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Radiation from the Two-Stream Instability*

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Coupling between longitudinal and transverse waves is examined for an unbounded homogeneous cold plasma in which the several plasma constituents are in relative drift with respect to one another. For mutual relativistic streaming along the direction of a homogeneous dc magnetic field B, a dispersion relation is derived. For the special case of two equal counterstreaming electron beams, the extraordinary wave-dispersion law is compared with Lee's nonrelativistic result. For $B=0$, the dispersion relation is analyzed in some detail, especially as regards the polarization of the unstable eigenmode, and compared with results of Neufeld. We show, both analytically and numerically, that radiation effects are relatively insignificant, except for highly relativistic streaming velocity and for a beam-plasma frequency comparable to that of the background plasma.

I. INTRODUCTION

Virtually all elementary plasma physics textbooks discuss the linear *electrostatic* stability of cold uniform plasmas with mutual streaming between the several constituent particle species.[1] Less frequently discussed is the *electromagnetic* stability of such a system, but a sizeable literature is accumulating.[2–5] Until recently, interest in the electromagnetic properties centered on attempts to identify mechanisms responsible for strong emissions of extraterrestrial radio noise, such as type-III solar bursts[6] since, under conditions of sizeable direct radiation from a two-stream instability, one would need not necessarily invoke nonlinear effects[7] or coupling at plasma inhomogeneities[8] to explain the observations. More recently, production of intense relativistic electron streams has been reported,[9] so that it may well be that a starting point in understanding the emission mechanisms operating in these inhomogeneous, nonlinear plasmas could involve the present work.

The present paper presents a concise derivation of the linearized dispersion relation for plane-wave propagation in a uniform unbounded cold plasma permeated by a uniform dc magnetic field. The plasma is taken to support relativistic streams moving along the direction of the dc magnetic field, but the plasma carries no net current, to ensure uniformity of the magnetic field.[10] Similarities and differences between present and published results will be identified as we proceed. Numerical results for the case of zero magnetic field are presented.

II. DISPERSION RELATION

Description of the field and particle variations is provided by Maxwell's equations and by particle and momentum conservation laws. We seek a perturbation solution in which the density $N_\alpha + n_\alpha \exp[i(\vec{k}\cdot\vec{r} - \omega t)]$ and the velocity $\vec{V}_\alpha + \vec{v}_\alpha \exp[i \times (\vec{k}\cdot\vec{r} - \omega t)]$ are performing small harmonic variations about their mean values; i.e., $n_\alpha \ll N_\alpha$ and $v_\alpha \ll V_\alpha$. The subscript α labels individual plasma species in the system. The coupling between the electric field $\vec{E}(\vec{r},t)=\vec{E}\exp[i(\vec{k}\cdot\vec{r}-\omega t)]$ and the particle motions is provided by solving Maxwell's equations for the current (MKS units):

$$\vec{J}=(i\epsilon_0/\omega)[c^2\vec{k}\times(\vec{k}\times\vec{E})+\omega^2\vec{E}]$$
$$=\sum_\alpha q_\alpha(N_\alpha\vec{v}_\alpha+n_\alpha\vec{v}'_\alpha)$$
$$=\sum_\alpha q_\alpha N_\alpha\left(\vec{v}_\alpha+\frac{(\vec{V}_\alpha\vec{k})\cdot\vec{v}_\alpha}{\omega-\vec{k}\cdot\vec{V}_\alpha}\right) . \quad (1)$$

The last step involving elimination of n_α was made using the continuity equation. The particle velocities obey the linearized relativistic momentum conservation

$$M_\alpha\frac{\partial\vec{v}_\alpha}{\partial t}+\vec{V}_\alpha\cdot\nabla\vec{v}_\alpha+\vec{V}_\alpha\frac{dM_\alpha}{dt}=q_\alpha(\vec{E}+\vec{V}_\alpha\times\vec{B}+\vec{v}_\alpha\times\vec{B}_0) . \quad (2)$$

This may be simplified by writing

$$\vec{V}_\alpha\frac{d}{dt}(M_\alpha)=\frac{\vec{V}_\alpha}{c^2}\frac{d}{dt}(M_\alpha c^2)=\frac{\vec{V}_\alpha}{c^2}q\vec{V}_\alpha\cdot\vec{E} , \quad (3)$$

$$(\overleftrightarrow{B}_0)_{\mu\nu}=-\epsilon_{\mu\nu\kappa}(B_0)_\kappa , \quad (4)$$

eliminating the perturbed magnetic field with Maxwell's equations to give

$$\left(\overleftrightarrow{I}+\frac{iq/M_\alpha}{(\omega-\vec{k}\cdot\vec{V}_\alpha)}\overleftrightarrow{B}_0\right)\cdot\vec{v}_\alpha$$
$$=\frac{iq/M_\alpha}{(\omega-\vec{k}\cdot\vec{V}_\alpha)}\left[\overleftrightarrow{I}\left(1-\frac{\vec{k}\cdot\vec{V}_\alpha}{\omega}\right)+\frac{\vec{k}\vec{V}_\alpha}{\omega}-\frac{\vec{V}_\alpha\vec{V}_\alpha}{c^2}\right]\cdot\vec{E} . \quad (5)$$

Equation (5) may be solved for $\vec{v}_\alpha$ by multiplying both sides by the inverse of the matrix on the left-hand side. Substituting the result into Eq. (1) and manipulating to obtain a result in the form

$$\overleftrightarrow{D}(\omega,\vec{k})\cdot\vec{E}=0 \quad (6)$$

yields

$$\overleftrightarrow{D}(\omega,\vec{k})=(c^2k^2-\omega^2)\overleftrightarrow{I}-c^2\vec{k}\vec{k}+\sum_\alpha\frac{\omega_{p\alpha}^2}{(\omega-\vec{k}\cdot\vec{V}_\alpha)^2-\omega_{c\alpha}^2}\Bigg[[\overleftrightarrow{I}(\omega-\vec{k}\cdot\vec{V}_\alpha)^2+(\omega-\vec{k}\cdot\vec{V}_\alpha)(\vec{k}\vec{V}_\alpha+\vec{V}_\alpha\vec{k})$$
$$+\frac{\vec{V}_\alpha\vec{V}_\alpha}{V_\alpha^2}\left(k^2V_\alpha^2-\omega^2\beta_\alpha^2-\frac{\omega^2(1-\beta_\alpha^2)\omega_{c\alpha}^2}{(\omega-\vec{k}\cdot\vec{V}_\alpha)^2}\right)-i\omega\overleftrightarrow{\omega}_{c\alpha}+i(\vec{\omega}_{c\alpha}\cdot\vec{V}_\alpha)\overleftrightarrow{k}] ,\Bigg] \quad (7)$$

where the following definitions have been made:

$$\beta_\alpha^2=V_\alpha^2/c^2,\quad \omega_{p\alpha}^2=N_\alpha q_\alpha^2/\epsilon_0 M_\alpha;\quad M_\alpha=(M_\alpha)_{\text{rest}}/(1-\beta_\alpha^2)^{1/2}$$

$$(\omega_{c\alpha})_i=(q_\alpha/M_\alpha)(\vec{B}_0)_i,\quad (\overleftrightarrow{\omega}_{c\alpha})_{ij}=-\epsilon_{ij\ell}(\vec{\omega}_{c\alpha})_\ell,\ (\overleftrightarrow{k})_{ij}=-\epsilon_{ij\ell}(\vec{k})_\ell .$$

The quantity $\omega_{p\alpha}^2$ is constant since both N_α and M_α are divided by $(1-\beta_\alpha^2)^{1/2}$.

The nonrelativistic form of the dyadic $\overleftrightarrow{D}$ can be obtained by setting $\beta_\alpha=0$, but leaving c^2 and $\vec{V}_\alpha$ as they appear. One immediate conclusion is that a nonrelativistic dispersion relation may cause important error unless $\omega^2\ll k^2c^2$; i.e., unless only very slow waves are being examined, even if the actual streaming velocities are nonrelativistic.

One example of this is for the case of two equal electron streams with plasma frequencies $\frac{1}{2}\omega_p^2$ counterstreaming with velocities $\pm V$. Neutralization by infinitely massive ions is assumed. For this case Eq. (7) yields a dispersion relation for the extraordinary wave propagating at right angles to $\vec{B}$, which is not dependent upon the streaming. The corresponding ordinary wave-dispersion relation, however, is $(\vec{k}\cdot\vec{B}=0)$

$$\omega^2-k^2c^2-\omega_p^2(1-\beta^2)+k^2V^2\omega_p^2/(\omega_c^2-\omega^2)=0 , \quad (8)$$

which predicts instability ($\omega^2 < 0$) for any nonzero k. The result published by Kai Fong Lee[5] from a nonrelativistic theory gives the same result with β set equal to zero. Omission of the β^2 is particularly significant for $\omega_c = 0$ and $\beta^2 \sim 1$.

We now specialize the general result given in Eqs. (6) and (7) to the situation of a neutralized plasma beam streaming through a stationary neutralized plasma. For the moving stream we have $\omega_{p\alpha} = \omega_b$ and $V_\alpha = V$, and for the stationary plasma, $\omega_{p\alpha} = \omega_p$, $V_\alpha = 0$. The frequencies of interest will be too high for the heavy ions of either component to participate. The external magnetic field is assumed zero. The adoption of a coordinate system such that $\vec{k} = (0, 0, k)$ and $\vec{V} = (V\sin\theta, 0, V\cos\theta)$ yields the dyadic relation

$$0 = \begin{pmatrix} D_{11} & 0 & D_{13} \\ 0 & D_{22} & 0 \\ D_{31} & 0 & D_{33} \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} , \tag{9}$$

where $D_{11} = \omega^2 - (\omega_p^2 + \omega_b^2 + k^2c^2) - [\omega_b^2/(\omega - kV\cos\theta)^2](k^2V^2 - \omega^2\beta^2)\sin^2\theta$,

$$D_{13} = D_{31} = [-\omega_b^2/(\omega - kV\cos\theta)^2](\omega kV - \omega^2\beta^2\cos\theta)\sin\theta ,$$

$$D_{22} = \omega^2 - (\omega_p^2 + \omega_b^2 + k^2c^2) , \tag{10}$$

and $D_{33} = \omega^2 - \omega_p^2 - \omega^2\omega_b^2(1 - \beta^2\cos^2\theta)/(\omega - kV\cos\theta)^2$.

An immediate result is that E_2 is decoupled from both E_1 and E_3. The dispersion relation (from $D_{22} = 0$) is $\omega^2 = k^2c^2 + \omega_p^2 + \omega_b^2$. This wave is purely transverse and is manifestly stable.

The other terms in Eq. (9) yield the dispersion relation $D_{11}D_{33} - D_{13}^2 = 0$, for a wave in which E_1 (transverse to $\vec{k}$) and E_3 (along $\vec{k}$) are coupled. This coupling can be measured in terms of a polarization ratio P defined as $P = E_1/E_3 = -D_{13}/D_{11}$:

$$P = \frac{(kV - \omega\beta^2\cos\theta)\,\omega\omega_b^2\sin\theta}{[\omega^2 - (\omega_p^2 + \omega_b^2 + k^2c^2)](\omega - kV\cos\theta)^2 - \omega_b^2\sin^2\theta(k^2V^2 - \beta^2\omega^2)} . \tag{11}$$

For a given root of the dispersion relation $\omega(k)$, $|P| \ll 1$ implies a longitudinal wave, and $|P| \gtrsim 1$ implies a hybrid or partly transverse wave.

The relation $D_{11}D_{33} - D_{13}^2 = 0$ leads to a polynomial of sixth degree in ω and fourth degree in k. Analysis of the structure of the polynomial led Neufeld[3] to conclude that the associated instabilities are convective. For our purposes, the properties of the dispersion relation are most easily seen by casting it into the form

$$[\omega_p^2 + \omega_b^2 + k^2c^2 - \omega^2]\left(\frac{\omega_p^2}{\omega^2} + \frac{\omega_b^2(1-\beta^2)}{(\omega - kV\cos\theta)^2} - 1\right)$$

$$\times\omega^2(\omega - kV\cos\theta)^2 + \omega_p^2\omega_b^2k^2V^2\sin^2\theta = 0 . \tag{12}$$

Due to the aforementioned mixed origins of the c's in these equations, it is not possible to obtain a nonrelativistic dispersion relation by merely setting $\beta = 0$ in Eq. (12). Comparison of Eq. (12) with Neufeld's dispersion relation, which can be cast in a similar form, shows that the latter omits the ω_b^2 term in square brackets, since he specializes to the case of a weak beam.

Equation (12) shows that the stable transverse wave, given by $\omega^2 = k^2c^2 + \omega_p^2 + \omega_b^2$, is coupled to the unstable longitudinal wave, given by

$$1 = \omega_p^2/\omega^2 + \omega_b^2(1 - \beta^2)/(\omega - kV\cos\theta)^2 . \tag{13}$$

The coupling is weak if $R \equiv \omega_b^2\tan^2\theta/(k^2c^2 + \omega_p^2 + \omega_b^2)$ is much smaller than unity, as can be seen by comparing the term not involving ω in the first term of Eq. (12) with the last term. For $R \ll 1$, the longitudinal wave-dispersion relation is identical with that obtained for $\vec{k}\times\vec{V} = 0$, except that kV is replaced by $\vec{k}\cdot\vec{V}$. This logical result is that which obtains using Poisson's equation in the solution, rather than the full Maxwell equations. As R becomes comparable with unity, it is difficult to estimate the polarization without numerical computation. Figure 1 shows the values of R *versus*

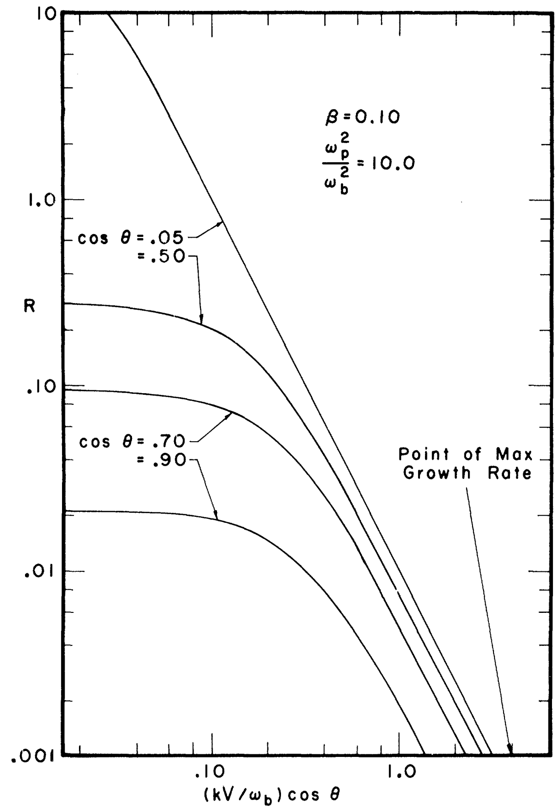

FIG. 1. A graph of the parameter R as a function of normalized wave number. If $R \ll 1$, the instability is largely longitudinal, but if $R \gtrsim 1$, the instability has an appreciable transverse part. The angle between wave propagation and the streaming velocity is θ.

$kV\cos\theta / \omega_b$ for one value of $\epsilon \equiv \omega_p{}^2/\omega_b{}^2$ and for $\beta = 0.1$. These plots indicate the region where Eq. (13) is a good approximation; this is seen to apply for all values of $kV\cos\theta/\omega_b$, save the very small ones, depending upon θ. These observations are not inconsistent with Neufeld's.

The conjecture in the literature that a single plasma stream might be unstable[4] can be examined by setting $\omega_p = 0$ in Eq. (12). The solution is then seen to be stable since

$$\omega^2 = \omega_b{}^2 + k^2c^2 \quad ,$$

$$\omega = kV\cos\theta \pm \omega_b(1 - \beta^2)^{1/2} \quad .$$

III. NUMERICAL RESULTS

This section presents some of the numerical results which have been obtained for the roots of Eq. (12) using an IBM 7094 computer. Two basic examples have been chosen for illustration; a weakly relativistic beam [$\beta^2 = 0.01$ (Figs. 2 and 3)] and a strongly relativistic beam [$\beta^2 = 0.50$ (Figs. 4 and 5)].

Since the deviation between results for $\cos\theta = 0.05$ and $\cos\theta = 1.0$ in Fig. 2 is so small, we show the lower left-hand corner of the graph (i.e., that regime where R becomes large) on an expanded scale in Fig. 3. Also shown here are the polarizations as defined by Eq. (11). The results suggest that there is no distinct division between the longitudinal and hybrid regimes, except insofar as the polarization is clearly longitudinal outside of the hybrid zone. This lack of distinction can be shown to be especially pronounced in the case $\epsilon = 0.001$, for example, since $|P|$ becomes comparable to unity only for quite small values of $kV\cos\theta / \omega_b$. For $\epsilon = 10.0$, $|P|$ rises to values near unity for values of $\vec{k} \cdot \vec{V}$ that are approximately $\frac{1}{20}$ its value at maximum growth; at this point the growth rate has dropped to $\sim \frac{1}{8}$ of the maximum.

For the singular case of $\cos\theta = 0$, the dispersion relation [Eq. (12)] is cubic in ω^2; all three roots are real, with two positive and one negative. The latter gives a pure imaginary value for ω which changes, but slowly, with kV/ω_b. It is this value that the knees of the large-angle curves of Fig. 3 are approaching. Since the real part of this eigen-

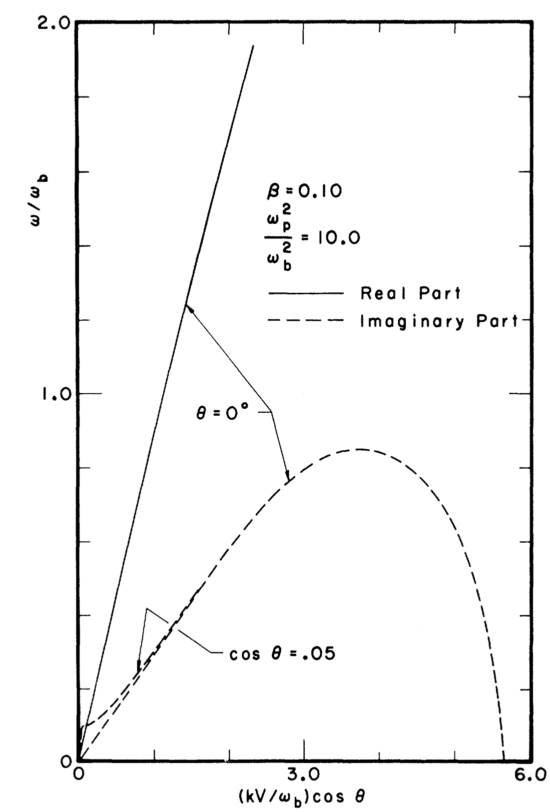

FIG. 2. Real and imaginary parts of the frequency of an unstable wave as a function of wave number. The effect on the growth rate when the propagation is nearly perpendicular to the streaming is shown.

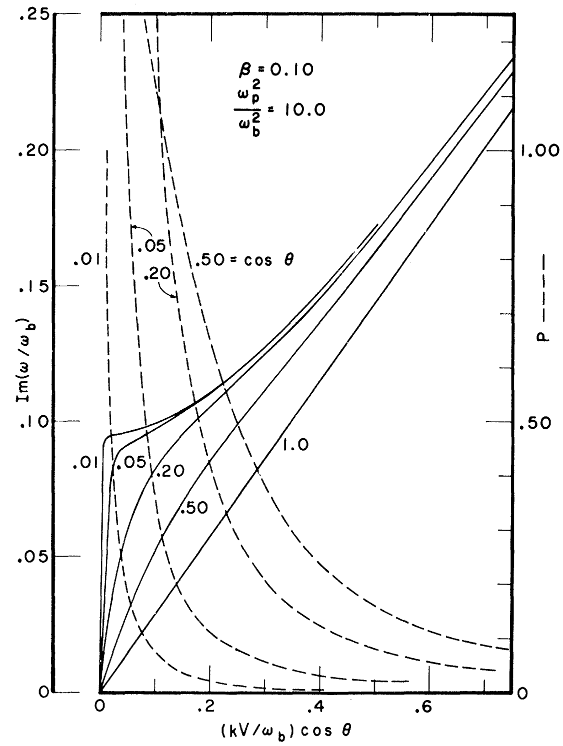

FIG. 3. Polarization ratio and growth rate of the instability for the long wavelengths at which the transverse part of the instability is significant. If $P \ll 1$, the instability is longitudinal for practical purposes. (The ordinate has been displaced for clarity.)

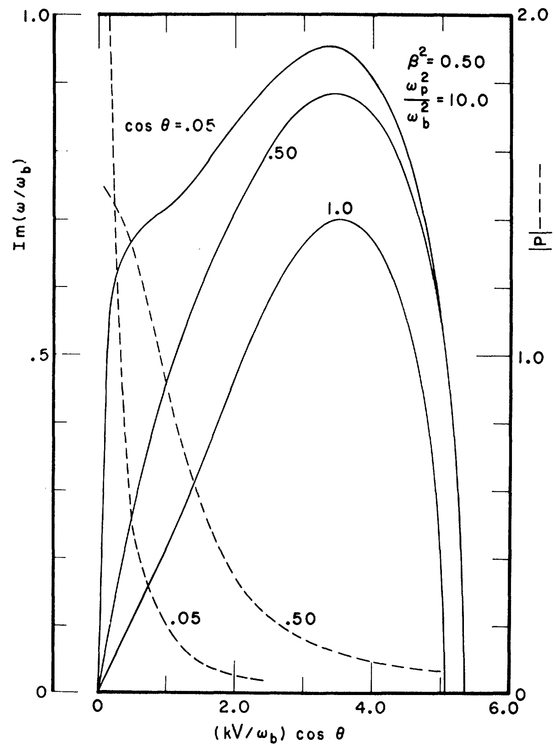

FIG. 4. Growth rate and polarization for a relativistic streaming velocity corresponding to $\beta^2 = 0.5$. Note that the polarization can be appreciably transverse at growth rates which are a significant fraction of the maximum growth rate.

value is zero, the phase velocity of the unstable wave vanishes for $\cos\theta = 0$; therefore, no contribution to radiation is found here.

As β^2 increases, the coupling ratio R becomes larger for the fixed value of $\vec{k}\cdot\vec{V}$. This results in enhanced values of the polarization $|P|$ and in a

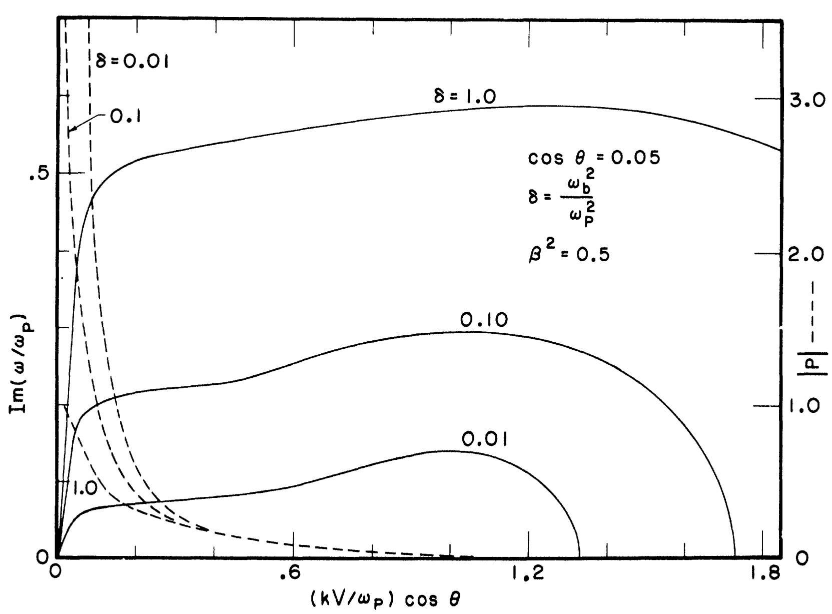

FIG. 5. The effect of changing the beam density is indicated for propagation nearly perpendicular to the streaming.

greater region in which the full dispersion relation must be used to give accurate results. The eigenvalues shown in Figs. 4 and 5, for $\beta^2 = 0.50$, were obtained from solution of the full equation.

In Fig. 4 is shown the behavior of $\mathrm{Im}\omega/\omega_b$ and of $|P|$ over the entire range of the instability for various angles of propagation and for the case of $\epsilon = 10.0$. One notes that the results are remarkably sensitive to the propagation angle and that the $\cos\theta = 1.0$ result is a poor approximation to the large-angle cases for this highly relativistic velocity. Perhaps more significant is the observation that for $\cos\theta = 0.05$, for example, the growth rate of the instability is $\sim\frac{2}{3}$ the maximum growth rate near the point where $|P| \simeq 1$. This indicates that the transverse-wave part of the instability for nearly perpendicular propagation is not overwhelmed by the longitudinal-wave part. It is for examples of this sort that we conclude that radiation effects would be significant.

In Fig. 5 some results are presented for nearly perpendicular propagation ($\cos\theta = 0.05$) with $\beta^2 = 0.5$ and for several values of $\delta \equiv \epsilon^{-1}$. The independent variable is $kV\cos\theta/\omega_p$ in this graph. One notes that, over a large range of wave numbers, the growth rate is essentially constant, a behavior in considerable contrast to the customary two-stream results. As the parameter δ $(\equiv \omega_b^2/\omega_p^2)$ increases beyond unity, values of the polarization for fixed $\vec{k}\cdot\vec{V}$ become small, relative to unity. The results shown in Fig. 5 show the best combinations which we have found between polarizations and large growth rates.

One concludes from this analysis that the linearized theory of the two-stream instability for a homogeneous, cold, unbounded plasma predicts significant radiation effects via a direct coupling between the transverse- and longitudinal-wave components of the eigenmode. The radiation effects have been found to be significant under the following conditions: (i) The streaming velocity is highly relativistic; (ii) the electron plasma frequencies of the beam and the background are within about an order of magnitude of one another; and (iii) the direction of propagation of the unstable eigenmode is nearly perpendicular to the streaming direction.

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