Distortion of the Electron Wave Function in $\alpha\pi^-e^-$, αK^-e^-, and $\alpha\bar{p}e^-$ Atoms

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An investigation is made of the extent to which the binding energies and wave functions for some states of the $\alpha\pi^-e^-$, αK^-e^-, and $\alpha\bar{p}e^-$ atoms are affected by the dipole and higher multipole interactions of the electron with the meson or the antiproton. The meson or antiproton is treated as a classical point charge. Perturbation theory is employed to determine the admixture of discrete excited states of the unperturbed electron Hamiltonian. The unperturbed electron Hamiltonian takes into account only the monopole interaction of the electron with the meson. A variational method is employed to determine approximately the change in the binding energy as well as some of the more important features of the distortion of the electron wave function. It is found, in particular, that the binding energies of those states of the αK^-e^- atom in which the meson is in a circular orbit with principal quantum number $n=27$, 28, or 29 are lowered by slightly less than 1 eV. The maximum relative change in the amplitude of the electron wave function for one of these states is found to be roughly 20%.

I. INTRODUCTION

In the preceding paper,[1] the wave functions and binding energies are estimated for a large number of states of the $\alpha\pi^-e^-$, αK^-e^-, and $\alpha\bar{p}e^-$ atoms. In each of these instances the meson is assumed to be in a circular or nearly circular orbit, and the electron is assumed to be in a $1s$ orbit. The calculations which are presented in Ref. 1 do not, in effect, take into account the dipole and higher multipole interactions of the electron with the meson.

The purpose of the present paper is to estimate the effects which these multipole interactions have on the atomic wave functions and binding energies. The estimates made here of the extent to which these interactions distort the electron wave function are used elsewhere[2,3] to ascertain that the effects of the interaction of an αK^-e^- atom with a nearby helium atom, which are calculated in Ref. 2, and the Auger rates for circular orbits of $\alpha\pi^-e^-$, αK^-e^-, and $\alpha\bar{p}e^-$ atoms, which are calculated in Ref. 3, are determined with sufficient accuracy. The calculations made here of the correction to the binding energy provide the basis for the estimate made in Ref. 1 of the extent to which the dipole and higher multipole interactions of the electron with the meson affect the energy difference between a circular orbit with principal quantum number n and a nearly circular orbit with the same principal quantum number, but with orbital angular momentum $(n-2)$. Since all of the effects of these interactions are found to be relatively minor in the instances which are of any interest, some of the calculations which are presented here are limited to those states of the αK^-e^- atom in which the K^- meson is in a circular orbit with $n=27$, 28, or 29. These are the states which are studied in considerable detail in Ref. 2.

Energy is expressed in rydbergs throughout this paper. Likewise, the unit of distance is the hydrogen Bohr radius, and the unit of mass is the electron mass.

The calculations which are presented here are based on some of the results which are obtained in Ref. 1. These results may be summarized as follows. The mesonic atom is found to be adequately described by solutions of the wave equation

$$H_b\Psi_b(\vec{r}_1,\vec{r}_\mu)=E_b\Psi_b(\vec{r}_1,\vec{r}_\mu)\ , \tag{1.1a}$$

$$H_b=-\nabla_1^2-\frac{4}{r_1}-\frac{1}{M}\nabla_\mu^2-\frac{4}{r_\mu}+\frac{2}{|\vec{r}_1-\vec{r}_\mu|}\ . \tag{1.1b}$$

In Eq. (1.1) the positions of the electron and the meson with respect to the α particle are denoted by $\vec{r}_1$ and $\vec{r}_\mu$, respectively, and the reduced mass of the meson and α particle is denoted by M. If only the monopole interaction between the electron and the meson is considered, Eq. (1.1) takes the form

$$H_c\Psi_c(\vec{r}_1,\vec{r}_\mu)=E_c\Psi_c(\vec{r}_1,\vec{r}_\mu)\ , \tag{1.2a}$$

$$H_c=-\nabla_1^2-\frac{4}{r_1}-\frac{1}{M}\nabla_\mu^2-\frac{4}{r_\mu}+\begin{pmatrix}2/r_\mu, & r_1<r_\mu\\ 2/r_1, & r_1>r_\mu\end{pmatrix}. \tag{1.2b}$$

The calculations which are given in Ref. 1 are to a considerable extent concerned with finding

approximate solutions to Eq. (1.2). It is shown there that the rms velocity of the electron is at least an order of magnitude larger than that of the meson if the principal quantum number of the meson is fairly large, as it is for the three levels of the $\alpha K^- e^-$ atom which are being given relatively detailed consideration in the present paper. It is also shown that a circular meson orbit with a fairly large value of n has a rather well-defined radius, since its wave function is sharply peaked. In particular, it is shown in Ref. 1 that for some purposes the meson may be regarded as a stationary point charge which is situated a distance $\bar{r}_\mu$ from the α particle. The distance $\bar{r}_\mu$ for a circular orbit is defined to be the value of the mean separation between the meson and the α particle if the solution to Eq. (1.1) or Eq. (1.2) is approximated with a two-parameter variational wave function which is the product of two hydrogenic functions, the variational parameters being the effective nuclear charges for the electron and the meson.

It therefore seems reasonable to suppose that the effects of the dipole and higher multipole interactions, which are given by

$$H_{bc} = H_b - H_c \ , \tag{1.3}$$

can be determined to a satisfactory degree of accuracy by treating the meson as a stationary point charge which is a distance $\bar{r}_\mu$ from the α particle, since the perturbation H_{bc} surely distorts the electron wave function far more than it does the wave function for the much more massive and rather more slowly moving meson. The problem is thereby reduced to determining what changes occur in the ground-state electron eigenfunction and eigenvalue if the interaction H_{bc} is added adiabatically to the unperturbed electron Hamiltonian

$$H_{de} = -\nabla_1^2 - \frac{4}{r_1} + \begin{pmatrix} 2/\bar{r}_\mu, & r_1 < \bar{r}_\mu \\ 2/r_1, & r_1 > \bar{r}_\mu \end{pmatrix} . \tag{1.4}$$

For the sake of mathematical simplicity, the meson is assumed to be on the positive z axis. The interaction H_{bc} can then be written in the form

$$H_{bc} = \sum_{L=1}^{\infty} V_{EL} \ , \tag{1.5}$$

where V_{EL} is defined by

$$V_{EL} = 2P_L(\cos\theta_1) \begin{pmatrix} r_1^{\,L}/\bar{r}_\mu^{\,L+1}, & r_1 < \bar{r}_\mu \\ \bar{r}_\mu^{\,L}/r_1^{\,L+1}, & r_1 > \bar{r}_\mu \end{pmatrix} . \tag{1.6}$$

In order to determine the effects of H_{bc} it is necessary to have reasonably accurate estimates of the eigenfunctions and eigenvalues of the operator H_{de}, particularly for the ground state. The ground-state eigenfunction and eigenvalue of H_{de}, which are denoted here by $u_{de}(\vec{r}_1, \bar{r}_\mu)$ and $E_{de}(\bar{r}_\mu)$, respectively, are calculated approximately in Ref. 1, although for a different purpose. Although it is possible to compute u_{de} and E_{de} exactly by numerical integration of the differential equation, it is assumed in Ref. 1 that it is sufficient to employ a $1s$ hydrogenic function with an effective charge z_v, which is determined by minimizing the expectation value E_{dev} of the Hamiltonian H_{de}. Values of z_v are shown in Fig. 3 of Ref. 1. It is assumed that these ground-state variational wave functions and energies are accurate enough to be employed in the present paper.

The perturbation H_{bc} introduces components of other eigenfunctions of H_{de} into the ground-state electron wave function. It is convenient, for two reasons, to consider the contributions from the discrete part of the spectrum of H_{de} separately. One reason is that these contributions may be readily estimated using perturbation theory. The other reason, as explained in more detail in Ref. 3, is that some of the low-lying excited states, even though they have very small amplitudes, are likely, because their wave functions have a large degree of overlap with the rather steeply rising wave function for the ejected low-energy electron, to affect the Auger rates to an appreciable extent. On the other hand, the interaction of the mesonic atom with a nearby helium atom might reasonably be expected to be affected significantly only by the distortion at points where the unperturbed ground-state wave function is not negligibly small. This contribution to the distortion, which is surely composed predominantly of eigenfunctions of H_{de} from the continuum, is estimated roughly in Sec. III with a variational method. The contribution from the discrete part of the spectrum of H_{de} is estimated in Sec. II.

II. COMPONENTS OF DISCRETE EXCITED STATE EIGENFUNCTIONS OF H_{de}

Since the ground state of the unperturbed Hamiltonian H_{de} is a $1s$ state, and since the meson is assumed to be on the z axis, the only eigenfunctions of H_{de} with nonvanishing components in the ground-state eigenfunction of the operator $(H_{de} + H_{bc})$ are states with magnetic quantum number $m = 0$. The amplitudes for the $2p$ and $3d$ states with $m = 0$ are the only ones which are computed in this section. These amplitudes are denoted by b_{2p} and b_{3d}. As explained in more detail in Ref. 3, it suffices to compute only these two amplitudes in order to be able to estimate roughly the

effect of the discrete excited-state eigenfunctions of H_{de} on the Auger rates.

Once the eigenfunctions and eigenvalues of H_{de} are known, the calculation of b_{2p} and b_{3d} becomes, in principle, straightforward. Some assumptions are now made concerning these states in order to simplify the calculation. As mentioned in Sec. I, the ground-state eigenfunction is assumed to be a $1s$ hydrogenic function which is characterized by the effective nuclear charge z_v, and the ground-state eigenvalue is assumed to be equal to the function $E_{dev}(\bar{r}_\mu)$. Values of z_v and E_{dev} are calculated in Ref. 1. The eigenfunctions of H_{de} for the discrete excited states are assumed to be given accurately enough by hydrogenic functions which are characterized by unit effective nuclear charge, because most of the values of $\bar{r}_\mu$ which are considered in this paper are, at most, not much larger than a hydrogen Bohr radius. The eigenvalues of H_{de} for the discrete excited states are assumed to be given accurately enough by $(-n_e^{-2})$, where n_e is the principal quantum number of the electron. If the mean radius $\bar{r}_1$ of the electron orbit in the ground state is appreciably larger than $\bar{r}_\mu$, as it is in many of the instances which are considered in this paper, including the circular orbits of the αK^-e^- atom with $n=27$, 28, or 29, then the overlap of the $1s$ wave function with either the $2p$ or the $3d$ wave function is probably given fairly accurately, and no very large error should be introduced into the calculation.

The amplitude b_{2p} is calculated using first-order perturbation theory. Therefore, only the dipole part of H_{bc} has to be taken into account. It is easily shown that b_{2p} is given by

$$b_{2p}=32(\tfrac{1}{2}z_v)^{3/2}\bar{r}_\mu^{\,3}y_1^{\,-5}\times[1-(1+y_1+\tfrac{1}{2}y_1^{\,2}+\tfrac{1}{8}y_1^{\,3})e^{-y_1}]\times(E_{dev}+\tfrac{1}{4})^{-1}, \tag{2.1}$$

where $$y_1=(z_v+\tfrac{1}{2})\bar{r}_\mu . \tag{2.2}$$

There are two significant contributions to the amplitude b_{3d}, which it is convenient to write as

$$b_{3d}=b_{3d}^{(1)}+b_{3d}^{(2)} . \tag{2.3}$$

The contribution $b_{3d}^{(1)}$ is calculated using first-order perturbation theory. Only the quadrupole part of H_{bc} has to be taken into account in the calculation of $b_{3d}^{(1)}$. It is easily shown that $b_{3d}^{(1)}$ is given by

$$b_{3d}^{(1)}=(\tfrac{8}{3})^3\sqrt{3}\,(\tfrac{1}{2}z_v)^{3/2}\bar{r}_\mu^{\,4}y_2^{\,-7}\times[1-(1+y_2+\tfrac{1}{2}y_2^{\,2}+\tfrac{1}{6}y_2^{\,3}+\tfrac{1}{24}y_2^{\,4}+\tfrac{1}{144}y_2^{\,5})e^{-y_2}]\times(E_{dev}+\tfrac{1}{9})^{-1}, \tag{2.4}$$

where $$y_2=(z_v+\tfrac{1}{3})\bar{r}_\mu . \tag{2.5}$$

The contribution $b_{3d}^{(2)}$ is calculated using second-order perturbation theory; but the only intermediate state which is taken into account is the $2p$ state. It is assumed that taking into account the $3p$ state and other intermediate states would not change the value of b_{3d} significantly, because there is good reason to believe that the $3p$-$3d$ dipole matrix element is anomalously small,[4] and because the first-order amplitudes for other possible states should be negligible, since their wave functions do not overlap to a very large extent with the $1s$ wave function. Only the dipole part of H_{bc} is considered in the calculation of $b_{3d}^{(2)}$, since the octupole contribution to the $2p$-$3d$ matrix element should be negligibly small. For the purpose of calculating the $2p$-$3d$ matrix element, it is assumed that the dipole part of H_{bc} is given accurately enough by

$$2(\bar{r}_\mu/r_1^{\,2})\cos\theta_1 ,$$

since the mean radii of both the $2p$ and the $3d$ states are, in every instance which is being considered, several times larger than $\bar{r}_\mu$. It is easily shown that $b_{3d}^{(2)}$ is given by

$$b_{3d}^{(2)}=(\tfrac{1}{24})(\tfrac{4}{5})^5\sqrt{3}\,\bar{r}_\mu b_{2p}(E_{dev}+\tfrac{1}{9})^{-1} . \tag{2.6}$$

Figures 1 and 2 show the probabilities $|b_{2p}|^2$ and $|b_{3d}|^2$ for circular orbits of the $\alpha\pi^-e^-$, αK^-e^-, and $\alpha\bar{p}e^-$ atoms. These probabilities are given in Figs. 1 and 2 as functions of the meson principal quantum number n. It should be remarked that the ratio $b_{3d}^{(1)}/b_{3d}^{(2)}$ is found to vary from approximately -1.5 to approximately -3.5 as $\bar{r}_\mu$ increases from 0.25 to 1.5. Although the probabilities $|b_{2p}|^2$ and $|b_{3d}|^2$ are quite small, it is shown in Ref. 3 that the components of the $2p$ eigenstates of H_{de} may very well account for an appreciable fraction of the Auger rate.

The presence of components of the discrete excited eigenstates of H_{de} has only a very slight effect on the energy levels. To first order in H_{bc} there is no correction to the binding energy of the electron. If the $2p$ state is the only excited eigenstate of H_{de} which is taken into account, the second-order correction to the energy is

$$(E_{dev}+\tfrac{1}{4})|b_{2p}|^2 .$$

This correction is found to lower the binding energy by only about 0.008 Ry at the very most. The correction to the binding energy which is

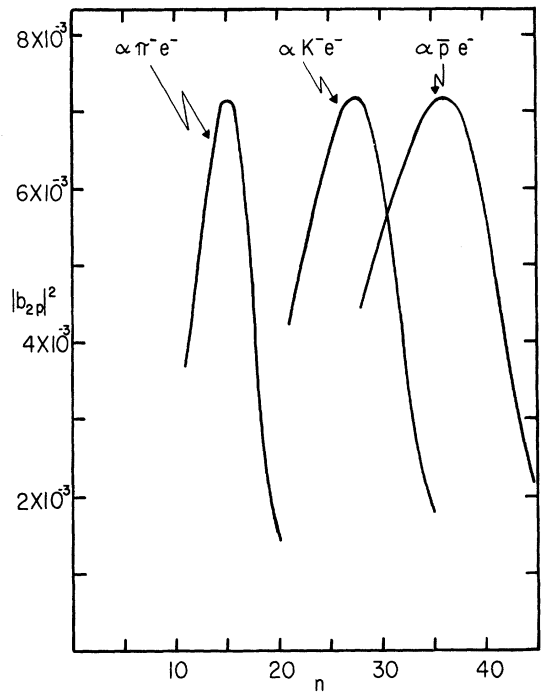

FIG. 1. The probability $|b_{2p}|^2$ as a function of the meson principal quantum number n.

due to the $3d$ state is several orders of magnitude lower than that which is due to the $2p$ state. It seems reasonable to assume that the energy correction which is due to the $2s$ eigenstate of H_{de} is also entirely negligible, because, to first order in H_{bc}, the $1s$ state is the only s state which has a nonvanishing component in the ground state of $H_{de}+H_{bc}$. The energy correction which is estimated by a variational method in Sec. III, and which is shown there to have an absolute value which is an order of magnitude larger than that of the correction which is due to the $2p$ state, is therefore due predominantly to the presence of components of continuum eigenfunctions of H_{de}.

III. VARIATIONAL CALCULATION

A. Estimates of Distortion of Wave Function and Correction to the Binding Energy

In this section, the distorting effect of the interaction H_{bc} on the electron wave function is estimated with a variational method. As in Sec. II, the meson is assumed to be a stationary point charge which is situated on the positive z axis a distance $\bar{r}_\mu$ from the origin. It is also assumed here that the distortion of the electron wave function is adequately described by a term with the angular dependence $P_1(\cos\theta_1)$. In this approximation, only the dipole and quadrupole parts of H_{bc} have to be considered. The Hamiltonian for the electron may then be regarded as being given by

$$H_e = H_{de} + V_{E1} + V_{E2} , \tag{3.1}$$

where H_{de} is the operator defined by Eq. (1.4), and where V_{E1} and V_{E2} are defined by Eq. (1.6).

The normalized ground-state eigenfunction of H_e is now approximated with a variational wave function of the form

$$u_e(\vec{r}_1, \bar{r}_\mu) = u_{de}(\vec{r}_1, \bar{r}_\mu)(1+ag)/(1+\zeta a^2)^{1/2} , \tag{3.2}$$

where u_{de} is the normalized ground-state eigenfunction of H_{de}. In Eq. (3.2) the quantity g is the product of $P_1(\cos\theta_1)$ and some suitable function of r_1; ζ is defined by

$$\zeta = \int |u_{de} g|^2 d\tau_1 ; \tag{3.3}$$

and a, which is a measure of the distortion, is a parameter which is determined by minimizing the energy

$$E_e = \int u_e^* H_e u_e \, d\tau_1 . \tag{3.4}$$

It is convenient to put the expression for E_e in the form

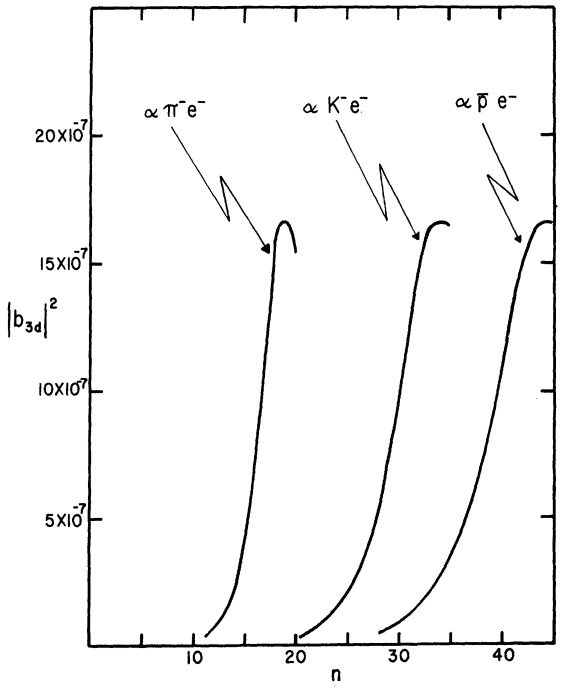

FIG. 2. The probability $|b_{3d}|^2$ as a function of the meson principal quantum number n.

$$E_e = [E_{de} + 2aT_2 + a^2(T_1 + T_3)]/(1 + \zeta a^2) , \quad (3.5)$$

where E_{de} is the ground-state eigenvalue of H_{de}, and where

$$T_1 = \int g^* u_{de}^* H_{de} u_{de} g \, d\tau_1 , \quad (3.6a)$$

$$T_2 = \int g |u_{de}|^2 V_{E1} \, d\tau_1 , \quad (3.6b)$$

$$T_3 = \int |u_{de} g|^2 V_{E2} \, d\tau_1 . \quad (3.6c)$$

It is easily shown that the parameter a is given by

$$a = \eta \{1 - [1 + (\zeta\eta^2)^{-1}]^{1/2}\} , \quad (3.7)$$

where $\eta = (T_1 + T_3 - \zeta E_{de})/(2\zeta T_2)$. (3.8)

The correction to the binding energy which arises from taking into account the interaction H_{bc} is given approximately by

$$E_{bc} = E_e - E_{de} . \quad (3.9)$$

It is now assumed that the approximations to u_{de} and E_{de} described in Sec. I are accurate enough to be employed in the computation of a and E_{bc}. The energy E_{de} which appears in Eqs. (3.5), (3.8), and (3.9) is replaced with the approximate energy E_{dev}, and the eigenfunction u_{de} which appears in Eqs. (3.6a)–(3.6c) is replaced with a $1s$ hydrogenic function which is characterized by the effective nuclear charge z_v.

The variational wave function u_e, which has the general form specified by Eq. (3.2), should satisfy several requirements. It should become spherically symmetric as $r_1 \to 0$ and also as $r_1 \to \infty$; for a given value of $\cos\theta_1$, the distortion should be largest near the points $r_1 = \bar{r}_\mu$; and $\vec{\nabla}_1 u_e$ must be continuous and single-valued at all points $r_1 \neq 0$. Calculations are described here for two choices of g which have somewhat different functional forms, but which are both such that these requirements are satisfied.

One choice of g, which is denoted by g_1, is

$$g_1 = V_{E1}, \quad |r_1 - \bar{r}_\mu| \geq \Delta , \quad (3.10)$$

where Δ is exceedingly small. Because V_{E1}, which is defined by Eq. (1.6), has a discontinuous gradient, it cannot be equated to g_1 in the immediate vicinity of the points $r_1 = \bar{r}_\mu$. For $|r_1 - \bar{r}_\mu| \leq \Delta$, it is only necessary to assume that g_1 is some finite and continuous function which has a continuous gradient. As $\Delta \to 0$ the contributions to the integrals ζ, T_2, and T_3 from the region $|r_1 - \bar{r}_\mu| < \Delta$ become vanishingly small, and the contribution to T_1 approaches

$$16 z_v^3 \bar{r}_\mu^{-1} e^{-2z_v \bar{r}_\mu} .$$

The evaluation of the contributions to these integrals from the regions with $|r_1 - \bar{r}_\mu| > \Delta$ is straightforward. Explicit expressions for ζ, T_1, T_2, and T_3 for this choice of g are given in the Appendix.

The other choice of g, which is denoted by g_2, is

$$g_2 = 2P_1(\cos\theta_1) \times \begin{pmatrix} (r_1/\bar{r}_\mu^{\,2})[1 - (3r_1)/(5\bar{r}_\mu)], & r_1 < \bar{r}_\mu \\ (\bar{r}_\mu/r_1^{\,2})[1 - (3\bar{r}_\mu)/(5r_1)], & r_1 > \bar{r}_\mu \end{pmatrix} . \quad (3.11)$$

This function has a continuous gradient at $r_1 = \bar{r}_\mu$. The maximum of g_2, which occurs at $r_1 = 5\bar{r}_\mu/6$, is 4% larger than its value at $r_1 = \bar{r}_\mu$. The expressions for the integrals ζ, T_1, T_2, and T_3 obtained with this choice of g are also given explicitly in the Appendix.

Some numerical results are presented in Tables I–III for those states of the αK^-e^- atom which are also studied in Ref. 2. These states are the circular orbits with principal quantum number $n = 27$, 28, and 29. The quantity $\bar{r}_1$ is the estimate of the mean radius of the undistorted electron orbit which is determined in Ref. 1. The values of $\bar{r}_\mu$, which are relatively small compared with the values of $\bar{r}_1$, favor, to some extent, the validity of approximating the distortion of the wave function with a dipole term. Depending on whether g_1 or g_2 is employed, the parameter a is denoted by a_1 or a_2, and the energy difference E_{bc} is denoted by $E_{bc},1$ or $E_{bc},2$. The values of $E_{bc},1$ and $E_{bc},2$, which are rather small compared with the values of E_{dev}, suggest that varying simultaneously both the parameter a and the effective nuclear charge for the undistorted hydrogenic wave function, instead of varying only a and employing the values of z_v

TABLE I. The values of z_v, $\bar{r}_1$, and E_{dev} for the undistorted $1s$ electron wave function which is associated with a circular orbit of the αK^-e^- atom with principal quantum number n and mean meson orbital radius $\bar{r}_\mu$. The unit of distance is the hydrogen Bohr radius, and the unit of energy is the rydberg.

n	$\bar{r}_\mu$	z_v	$\bar{r}_1$	E_{dev}
29	0.569	1.51	0.993	−1.4170
28	0.514	1.43	1.049	−1.3362
27	0.465	1.36	1.103	−1.2713

TABLE II. A summary of the results which are obtained for several circular orbits of the $\alpha K^- e^-$ atom by using an electron wave function with a distortion which is proportional to the function g_1. The unit of energy is the rydberg, and the unit of distance is the hydrogen Bohr radius.

n	a_1	$E_{bc,1}$	$a_1^2\zeta_1$	$2a_1/\bar{r}_\mu$
29	−0.0611	−0.0662	0.0041	−0.215
28	−0.0526	−0.0575	0.0030	−0.205
27	−0.0453	−0.0486	0.0022	−0.195

which are calculated in Ref. 1, would not lead to results which are substantially different. The approximate inequality

$$-0.20 \lesssim \frac{dE_{bc}}{d\bar{r}_\mu} \lesssim -0.14 , \qquad (3.12)$$

which is seen from the results listed in Tables I–III to hold for the three cases being considered, is referred to in Sec. V of Ref. 1, where the energy differences between some pairs of almost degenerate states of the $\alpha K^- e^-$ atom are estimated. The very small values of $a^2\zeta$ would seem to indicate that the distortion of the electron probability distribution is probably not large enough to cause the actual values of the mean radius of the meson orbit to differ very much from what they would be if there were only a monopole interaction between the electron and the meson. Also given in Tables II and III are the values of $2a_1/\bar{r}_\mu$ and $4a_2/(5\bar{r}_\mu)$, which are the expressions for $a_1 g_1$ and $a_2 g_2$ at $r_1 = \bar{r}_\mu$ on the positive z axis. Figure 3 shows the behavior of $a_1 g_1$ and $a_2 g_2$ on the negative z axis for the circular orbit of the $\alpha K^- e^-$ atom with $n = 28$. Apart from the difference in the behavior of $a_1 g_1$ and $a_2 g_2$ as r_1 becomes very large, none of the results given in Tables II and III and Fig. 3 depends dramatically on the choice of g.

TABLE III. A summary of the results which are obtained for several circular orbits of the $\alpha K^- e^-$ atom by using an electron wave function with a distortion which is proportional to the function g_2. The unit of energy is the rydberg, and the unit of distance is the hydrogen Bohr radius.

n	a_2	$E_{bc,2}$	$a_2^2\zeta_2$	$4a_2/(5\bar{r}_\mu)$
29	−0.1468	−0.0836	0.0067	−0.206
28	−0.1316	−0.0760	0.0055	−0.205
27	−0.1167	−0.0668	0.0043	−0.201

B. Calculations to First and Second Order in the Strength of the Distorting Interaction

In Ref. 2 an estimate is made of the effect which the distortion of the electron wave function has on the interaction of an $\alpha K^- e^-$ atom with a helium atom. This estimate is carried only to first order in the strength of the dipole interactions of the meson with the electrons. It is, therefore, appropriate to make a suitable modification of the preceding calculation of the parameter a.

Because it can be shown numerically for each of the three cases being considered that $(\zeta\eta^2)^{-1} \ll 1$, the expression for a given by Eq. (3.7) can be expanded in the rapidly converging series

$$a = -\tfrac{1}{2}\zeta^{-1}\eta^{-1} + (\tfrac{1}{8})\zeta^{-2}\eta^{-3} - \cdots . \qquad (3.13)$$

Of the five quantities ζ, E_{dev}, T_1, T_2, and T_3 which determine a, only T_2 and T_3 depend on the distorting interaction $(V_{E1} + V_{E2})$. The integral T_2 is proportional to the strength of the dipole interaction, and T_3 is proportional to the strength of the quadrupole interaction. Because it can also be shown numerically in each of the three instances being considered that

$$T_3(T_1 - \zeta E_{dev})^{-1} \ll 1 , \qquad (3.14)$$

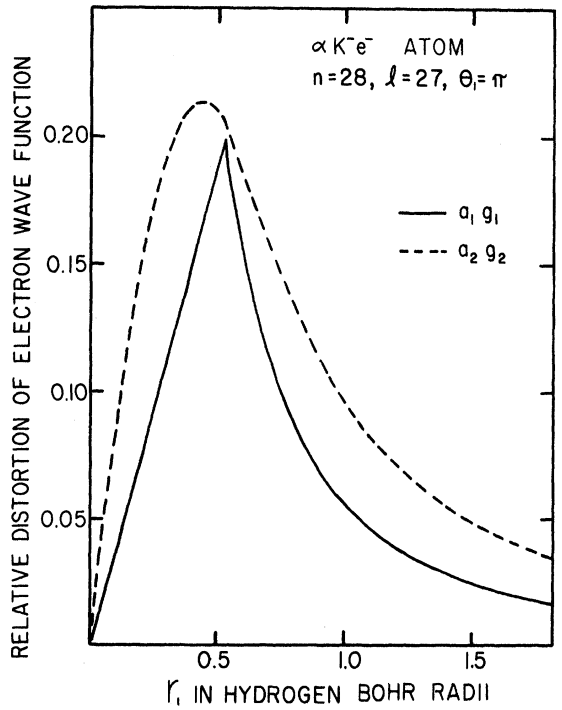

FIG. 3. The relative distortion of the electron wave function on the negative z axis for a circular orbit of the $\alpha K^- e^-$ atom with $n = 28$, as estimated using functions which are proportional to g_1 or g_2. The meson is situated on the positive z axis.

TABLE IV. The relative accuracy of the first- and second-order values of the distortion parameter a for several circular orbits of the αK^-e^- atom.

n	$a_1^{(1)}/a_1$	$(a_1^{(1)}+a_1^{(2)})/a_1$	$a_2^{(1)}/a_2$	$(a_2^{(1)}+a_2^{(2)})/a_2$
29	1.059	1.001	1.068	1.003
28	1.054	1.001	1.064	1.002
27	1.049	1.000	1.060	1.001

the quantity η which is defined by Eq. (3.8) may be computed rather accurately using the relation

$$\eta^{-1}\simeq 2\xi T_2(T_1-\xi E_{dev})^{-1}[1-T_3(T_1-\xi E_{dev})^{-1}]. \tag{3.15}$$

It then follows that to first order in the strength of $(V_{E1}+V_{E2})$ the parameter a is given by

$$a^{(1)}=-T_2(T_1-\xi E_{dev})^{-1}, \tag{3.16}$$

and to second order it is given by

$$a^{(1)}+a^{(2)}=-T_2(T_1-\xi E_{dev})^{-1}\times[1-T_3(T_1-\xi E_{dev})^{-1}]. \tag{3.17}$$

The effect of the quadrupole interaction first appears in the second-order expression. A comparison of the values of $a^{(1)}$ and $(a^{(1)}+a^{(2)})$ with the values of a calculated using Eq. (3.7) is given in Table IV for both choices of the function g. The first-order results are accurate to within 7%. The relative unimportance of the interaction V_{E2} in determining the distortion parameter a is interpreted as indicating that including a quadrupole term in the trial wave function and taking into account the multipole interactions V_{E3} and V_{E4} would not lead to significantly improved results.

APPENDIX

The following definitions are made:

$$y_0=2z_v\bar{r}_\mu, \tag{A1}$$

$$s_p=y_0^{-p}\int_0^{y_0}y^p e^{-y}dy,\quad p=0,1,2,\ldots, \tag{A2}$$

$$t_p=y_0^{\,p}\int_{y_0}^{\infty}y^{-p}e^{-y}dy,\quad p=1,2,3,\ldots. \tag{A3}$$

Values of the integrals s_p and t_p for fixed y_0 and successive values of p may be calculated using the relations

$$s_p=py_0^{-1}s_{p-1}-e^{-y_0},\quad p=1,2,3,\ldots, \tag{A4}$$

$$t_p=y_0(e^{-y_0}-t_{p-1})/(p-1),\quad p=2,3,4,\ldots. \tag{A5}$$

If $g=g_1$, where g_1 is the function defined by Eq. (3.10), the integral ξ, which is defined by Eq. (3.3), and the integrals T_1, T_2, and T_3, which are defined by Eqs. (3.6a)–(3.6c), are denoted by ξ_1, $T_{1,1}$, $T_{2,1}$, and $T_{3,1}$ and are given by

$$\xi_1=\tfrac{8}{3}z_v^{\,2}(s_4+t_2), \tag{A6}$$

$$T_{1,1}=\tfrac{32}{3}z_v^{\,4}y_0^{-1}\{2(1-z_v^{-1})s_3+[z_v^{-1}-\tfrac{1}{4}y_0]s_4-\tfrac{1}{4}y_0t_2-(1+z_v^{-1})t_3+3e^{-y_0}\}, \tag{A7}$$

$$T_{2,1}=\xi_1, \tag{A8}$$

$$T_{3,1}=\tfrac{64}{15}z_v^{\,3}y_0^{-1}(s_6+t_5). \tag{A9}$$

If $g=g_2$, where g_2 is the function defined by Eq. (3.11), the expressions for ξ, T_1, T_2, and T_3 are denoted by ξ_2, $T_{1,2}$, $T_{2,2}$, and $T_{3,2}$ and are given by

$$\xi_2=\tfrac{8}{3}z_v^{\,2}[s_4+t_2-\tfrac{6}{5}(s_5+t_3)+\tfrac{9}{25}(s_6+t_4)], \tag{A10}$$

$$T_{1,2}=\tfrac{32}{3}z_v^{\,4}y_0^{-1}\{[2(1-z_v^{-1})+\tfrac{12}{5}y_0^{-1}]s_3+(\tfrac{17}{5}z_v^{-1}-\tfrac{1}{4}y_0-3-\tfrac{36}{25}y_0^{-1})s_4+\tfrac{3}{50}(18+5y_0-32z_v^{-1})s_5+\tfrac{9}{100}(4z_v^{-1}-y_0)s_6-\tfrac{1}{4}y_0t_2-(1+z_v^{-1}-\tfrac{3}{10}y_0)t_3-\tfrac{3}{5}(\tfrac{3}{20}y_0-3-2z_v^{-1})t_4-\tfrac{3}{5}(\tfrac{6}{5}+\tfrac{3}{5}z_v^{-1}-4y_0^{-1})t_5-\tfrac{36}{25}y_0^{-1}t_6\}, \tag{A11}$$

$$T_{2,2}=\tfrac{8}{3}z_v^{\,2}[s_4+t_2-\tfrac{3}{5}(s_5+t_3)]\ , \tag{A12}$$

$$T_{3,2}=\tfrac{64}{15}z_v^{\,3}y_0^{\,-1}[s_6+t_5-\tfrac{6}{5}(s_7+t_6)+\tfrac{9}{25}(s_8+t_7)]. \tag{A13}$$

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Auger Rates for Circular Orbits of $\alpha\pi^-e^-$, αK^-e^-, and $\alpha\bar{p}e^-$ Atoms

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Auger rates are estimated for some circular orbits of the $\alpha\pi^-e^-$, αK^-e^-, and $\alpha\bar{p}e^-$ atoms. It is found, as conjectured by Condo several years ago, that the Auger rate for a circular orbit depends sensitively on the multipolarity of the transition. In some instances, the multipolarity may be required by energy conservation to be large enough to permit a π^- or a K^- meson to decay rather than to eject an electron. In particular, the estimated Auger rates for those circular orbits of the $\alpha\pi^-e^-$, αK^-e^-, and $\alpha\bar{p}e^-$ atoms with binding energies most nearly equal to that of a He atom are, respectively, 2×10^{12}, 4×10^{5}, and $\lesssim10^{3}$ sec^{-1}. Radiative rates are also computed.

I. INTRODUCTION

Estimates of Auger rates for some states of the $\alpha\pi^-e^-$, αK^-e^-, and $\alpha\bar{p}e^-$ atoms are obtained in the present paper. These atoms are formed when π^- and K^- mesons and antiprotons are stopped in helium. In the particular states which are being considered here, the electron is in a $1s$ orbit, and the meson or the antiproton is in a circular orbit with large principal quantum number n. In a circular orbit, the orbital angular momentum l is given by $l=(n-1)$. For the sake of brevity, the particular atomic states which are being considered here are frequently referred to simply as circular orbits, and an antiproton is frequently referred to loosely as a meson.

The purpose of this investigation is to determine the extent to which the Auger rate for a circular orbit depends on the multipolarity of the transition. The circular orbits are distinguished by the fact that the magnitude $|\Delta n|$ of the change in the meson principal quantum number is equal to both the multipolarity of the Auger transition and the orbital angular momentum of the ejected electron. It was argued several years ago by Condo[1] that the Auger rate for a circular orbit would be extremely sensitive to the value of $|\Delta n|$, which in some instances might be required by energy conservation to be large enough to allow a π^- or a K^- meson to decay rather than to eject the electron, thereby accounting for the large values of the measured mean cascade times[2–4] for these particles in atomic orbits in liquid helium. Since the antiproton does not decay, radiative rates are also computed in the present paper.

The computation of the Auger rates, which is quite straightforward, is carried out in Sec. II by making use of some approximate wave functions and binding energies which are determined elsewhere.[5] The allowed values of $|\Delta n|$ are also determined in Ref. 5. The results obtained in Sec. II are found to be indeed sensitive to the values of $|\Delta n|$. In Sec. III, a detailed investigation is made of the extent to which inaccuracies in the wave func-