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Comments on "Qualitative explanation of Pellam's helium paradox"*

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The paper of Penney, concerning the role of "heat-exchange" torques in the "λ-point anomaly" of Pellam, is examined. It is shown, using only the equations and assumptions of Penney's paper, that the heat-exchange torque reduces to simply a torque produced by the inertia of counterflow. His calculations of the sense and temperature dependence of the torque are shown to be in error. It is shown in the light of these results that little effect on the "λ-point anomaly" would be expected for an assumed approximately isotropic room-temperature radiation.

Penney[1] has developed a theory of "heat-exchange" torques on heated bodies in a flow of superfluid helium. Central to this theory is the existence of a *tangential* force (totally separate from any viscous forces) acting on the surfaces of such bodies, arising from the two-fluid nature of superfluid helium; this was first pointed out by Landau and Lifshitz.[2] The theory was later elaborated by Penney and Overhauser,[3] and experimentally checked by Hunt,[4] and Payne.[5]

Subsequently, Penney[6] conjectured that the "λ-point anomaly" of Pellam[7] might be due to such a heat-exchange torque, the heating of the disk being caused by absorbed room-temperature radiation falling on the disk. In his paper[6] Penney derived an expression for the heat-exchange torque on a Rayleigh disk; stating that evaluation of the expression would be "exceedingly difficult," he instead gave estimates of its sense and magnitude for $T \sim T_\lambda$ and $T \to 0$. He found, for the parameters of Pellam's experiment (assuming a rough estimate of the room-temperature radiation), a heat-exchange torque of approximately the correct order of magnitude to explain the "λ-point anomaly." However, Pellam[8] found no significant effect on the anomaly when the "bucket" containing the disk was shielded from exterior radiation.

In fact, for the case of an infinitely thin disk, using nothing more than simple vector algebra and the assumptions contained in Ref. 6, it is easy to show that Penney's expression reduces to a torque produced by the reaction of counterflow, easily calculable given the heat distribution over the disk. The tangential forces of the heat-exchange torque theory do not appear in the expression at all.[9] This enables one to show that the sense and temperature dependence of the torque were incorrectly calculated in Ref. 6.

To see this, consider the expression developed by Penney for the heat-exchange torque, $\vec{\tau}_{\text{heat}}$ (but written with a minus sign[10]):

$$\vec{\tau}_{\text{heat}} = -\int \vec{r} \times \frac{\rho_n \kappa (\vec{v}_n - \vec{v}_s)\, dS}{\rho s T + \rho_n \vec{v}_n \cdot (\vec{v}_n - \vec{v}_s)}, \tag{1}$$

where $\vec{r}$ is the vector from the axis (along a disk diameter) about which the torque is being computed, to the area element $d\vec{S}$. κ is the heat flow from the disk, which is taken to satisfy the equation

$$\begin{aligned}\kappa\, dS &= \vec{Q} \cdot d\vec{S} \\ &= \{(\mu + \tfrac{1}{2} v_s^2)\vec{J} + \rho s T \vec{v}_n + \rho_n \vec{v}_n [\vec{v}_n \cdot (\vec{v}_n - \vec{v}_s)]\} \cdot d\vec{S} \\ &= \{\rho s T + \rho_n [\vec{v}_n \cdot (\vec{v}_n - \vec{v}_s)]\} \vec{v}_n \cdot d\vec{S}, \end{aligned} \tag{2}$$

where $\vec{Q}$ is the energy-flux density; the second expression follows, since $\vec{J}$, the mass current density, satisfies the equation

$$\vec{J} \cdot d\vec{S} = (\rho_n \vec{v}_n + \rho_s \vec{v}_s) \cdot d\vec{S} = 0 \tag{3}$$

on the surface of the disk.

Using the definitions of $\vec{r}$ and $d\vec{S}$, Eq. (2) and Eq. (3) to rewrite the denominator in Eq. (1), one easily finds

$$\vec{\tau}_{\text{heat}} = -(\rho \rho_n / \rho_s) \int v_{n\perp}^2 (\vec{r} \times d\vec{S}), \tag{4}$$

where $v_{n\perp}$ is the component of the normal fluid flow perpendicular to the surface of the disk. This is the desired expression, written solely in terms of normal forces. Assuming, as did Penney, that the normal fluid flow is viscous, so that the component of $\vec{v}_n$ parallel to the disk vanishes on the disk, $v_{n\parallel} = 0$, Eq. (2) can be rewritten [using Eq. (3)] as

$$\kappa = v_{n\perp} [\rho s T + (\rho_n \rho / \rho_s) v_{n\perp}^2]. \tag{5}$$

This equation indicates that $v_{n\perp}$, hence $\vec{\tau}_{\text{heat}}$, is a function only of the heat distribution on the disk and the temperature, *independent of the external mass-flow conditions*. Thus, depending on the heat distribution over the disk, $\vec{\tau}_{\text{heat}}$ would add or subtract to the usual Bernoulli torque on a Rayleigh disk, rather than having a $\sin 2\theta$ dependence (where θ is the angle of attack with respect to the external mass flow) incorrectly found by Penney.[6]

The temperature dependence of the torque is easily found. If, in Eq. (5) one neglects the second term on the right,[11] then the temperature dependence of $\vec{\tau}_{\text{heat}}$ is found to be $(\rho_n/\rho_s)(1/\rho s^2 T^2)$. This goes as $1/\rho_s$ near T_λ, and as $1/T^4$ as $T \to 0$. This is what one would expect from the form of Eq. (4), which is that of a heat-flow force acting on the disk to produce a torque. Hall[12] has experimentally verified that the temperature dependence of the heat-flow force is that stated above. If one includes the $v_{n\perp}^3$ term in Eq. (5), the temperature dependence becomes $1/\rho_s^{1/3}$ near T_λ, $\rho_n^{1/3} \sim T^{4/3}$ as $T \to 0$. Both sets of temperature dependence are different from those incorrectly estimated by Penney, finite constant at T_λ, and going as $\rho_n \sim T^4$ as $T \to 0$.

It is interesting to note that for uniform heating, Eq. (5) indicates that $v_{n\perp}$ is a function of temperature alone, and constant over the disk; in this case the torque vanishes:

$$\vec{\tau}_{\text{heat}} = -\int \text{const} \times (\vec{r} \times d\vec{S}) = 0 \qquad (6)$$

(note that the constant may be different for each face of the disk). It would appear then, that aside from Pellam's[8] experimental null result on the role of room-temperature radiation in the "λ-point anomaly," one would expect little effect of such radiation in light of Eq. (6), since one would expect the radiation to be approximately isotropic, thus giving rise to approximately uniform heating.

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[9]Physically this comes about because although $\vec{\pi}$, the momentum flux-density tensor which Penney uses in his calculations, contains terms representing both tangential and normal forces, only the normal forces contribute to the torque, the tangential forces having no lever arm.
[10]The minus sign appears because Eq. (1) and subsequent derived expressions of Ref. 6 are incorrect unless $d\vec{S}$ is consistently taken normal *into* the disk; throughout this paper $d\vec{S}$ is normal *outward* from the disk.
[11]Such nonlinear terms in the theory of superfluid dynamics are necessary for a consistent theory, but it is controversial whether or not superfluidity is destroyed for large enough velocities to warrant such terms; see, for example, Ref. 2, p. 511; Ref. 12, p. 485; and I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965), pp. 59–61.
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