitting the k index for simplicity)

$$\begin{aligned} \delta E^0_e &= k_i A^0_i + (\xi/E)V^0 + \eta\,|\Delta|^2/E , \\ \delta\vec{E}_e &= k_i\vec{A}_i + (\xi/E)\vec{V} - (1-\xi/E)(\vec{d}/|\Delta|^2) \\ &\quad\times[\vec{d}^*(k_i\vec{A}_i - \vec{V})] , \end{aligned} \quad (21)$$

where, for example, for the matrix $V_{\alpha\beta}$ and for each component of the spatial vector $\vec{A}_{\alpha\beta}$, we have written

$$V = V^0\sigma_0 + \vec{V}\cdot\vec{\sigma} . \quad (22)$$

In Eq. (21), the index i refers to components in the real space.

As one will see, η does not appear in the final equations, owing to the electron-hole symmetry. However, we can still indicate the way to derive an equation for η. This is done by putting Eq. (19) into the self-consistent relation, Eq. (13). On the right-hand side of Eq. (13); through Eqs. (18), $\delta n_\pm$ is expressed in terms of $\delta\nu_{e,t}$ and $\delta\nu_\pm$. With Eq. (17c), $\delta\nu_\pm$ is expressed in terms of $\delta E_\pm$, which is converted back in terms of $\delta\epsilon_{e,t}$ and $\delta\epsilon_\pm$ through Eqs. (18). Because of the electron-hole symmetry, $\delta\epsilon_{e,t}$ disappears and we are left with

$$\eta\Delta_k = \sum_{k'} V_{kk'}\left(\frac{\Delta\delta\nu_e - \delta\nu_t\Delta}{2E} + \varphi\frac{\xi^2}{E^3}\eta\Delta\right)_{k'} . \quad (23)$$

Using the gap equation,

$$\Delta_k = \sum_{k'} V_{kk'}\frac{\varphi_{k'}}{E_{k'}}\Delta_{k'} , \quad (24)$$

and the explicit p-wave dependence of Δ_k, and $V_{kk'}$ on $\vec{k}$ and $\vec{k}'$, and then taking the trace in the spin space, we obtain

$$\eta\sum_k \frac{\varphi_k}{E_k^3}|\Delta_k|^4 = \sum_k \delta\nu^0_{e,k}\frac{|\Delta_k|^2}{E_k} , \quad (25)$$

which is the desired result. Calculating $\delta\nu^0_e$ by Eq. (17) and Eq. (21) and using electron-hole and angular symmetries, we can reduce Eq. (25) to

$$\eta\left(\sum_k \frac{\varphi_k}{E_k^3}|\Delta_k|^4 + \sum_k \varphi_k'\frac{|\Delta|^4}{E_k^2}\frac{(\vec{q}\cdot\vec{\nabla}_k E_k)^2}{\omega^2 - (\vec{q}\cdot\vec{\nabla}_k E_k)^2}\right) = 0 . \quad (26)$$

The expression in parentheses may be shown to be not equal to zero, first by verifying that it is always negative for $\omega = 0$ and then by verifying that it is a decreasing function of ω. We will not go into the details of demonstrating this, but merely take the conclusion that $\eta = 0$. This means that there is no order-parameter amplitude fluctuation associated with zero sound or spin waves in the collisionless regime. This also shows that there is no mode associated with order-parameter amplitude fluctuations in this regime.

Returning to Eqs. (17a) and (21), we see that there is no equation to determine $\vec{A}$ and V; we have more unknowns than equations. In principle, the self-consistency relation (13) should play the role of the missing equation, but, as we have seen, it provides only information about η. What has happened is that we have lost some information by performing our lowest-order expansion of the kinetic equation. A careful treatment of the higher-order terms together with the use of the gap equation (allowing cancellation of lowest-order terms) would show that it is possible to get an equation of the same order in ω and $\vec{q}$ as Eqs. (17). This circumstance must be related to the singularity which appears in the vertex part at low ω and $\vec{q}$.[20] This singularity is related to the Bogoliubov mode, which is of the same kind of mode that we are dealing with. As in our case, this singularity appears only through higher-order terms, the lowest-order term being cancelled by the use of the gap equation. This mode is known to be necessary to restore the Ward's identity, that is, to satisfy charge conservation. In a similar manner, we may expect that the missing higher-order terms in the kinetic equation expansion make some conservation law not satisfied. And indeed, the mass and spin conservation laws, which are included in the original kinetic equation, are no longer implied by Eqs. (17). So we must add these conservation laws to our set of equations, and in this way we get a closed set of equations. These new equations restore the information which has been lost in our expansion of the kinetic equation.

The mass and spin conservation laws are

$$\frac{\partial\rho^0}{\partial t}+\frac{\partial}{\partial x_i}\,j_i^0=0,\qquad \frac{\partial\vec{\rho}}{\partial t}+\frac{\partial}{\partial x_i}\,\vec{j}_i=0, \tag{27}$$

where ρ^0 and j_i^0 are the usual density and current density,

$$\rho^0(\vec{r})=\psi_\alpha^\dagger(\vec{r})\psi_\alpha(\vec{r}),$$
$$j_i^0(\vec{r})=\frac{1}{2mi}\left[\psi_\alpha^\dagger(\vec{r})\left(\frac{\partial}{\partial x_i}\,\psi_\alpha(\vec{r})\right)-\left(\frac{\partial}{\partial x_i}\,\psi_\alpha^\dagger(\vec{r})\right)\psi_\alpha(\vec{r})\right], \tag{28}$$

and $\vec{\rho}$ and $\vec{j}_i$ are the spin density and spin-current density,

$$\vec{\rho}(r)=\psi_\alpha^\dagger(\vec{r})\vec{\sigma}_{\alpha\beta}\psi_\beta(\vec{r}),$$
$$\vec{j}_i(\vec{r})=\frac{1}{2mi}\left[\psi_\alpha^\dagger(\vec{r})\vec{\sigma}_{\alpha\beta}\left(\frac{\partial}{\partial x_i}\,\psi_\beta(\vec{r})\right)-\left(\frac{\partial}{\partial x_i}\,\psi_\alpha^\dagger(\vec{r})\right)\vec{\sigma}_{\alpha\beta}\psi_\beta(\vec{r})\right]. \tag{29}$$

The arrows above the symbols ρ and j_i indicate spin vectors. Space vectors are indicated by the index i. The spin conservation law means that the spin density $\vec{\rho}(\vec{r})$ commutes with the BCS interaction. This commutation is exact if this interaction is a density-density or a spin-density–spin-density interaction. If this is not the case, the mass and spin conservation laws are only approximate; but compared to the other terms the additional term is of order of the BCS energy over the kinetic energy and may be safely neglected.

Performing the canonical transformation and taking the Fourier transform and the statistical average, we obtain the following for the $\vec{q}$ component of the current and of the equilibrium departure of the densities,

$$\delta\rho^0=\sum_k \delta n_{e,k}^0\quad \delta\vec{\rho}=\sum_k \delta\vec{n}_{e,k},$$
$$j_i^0=\sum_k\frac{k_i}{m}\,\delta n_{e,k}^0+\rho A_i^0\quad \vec{j}_i=\sum_k\frac{k_i}{m}\,\delta\vec{n}_{e,k}+\rho\vec{A}_i\,. \tag{30}$$

The summations over k include a factor 2, which means that the corresponding density of states is for up and down spins. ρ is the density of ^{3}He. We now express these quantities in terms of $\delta\nu_{e,k}$. Using Eq. (18), we obtain δn_e in terms of $\delta\nu_{e,t}$ and $\delta\nu_\pm$. Through the relation $\delta\nu_{t,k}=-{}^{tr}\delta\nu_{e,-k}$, we eliminate $\delta\nu_t$, and expressing $\delta\nu_\pm$ in terms of $\delta E_\pm$ by Eq. (17c) and $\delta E_\pm$ in terms of $\delta\epsilon_{e,t}$ and $\delta\epsilon_\pm$ by Eq. (18), we obtain

$$\delta\rho^0=\sum_k\frac{\xi_k}{E_k}\,\delta\nu_{e,k}^0+\sum_k\varphi_k\frac{|\Delta_k|^2}{E_k^3}\,V^0,$$
$$j_i^0=\sum_k\frac{k_i}{m}\,\delta\nu_{e,k}^0+\rho A_i^0,$$
$$\delta\vec{\rho}=\sum_k\frac{\xi_k}{E_k}\,\delta\vec{\nu}_{e,k}+\sum_k\left(1-\frac{\xi_k}{E_k}\right)\frac{\vec{d}_k(\vec{d}_k^*\cdot\delta\vec{\nu}_{e,k})}{|\Delta_k|^2}+\sum_k\frac{\varphi_k}{E_k^3}\left[|\Delta_k|^2\vec{V}-\vec{d}_k(\vec{d}_k^*\cdot\vec{V})\right], \tag{31}$$
$$\vec{j}_i=\sum_k\frac{k_i}{m}\,\delta\vec{\nu}_{e,k}-\sum_k\frac{k_i}{m}\left(1-\frac{\xi_k}{E_k}\right)\frac{\vec{d}_k(\vec{d}_k^*\cdot\delta\vec{\nu}_{e,k})}{|\Delta_k|^2}+\sum_k\frac{k_ik_j}{m}\frac{\varphi_k}{E_k^3}\,\vec{d}_k(\vec{d}_k^*\vec{A}_j)+\rho\vec{A}_i,$$

where the $\delta\epsilon_\pm$ terms have disappeared by electron-hole symmetry. Together with the conservation laws which can be rewritten

$$\omega\delta\rho^0=q_ij_i^0,\quad \omega\delta\vec{\rho}=q_i\vec{j}_i, \tag{32}$$

and with the relations between $\vec{A}$ and V which result from the definition (4), we have

$$m\omega A_i^0+q_i\,V^0=0,\quad m\omega\vec{A}_i+q_i\vec{V}=0. \tag{33}$$

Equations (17a), (21), and (31) are a closed set of equations describing the coupled motion of the collective modes and the quasiparticle distribution.

We observe that the equations for the density part and the spin part of the quasiparticle distribution are decoupled. The density part is coupled to the phase fluctuation of the order parameter which gives rise to the zero-sound mode. The spin part is coupled to the rotations of the order parameter in spin space, giving rise to spin waves. The equations obtained for the first case are identical to the ones obtained in the s-wave case, except that the gap may now be anisotropic. This confirms that Wölfle's approach,[19] treating zero sound in ^{3}He as zero sound in an anisotropic s-wave superfluid is correct, as could be expected. Our A_i^0 must be identified with the usual superfluid velocity, and V^0 with the superfluid chemical potential change (with a change in the sign). In the same way, $\vec{A}_i$ could be called a "spin superfluid velocity" and $\vec{V}$ a "spin superfluid chemical potential."

In fact the similarity between the density equations and the spin equations is more striking if we specialize to states where the direction of $\vec{d}_k$ in spin space is independent of $\vec{k}$. This is the case for the axial state, which is presently believed to be the A phase, and for the polar state. In these cases, we get three sets of decoupled equations. Two of them, corresponding to spin directions perpendicular to $\vec{d}$, are completely identical to sets describing the coupling between density fluctuations and zero sound. In this way, the above designations for $\vec{A}_i$ and $\vec{V}$ become quite natural. The third set of equations may be written (taking the spin component parallel to $\vec{d}$),

$$(\omega-\vec{q}\cdot\vec{\nabla}_k E_k)\delta\nu_{e,k}=-\vec{q}\cdot\vec{\nabla}_k\varphi_k'\delta E_{e,k},$$

$$\delta E_e=(\xi/E)k_iA_i+V,\quad \delta\rho=\sum_k\delta\nu_{e,k},\tag{34}$$

$$j_i=\sum_k\frac{k_i}{m}\frac{\xi}{E}\delta\nu_{e,k}+\sum_k\frac{k_ik_j}{m}\frac{\varphi}{E^3}|\Delta|^2A_j+\rho A_i,$$

$$\omega\delta\rho=q_ij_i,\quad m\omega A_i+q_iV=0.$$

Summing the kinetic equation over k and comparing it to the formulas for $\delta\rho$ and j_i, we find

$$\omega\delta\rho-q_ij_i=-q_i\left[\sum_k\left(\frac{\xi^2}{E^2}\varphi'+\frac{|\Delta|^2}{E^3}\varphi\right)\times\frac{k_ik_j}{m}A_j+\rho A_i\right].\tag{35}$$

But, integrating by parts, one finds that the right-hand side of the above is zero so that the conservation law is identically satisfied. This means that A_i is indeterminate. However, this is quite natural, since in this polarization, our canonical transformation corresponds to a spin rotation around the direction of $\vec{d}$ which has no physical effect on $\vec{d}$. In this polarization, spin-density fluctuations are not coupled to order-parameter fluctuations.

In concluding Sec. II, we examine the equations for spin waves. This is done by introducing the kinetic equation into the formulas for $\delta\vec{\rho}$ and $\vec{j}_i$, writing the spin conservation law, and eliminating $\vec{A}_i$ by Eq. (33). After some algebraic manipulations and integration by parts, we obtain

$$\sum_k\left(\frac{\varphi_k}{E_k}+\varphi_k'\frac{\vec{q}\cdot\vec{\nabla}_kE_k}{\omega-\vec{q}\cdot\vec{\nabla}_kE_k}\right)\frac{\omega^2-(\vec{q}\cdot\vec{v}_k)^2}{E_k^2}\vec{d}_k\times(\vec{d}_k^*\times\vec{V})=0.\tag{36}$$

If the direction of $\vec{d}_k$ is independent of $\vec{k}$, as for the axial or the polar state, we see immediately that only modes with spin polarization perpendicular to $\vec{d}$ exists and that these modes are degenerate. Another interesting case is the isotropic state where $\vec{d}_{\vec{k}}$ is found by applying some rotation R to $\vec{k}$,

$$\vec{d}_{\vec{k}}=R(\vec{k}).$$

This state is a possible candidate for the B phase in ^{3}He. We see that there are three spin-waves modes; one has a spin polarization parallel to $R(\vec{q})$, the other two have polarization perpendicular to this direction and are degenerate. This result for the spin polarization is in fact straightforward for symmetry reasons. We remark that looking experimentally at the polarization of the spin waves in a possible isotropic state would allow one to know exactly what is the rotation R, which is not a physical quantity very easy to measure otherwise.

We will not study further the velocity of these modes since Fermi-liquid effects are known to be very important in ^{3}He and must be included in our calculations.

III. INCLUSION OF FERMI-LIQUID EFFECTS

The inclusion of Fermi-liquid effects will follow the work of Betbeder and Nozières.[18] The essence of their method is to build renormalized operators in the presence of the Fermi-liquid interaction. Then the weak BCS interaction is taken into account; owing to its weakness, it will not perturb the Fermi-liquid renormalization. One is led to a simple superposition of Fermi-liquid effects and pairing effects. Consequently, Sec. II must be reformulated in terms of renormalized quantities. In addition, we have to include in the $\delta\epsilon_k$ matrix a Landau term corresponding to the modification of the energy matrix because of the change in the quasiparticle distribution. We have

$$\delta\epsilon_k=\begin{pmatrix}M_{k,\beta\alpha}+\delta\epsilon_{k,F} & \delta\Delta^\dagger_{k,\alpha\beta}\\ \delta\Delta_{k,\alpha\beta} & -M_{-k,\alpha\beta}-{}^{\text{tr}}\delta\epsilon_{-k,F}\end{pmatrix},\tag{37}$$

where

$$\delta\epsilon_{k,F}=\sum_{k'}f^s_{k,k'}\delta n^0_{e,k'}+\sum_{k'}f^a_{k,k'}\,{}^{\text{tr}}\vec{\sigma}\cdot\delta\vec{n}_{e,k'},\tag{38}$$

and f_s and f_a are, respectively, the symmetric and antisymmetric Landau kernel. However, the mass in Eq. (33) must be modified. This is most easily seen from the kinetic equation (11) in the normal state, in the presence of Fermi-liquid effects, but without performing our canonical transformation. This is the Landau kinetic equation (see Nozières[21] for example),

$$\omega\delta n_{e,k}=\vec{q}\cdot\vec{v}_k\left(\delta n_{e,k}+\delta(\xi_k)\sum_{k'}f_{kk'}\,\delta n_{e,k'}\right). \tag{39}$$

We now perform the canonical transformation. First we see that $\delta n_{e,k}$ becomes

$$\delta n_{e,k}+i^{\text{tr}}\sigma_\lambda\theta_\lambda(n^0_{k-q/2}-n^0_{k+q/2})\simeq\delta n_{e,k}+i^{\text{tr}}\sigma_\lambda\theta_\lambda\,\vec{q}\cdot\vec{v}_k\,\delta(\xi_k), \tag{40}$$

and the kinetic equation becomes

$$\omega\delta n_{e,k}=\vec{q}\cdot\vec{v}_k\delta n_{e,k}+\vec{q}\cdot\vec{v}_k\delta(\xi_k)\times\left({}^{\text{tr}}V+k_i\,{}^{\text{tr}}A_i+\sum_{k'}f_{kk'}\,\delta n_{e,k'}\right), \tag{41}$$

where

$$V=-i\,\omega\theta_\lambda\sigma_\lambda,\quad A^0_i=iq_i\theta_0(1/m^*)(1+\tfrac{1}{3}F^s_1),\quad \vec{A}_i=iq_i\vec{\theta}(1/m^*)(1+\tfrac{1}{3}F^a_1), \tag{42}$$

and

$$F^s_1=N_0f^s_1,\quad F^a_1=N_0f^a_1,$$

with f^s_1 and f^a_1 the p-wave coefficients in the Legendre expansion of $f^s_{k,k'}$ and $f^a_{k,k'}$, respectively.

We see that the kinetic equation (11) retains its form. $\delta\epsilon_k$ is still defined by Eq. (6) and Eq. (37), but Eq. (33) must be replaced by

$$m\omega A^0_i+q_i\,V^0=0,\quad m_s\omega\vec{A}_i+q_i\vec{V}=0, \tag{43}$$

where

$$1/m_s=(1/m^*)(1+\tfrac{1}{3}F^a_1), \tag{44a}$$

where we have used the Landau relation between the bare and the renormalized mass,

$$1/m=(1/m^*)(1+\tfrac{1}{3}F^s_1). \tag{44b}$$

In other words, in the effective Hamiltonian corresponding to Eq. (39), the interaction is not a density-density one, if we retain nonzero momentum coefficients in the Legendre expansion of the Landau kernel. The canonical transformation then produces additional terms coming from this interaction, which can never be neglected. The result is the F^s_1 and F^a_1 terms in Eq. (44).

We must also modify the expression (30) for the currents. This again may be found from the kinetic equation (41). Summing over $\vec{k}$, we find

$$\omega\delta\rho^0=q_i\left(\sum\frac{k_i}{m}\delta n^0_{e,k}+\rho A^0_i\right),\qquad \omega\vec{\delta\rho}=q_i\left(\sum\frac{k_i}{m_s}\,\delta\vec{n}_{e,k}+\rho\vec{A}_i\right), \tag{45}$$

where we have made use of Eq. (44). Accordingly, Eq. (30) for the density current does not need to be modified. This is a well known result, which is related to the fact that the density current commutes with the bare interaction. On the other hand, the spin current does not commute with the bare interaction and, in the spin current in the Eq. (30) we must replace the bare mass m by the "spin mass" m_s.

Now all the results in the Sec. II, obtained without using the explicit form of $\delta\epsilon_k$, are conserved [since ${}^{\text{tr}}\delta\epsilon_{e,-k}=-\delta\epsilon_{t,k}$ is also conserved by Eq. (37)]. We have only to modify Eqs. (21) and (31). For simplicity, we will only keep the s-wave and the p-wave part of the Landau kernel, the higher-order angular momentum likely being negligible in ^{3}He. With this simplification, Eq. (38) reads

$$\delta\epsilon_{k,F}=f^s_0\delta\rho^0+f^s_1(m/k^2_F)k_i(j^0_i-\rho A^0_i)+{}^{\text{tr}}\vec{\sigma}[f^a_0\vec{\delta\rho}+f^a_1(m_s/k^2_F)k_i(\vec{j}_i-\rho\vec{A}_i)], \tag{46}$$

where f^s_0 and f^a_0 are the s-wave parts of the Landau kernel. Using Eq. (18), we get a new formula for δE_k,

$$\delta E^0_e=(\xi/E)(V^0+f^s_0\delta\rho^0)+(m/\rho m^*)k_i(\rho A^0_i+\tfrac{1}{3}F^s_1j^0_i),$$

$$\delta\vec{E}_e=(\xi/E)(\vec{V}+f^a_0\vec{\delta\rho})+(m_s/\rho m^*)k_i(\rho\vec{A}_i+\tfrac{1}{3}F^a_1\vec{j}_i)+[1-(\xi/E)](\vec{d}^*/|\Delta|^2)(\vec{d}\{(\vec{V}+f^a_0\vec{\delta\rho})-(m_s/\rho m^*)k_i(\rho\vec{A}_i+\tfrac{1}{3}F^a_1\vec{j}_i)\}), \tag{47}$$

where we have used

$$f^s_1(m/k^2_F)\rho=(m/m^*)\tfrac{1}{3}F^s_1,\quad f^a_1(m_s/k^2_F)\rho=(m_s/m^*)\tfrac{1}{3}F^a_1.$$

In order to find the new expressions for the currents and the density fluctuations, we follow the derivation of Eq. (31), but we have to use now the corrected value for $\delta\epsilon_e$, given by Eq. (37) and Eq. (46). Using, as we have done before, symmetries with respect to $\vec{k}$ and ξ, we obtain (we omit the k index for clarity),

$$
\begin{aligned}
\delta\rho^0 &= \sum \frac{\xi}{E}\delta\nu_e^0 + \sum \varphi \frac{|\Delta|^2}{E^3} V^0 + f_0^s \sum \frac{\varphi}{E^3} |\Delta|^2 \delta\rho^0 ,\\
j_i^0 &= \sum \frac{k_i}{m}\delta\nu_e^0 + \rho A_i^0 ,\\
\delta\vec{\rho} &= \sum\left[\frac{\xi}{E}\delta\vec{\nu}_e + \left(1-\frac{\xi}{E}\right)\frac{\vec{d}(\vec{d}^*\cdot\delta\vec{\nu}_e)}{|\Delta|^2}\right]\\
&\quad + \sum \frac{\varphi}{E^3}\vec{d}^*\times[(\vec{V}+f_0^a\delta\vec{\rho})\times\vec{d}\,] ,\\
\vec{j}_i &= \sum \frac{k_i}{m_s}\left[\delta\vec{\nu}_e - \left(1-\frac{\xi}{E}\right)\frac{\vec{d}(\vec{d}^*\cdot\delta\vec{\nu}_e)}{|\Delta|^2}\right]\\
&\quad + \sum \frac{k_i k_j}{\rho m^*}\frac{\varphi}{E^3}\vec{d}[\vec{d}^*(\rho\vec{A}_j + \tfrac{1}{3}F_1^a\vec{j}_j)] + \rho\vec{A}_i .
\end{aligned}
\tag{48}
$$

The first of these equations is easily solved giving

$$
\delta\rho^0 = \left(\sum \frac{\xi}{E}\delta\nu_e^0 - V^0 N_0 \chi\right)\Big/(1+F_0^s\chi)
$$

with

$$
\chi = -\frac{1}{N_0}\sum \frac{\varphi}{E^3}|\Delta|^2 . \tag{49}
$$

The equations obtained for the density part are identical to the results of Betbeder and Nozières[18] for s-wave pairing, except for a possible anisotropy of the gap. These are the equations used by Wölfle[19] to derive the zero-sound velocity in an anisotropic (s-wave) superfluid. We will not go further into the study of the density part. We refer the reader to Wölfle's paper for the derivation of the zero-sound velocity.

Equations (48) for the spin-density part need an explicit knowledge of $\vec{d}_k$ to be solved explicitly in $\delta\vec{\rho}$ and $\vec{j}_i$. Therefore, we will specialize to the two cases of most physical importance: the axial state (the polar state could be treated exactly along the same line) and the isotropic state.

In the axial state, $\vec{d}_k$ has a fixed direction, independent of $\vec{k}$. For spin polarization perpendicular to $\vec{d}$, we find again that our equations reduce to a form which is formally identical to the one obtained for the density part, except that the symmetric Landau coefficients must be replaced by the antisymmetric ones and the bare mass replaced by m_s in the expressions for $\delta\vec{E}_e$ and $\vec{j}_i$.

For spin polarization parallel to $\vec{d}$, we can write corresponding equations and verify, following exactly the same steps as going from Eq. (34) to Eq. (35), that the spin conservation law is identically satisfied; the physical meaning of this is exactly the same as before so that we can choose $\vec{V}=0$ and $\vec{A}_i=0$ to study this polarization. The resulting equations are very simple,

$$
\begin{aligned}
&\omega\delta\nu_e - (q_i k_i/m^*)(\xi/E)(\delta\nu_e - \varphi'\,\delta E_e) = 0 ,\\
&\delta E_e = f_0^a\delta\rho + (\xi/E)(m_s/\rho m^*)\tfrac{1}{3}F_1^a j_i k_i ,\\
&\delta\rho = \sum_k \delta\nu_e ,\\
&j_i = \sum \frac{\xi}{E}\frac{k_i}{m_s}\delta\nu_e + \frac{F_1^a}{3\rho m^*}\sum \varphi\frac{|\Delta|^2}{E^3}k_i k_j j_j .
\end{aligned}
\tag{50}
$$

Expressing the current into components parallel and perpendicular to the gap axis (the direction along which the gap is zero in the axial state), we can easily solve Eq. (50) for j_i and hence, we could write an equation for the velocity of this spin wave. However, this equation is not simple. Moreover, the Landau coefficient F_1^a is likely to be very small (F_0^a itself is only of order of unity). Therefore, we will neglect F_1^a in what follows.

We now can write

$$
\begin{aligned}
\delta\rho = \sum_k \delta\nu_e &= \sum(-\varphi')\delta E_e \frac{(q_i k_i/m^*)(\xi/E)}{[\omega - (q_i k_i/m^*)(\xi/E)]}\\
&= f_0^a\delta\rho \sum \frac{(-\varphi')(q_i k_i/m^*)(\xi/E)}{[\omega-(q_i k_i/m^*)(\xi/E)]} ,
\end{aligned}
\tag{51}
$$

and we are left with the following equation for s, which is the ratio of the spin-wave velocity divided by the Fermi velocity,

$$
\frac{1}{F_0^a} = \int \frac{d\Omega_k}{4\pi} d\xi(-\varphi')\frac{[\hat{k}\cdot\hat{q}(\xi/E)]^2}{s^2 - [\hat{k}\cdot\hat{q}(\xi/E)]^2} . \tag{52}
$$

We have strong arguments indicating that this equation has no solution. Indeed, in ^{3}He, F_0^a is known to be negative with an absolute value less than one. For $s=0$, the right-hand side is

$$
\int \frac{d\Omega_k}{4\pi} d\xi\,\varphi' ,
$$

which is negative and has an absolute value that may be shown to be less than one for all temperatures (for $T=T_c$, this absolute value is exactly one). For $T=0$, the right-hand side is zero. For $T=T_c$, Eq. (52) is known to have no solution. Finally, for $s>1$, the right-hand side is positive. Therefore, in the s-T plane only solutions inside the rectangle $s<1$, $T<T_c$ can exist, but there is no solution on the perimeter of this rectangle. So we are fairly confident that Eq. (52) has no solution for any direction of $\vec{q}$ with respect to the gap axis. This result has been checked numerically for $\vec{q}$ parallel to the gap axis.

We now examine the spin polarization perpendicular to $\vec{d}$. We still neglect F_1^a. Equations (48) become

$$
\begin{aligned}
(1+F_0^a\chi)\delta\vec{\rho} &= \sum_k \frac{\xi}{E}\delta\vec{\nu}_e - \vec{V}N_0\chi ,\\
\vec{j}_i &= \rho\vec{A}_i + \sum \frac{k_i}{m^*}\delta\vec{\nu}_e ,
\end{aligned}
\tag{53}
$$

since $m_s=m^*$ if $F_1^a=0$. We also have, from Eqs. (47) and (17),

$$
\delta\vec{\nu}_e = (-\varphi')\frac{q_i v_{ki}(\xi/E)}{\omega - q_i v_{ki}(\xi/E)}[k_i\vec{A}_i + (\xi/E)(\vec{V}+f_0^a\delta\vec{\rho})] . \tag{54}
$$

Together with Eq. (43) and Eq. (32), Eq. (53) become

$$(1+F_0^a\chi)\delta\vec{\rho} = F_0^a I_1\delta\vec{\rho} + N_0\vec{V}(I_1 - I_2/s - \chi),$$
$$s\delta\vec{\rho} = F_0^a I_2\delta\vec{\rho} + N_0\vec{V}(I_2 - I_3/s - \tfrac{1}{3}s), \quad (55)$$

with the definitions

$$N_0 I_1 = \sum_k (-\varphi')\frac{\hat{k}\cdot\hat{q}(\xi/E)^3}{s-\hat{k}\hat{q}(\xi/E)},$$
$$N_0 I_2 = \sum_k (-\varphi')\frac{(\hat{k}\cdot\hat{q})^2(\xi/E)^2}{s-\hat{k}\cdot\hat{q}(\xi/E)}, \quad (56)$$
$$N_0 I_3 = \sum_k (-\varphi')\frac{(\hat{k}\cdot\hat{q})^3(\xi/E)}{s-\hat{k}\cdot\hat{q}(\xi/E)}.$$

From Eq. (55) we get the following equation for the velocity of the spin waves,

$$s^2(I_1-\chi) - 2I_2 s + (I_3+\tfrac{1}{3}) - F_0^a[(I_3+\tfrac{1}{3})(I_2-\chi)-(I_2)^2] = 0. \quad (57)$$

[In this equation and in similar ones, we will neglect the imaginary part of I_1, I_2, and I_3, and take s to be real. This is justified near $T=0$ because, owing to the $(-\varphi')$ factor, these imaginary parts fall off very rapidly with the temperature. However this does not mean that our approximation is only correct for $T\to 0$, because it may be seen that the correction to Eq. (57) coming from the imaginary parts of I_1, I_2, I_3 are of second order in these quantities, while the imaginary part of s (that is the spin-wave attenuation) is of first order. Therefore, Eq. (57) remains correct even with noticeable spin-wave attenuation. Since, as seen from Fig. 1, the $T\simeq 0$ regime extends rather far, Eq. (57) should be valid in most of the temperature range (say $T_c - T \gtrsim 0.1T_c$) but it clearly fails near T_c. In any case, when the imaginary parts are taken into account, Eq. (57) can be used to study spin-wave attenuation. Note, also, that our proof that Eq. (57) has a root can be extended when imaginary parts are taken into full account.] This equation is identical to that obtained for zero-sound velocity (assuming $F_1^s=0$), except that we have now F_0^a instead of F_0^s. The zero-sound equation is easy to study since F_0^s is large, which makes possible an expansion in $1/s$. Here, we know that F_0^a is of order of unity.

For $T=0$, we have $I_1=I_2=I_3=0$ and $\chi=1$ and we obtain,

$$s^2 = \tfrac{1}{3}(1+F_0^a), \quad (58)$$

which agrees with Maki and Ebisawa's results.[17] [This result becomes $s^2=\frac{1}{3}(1+F_0^a)(1+\frac{1}{3}F_1^a)$ if F_1^a is nonzero.]

For $s=0$, we have $I_2=0$ and $\chi - I_1 = 1$, so that Eq. (57) reads,

$$(1+F_0^a)\left(\tfrac{1}{3}+\int d\Omega_k\, d\xi(-\varphi')(\hat{k}\cdot\hat{q})^2\right)=0. \quad (59)$$

The first factor on the left-hand side is positive and the second one may easily be shown to be strictly positive, except at $T=T_c$, where it is zero. So that $s=0$ is the solution of Eq. (57) only at $T=T_c$. Now we can show that Eq. (57) always has a solution for $T\leq T_c$. We first note that the left-hand side of Eq. (57) is positive for $s=0$, as we have just seen. On the other hand, as $s\to\infty$, I_1, I_2, $I_3\to 0$ so that the left-hand side of Eq. (57) is or order of $-\chi s^2$ which is negative. Since I_1, I_2, and I_3 have no singularities and are clearly continuous functions of s, there is necessarily a solution. We have studied this solution numerically for $\vec{q}$ parallel to the gap axis, with $F_0^a=-0.8$. The result is shown on Fig. 1.

We now investigate the isotropic state. Again we will neglect F_1^a for physical reasons and for simplicity, although there is no problem taking F_1^a into account. As in Sec. I, we will find three spin-wave modes: one with a spin polarization parallel to $R(\vec{q})$, the other two having a spin polarization perpendicular to $R(\vec{q})$. For symmetry reasons, j_i and A_i are either parallel or perpendicular to $\vec{q}$. We first look at the case where they are parallel to $\vec{q}$.

We put Eq. (47) into Eq. (48) and we use Eqs. (43) and (32). We know the directions of the various vectors in spin space as well as in real space. Using the fact that the gap is isotropic in performing some of the angular averages, we obtain, for the spin polarization parallel to $R(\vec{q})$,

$$\delta\rho[1+F_0^a(\tfrac{2}{3}\chi+I_1'-I_1-I_3)] = (N_0V/s)[(I_1-I_1'-\tfrac{2}{3}\chi+I_3)s-I_2], \quad (60)$$
$$\delta\rho(s-F_0^aI_2) = (N_0V/s)(sI_2+\tfrac{1}{5}\chi-I_1''+I_3'-I_3-\tfrac{1}{3}),$$

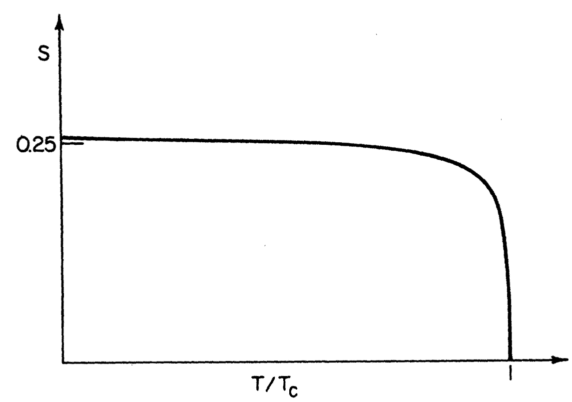

FIG. 1. Spin-wave velocity divided by the Fermi velocity vs the reduced temperature in the axial state for a direction of propagation parallel to the gap axis.

where I_1, I_2, and I_3 have been defined previously and

$$N_0 I_1' = \sum_k (-\varphi') \frac{(\hat{k}\cdot\hat{q})^3(\xi/E)^3}{s-\hat{k}\hat{q}(\xi/E)},$$

$$N_0 I_1'' = \sum_k (-\varphi') \frac{(\hat{k}\cdot\hat{q})^5(\xi/E)^3}{s-\hat{k}\hat{q}(\xi/E)}, \tag{61}$$

$$N_0 I_3' = \sum_k (-\varphi') \frac{(\hat{k}\cdot\hat{q})^5(\xi/E)}{s-\hat{k}\cdot\hat{q}(\xi/E)}.$$

The equation for s is obtained by requiring that the determinant of the system (60) is zero.

For $T=0$, each I is zero and $\chi=1$, so that

$$s^2 = \tfrac{1}{5}(1+\tfrac{2}{3}F_0^a). \tag{62}$$

For $s=0$, we have

$$I_2 = 0, \quad I_1 = \chi - 1, \quad I_1' = \tfrac{1}{3}I_1,$$
$$I_1'' = \tfrac{1}{5}I_1, \quad I_3 = -\tfrac{1}{3}\phi, \quad I_3' = -\tfrac{1}{5}\phi, \tag{63}$$

where $N_0\phi = -\sum_k(-\varphi')$, and the determinant is simply

$$-\tfrac{2}{15}(1-\phi)[1+\tfrac{1}{3}F_0^a(\phi+2)].$$

Since clearly $\phi<1$ except at $T=T_c$, where $\phi=1$, the determinant is negative for $s=0$ except at $T=T_c$, where $s=0$ is a solution. Now, for $s\to\infty$, the determinant is of order of $\frac{2}{3}\chi s^2$, which is positive. As in Sec. III, we conclude that spin waves, with the present polarization, exist in the isotropic state; the shape of s as a function of T is very likely similar to the one found in Sec. III.

In a similar manner, we obtain for the spin polarization perpendicular to $R(\vec{q})$,

$$\delta\rho[1+F_0^a(\tfrac{2}{3}\chi + J_1' - I_1 - J_3)] = (N_0V/s)[(I_1 - J_1' - \tfrac{2}{3}\chi + J_3)s - I_2],$$
$$\delta\rho(s - F_0^a I_2) = (N_0 V/s)[sI_2 + \tfrac{1}{15}\chi - \tfrac{1}{3} + J_3' - J_1'' - I_3], \tag{64}$$

where

$$N_0 J_1' = \sum_k (-\varphi') \frac{\hat{k}_x^2\hat{k}_z(\xi/E)^3}{s-\hat{k}_z\xi/E},$$

$$N_0 J_1'' = \sum_k (-\varphi') \frac{\hat{k}_x^2\hat{k}_z^3(\xi/E)^3}{s-\hat{k}_z\xi/E},$$

$$N_0 J_3 = \sum_k (-\varphi') \frac{\hat{k}_x^2\hat{k}_z(\xi/E)}{s-\hat{k}_z\xi/E}, \tag{65}$$

$$N_0 J_3' = \sum (-\varphi') \frac{\hat{k}_x^2\hat{k}_z^3(\xi/E)}{s-\hat{k}_z\xi/E},$$

and $\vec{q}$ has been chosen along the z axis.

As before, we obtain at $T=0$,

$$s^2 = \tfrac{2}{5}(1+\tfrac{2}{3}F_0^a). \tag{66}$$

For $s=0$, we have, in addition to Eq. (63), $J_3 = -\frac{1}{3}\phi$, $J_3' = -\frac{1}{15}\phi$, $J_1' = \frac{1}{3}I_1$, and $J_1'' = \frac{1}{15}I_1$. The determinant,

$$-\tfrac{4}{15}(1-\phi)[1+F_0^a(\tfrac{2}{3}+\tfrac{1}{3}\phi)],$$

is negative, except that for $T=T_c$, $s=0$ is the solution. Since, for $s\to\infty$, the determinant is of order of $\frac{2}{3}\chi s^2$ and positive, we have also spin waves with spin polarization perpendicular to $R(\vec{q})$ in the isotropic state at any temperature.

Finally, we look at the space polarization perpendicular to $\vec{q}$. Since, from Eq. (43), A_i is parallel to q_i, we must have $A_i=0$, which implies $V=0$. So the possible modes are decoupled from the fluctuations in the order parameter. Moreover, Eq. (32) gives $\delta\vec{\rho}=0$, and as a result $\vec{j}_i$ is zero from Eqs. (47) and (48). Thus, we find that there is no mode with transverse space polarization. In fact this result is clearly related to our approximation $F_1^a=0$. Taking a nonzero F_1^a into account may give rise to new modes. However, because F_1^a is small, they are unlikely to actually appear. We also note that a nonzero F_1^a would couple the space polarizations parallel and perpendicular to $\vec{q}$, together with a mixing of the spin polarizations parallel and perpendicular to $R(\vec{q})$.

IV. CONCLUSION

We have developed a formalism to study spin waves in superfluid ^{3}He, in the collisionless regime. In this formalism, superfluid parameters are introduced explicitly, and we obtain a set of equations describing the coupled motion of the superfluid and the quasiparticles distribution. Using these equations, we have studied spin waves in the axial and the isotropic state. In the axial state, only two degenerate modes with spin polarization perpendicular to $\vec{d}_k$ can exist. In the isotropic state, with respect to spin polarization parallel to $R(\vec{q})$, we obtain one longitudinal mode and two degenerate transverse modes. We have written the equations for the velocity of these modes. In general, they must be studied numerically. Our formalism may easily be generalized to include magnetic field effects and the dipole-dipole interaction and to study the hydrodynamic regime. These generalizations are being developed and will be presented elsewhere.

ACKNOWLEDGMENTS

I am very grateful to V. Ambegaokar for introducing me to this problem and for useful advice. I am indebted to D. Rainer for many useful and instructive discussions and thank H. Ebisawa for the communication and discussion of his results prior to publication. I am also grateful to N. D. Mermin and to J. Serene for interesting conversations. Much benefit was derived from stimulating conversations on experimental problems in spin-wave studies with the members of the experimental group at Cornell University. J. Gubernatis is thanked for his kind help in the preparation of the manuscript.

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*Work supported in part by the National Science Foundation under Grant No. GH-36457.

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