# Structure of Neutral Mesonic Atoms Formed in Liquid Helium

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Wave functions and binding energies are calculated for some states of the $\alpha\pi^-e^-$, $\alpha K^-e^-$, and $\alpha\bar{p}e^-$ atoms in which the electron is in a 1s orbit and in which the meson or the antiproton is in a circular or a nearly circular orbit with large principal quantum number $n$. The results which are obtained in the present paper provide a basis for an estimate, described elsewhere, of the probability that atoms in some of these states can exist for an anomalously long time after being formed in liquid helium. Two methods are employed to calculate the wave functions and binding energies. A variational method which employs hydrogenic functions is used for states in which the meson is in a circular orbit. A more elaborate method, which makes use of the Born-Oppenheimer approximation and which is applicable to states with meson orbits which are not necessarily circular, is used for circular and nearly circular orbits of a few levels of the $\alpha\pi^-e^-$ and $\alpha K^-e^-$ atoms. It is found, in particular, that an $E4$ or higher multipole transition is required for an Auger electron to be ejected by a $K^-$ meson in a circular orbit of the $\alpha K^-e^-$ atom with $n=27$, 28, or 29. It is found that for levels of the $\alpha K^-e^-$ atom with $n=27$, 28, or 29 the magnitude of the energy difference between two states with the same value of $n$, but with orbital angular momenta $(n-2)$ and $(n-1)$, is roughly 0.5 eV. It is also found that, in some instances, replacing the absolute square of the meson wave function with a $\delta$ function does not significantly affect the accuracy of a calculation.

## I. INTRODUCTION

The purpose of this paper is to calculate reasonably accurate approximations to the wave functions and binding energies for some states of the neutral mesonic atoms which are formed when $\pi^-$ and $K^-$ mesons and antiprotons are brought to rest in liquid helium. For the sake of brevity, an antiproton is frequently referred to in this paper as a meson. In each instance considered in the present paper, the electron in the mesonic atom may be thought of as being in a 1$s$ orbit, and the meson may be thought of as being in an orbit which is either circular or very nearly circular. A circular atomic orbit is one for which $l=(n-1)$, where $l$ is the orbital angular momentum, and $n$ is the principal quantum number. For the sake of brevity, a state of the mesonic atom in which the meson is in a circular or a nearly circular orbit, and in which the electron is in a 1$s$ orbit, is frequently referred to in this paper simply as a circular or a nearly circular orbit. Since actually the electron and the meson share, to some extent, the total orbital angular momentum of the atom, it would be more precise to say that the so-called circular orbits being considered here are states of the mesonic atom in which, for a given value of the total orbital angular momentum, the system has the lowest possible energy.

The results which are obtained in this paper are used in some related papers to calculate Auger rates[1] for circular orbits of isolated $\alpha\pi^-e^-$, $\alpha K^-e^-$, and $\alpha\bar{p}e^-$ atoms and to estimate the rate with which Stark-effect transitions[2] occur between some circular and almost circular orbits of the $\alpha K^-e^-$ atom during a collision with a neutral helium atom. These investigations are motivated by a suggestion of Condo,[3] who sought to account for the large experimental values of the mean cascade times for $\pi^-$ and $K^-$ mesons in liquid helium[4,5] by supposing that some circular orbits of the $\alpha\pi^-e^-$ and $\alpha K^-e^-$ atoms may have anomalously long lifetimes.

In order to calculate the Auger rates and to investigate the interaction with a helium atom, it is convenient to be able to describe the mesonic atom with an approximate wave function which has a fairly tractable analytic form. For that reason, some simple variational wave functions are calculated in Sec. II using hydrogenic functions. However, the Stark rate for a circular orbit depends not only on the atomic transition matrix element, but also on the energy difference $\epsilon$ between two states of the mesonic atom which have the same principal quantum number $n$, and which have orbital angular momentum $(n-1)$ in one state and $(n-2)$ in the other.[3,6] In order to provide a basis for determining the energy difference $\epsilon$, and also to determine the extent to which the variational wave functions obtained in Sec. II are reliable, some rather more elaborate calculations are performed in Sec. III, and also in an accompa-

nying paper,[7] for a few cases of special interest. In Sec. IV of the present paper an investigation is made of the extent to which the meson may be regarded as a classical particle, since in a number of instances in this paper, and also in Refs. 2 and 7, calculations are simplified by replacing the absolute square of the meson wave function with a $\delta$ function. Some of the results which are obtained in Secs. III and IV and in Ref. 7 are employed in Sec. V of the present paper to estimate the values of the energy difference $\epsilon$ for a few levels of the $\alpha K^- e^-$ and $\alpha \bar{p} e^-$ atoms. Energy is expressed in rydbergs throughout this paper; the unit of distance is the hydrogen Bohr radius; and the unit of mass is the electron mass.

## II. VARIATIONAL CALCULATION

### A. Introduction

It is assumed that all of the constituent particles of the mesonic atom are nonrelativistic and spinless. The states of the atom are then represented by solutions of the equation

$$H_a \Psi_a(\vec{r}_1, \vec{r}_\mu) = E_a \Psi_a(\vec{r}_1, \vec{r}_\mu) , \tag{2.1a}$$

$$H_a = -(1-\xi)^{-1}\nabla_1^2 - \frac{4}{r_1} - \frac{1}{M}\left[1+\xi\left(\frac{M}{m_\mu}-1\right)\right]\nabla_\mu^2 - \frac{4}{|\vec{r}_\mu + \xi\vec{r}_1|} + \frac{2}{|(1-\xi)\vec{r}_1 - \vec{r}_\mu|} . \tag{2.1b}$$

In Eq. (2.1), the position of the electron with respect to the $\alpha$ particle is denoted by $\vec{r}_1$; the position of the meson with respect to the c.m. of the electron and $\alpha$ particle is denoted by $\vec{r}_\mu$; the mass of the meson is denoted by $m_\mu$; and the reduced mass of the meson and $\alpha$ particle is denoted by $M$. The quantity $\xi$, which is a measure of the displacement of the $\alpha$ particle from the c.m. of the electron and $\alpha$ particle, is defined by

$$\xi = (1+m_\alpha)^{-1} , \tag{2.2}$$

where $m_\alpha$ is the mass of the $\alpha$ particle. The purpose of the subscript $a$ is to distinguish the Hamiltonian, the wave function, and the binding energy in Eq. (2.1) from those which appear in the approximate calculations presented below.

For the sake of mathematical simplicity, the quantity $\xi$, which is quite small, is first set equal to zero, and the somewhat simpler equation

$$H_b \Psi_b(\vec{r}_1, \vec{r}_\mu) = E_b \Psi_b(\vec{r}_1, \vec{r}_\mu) , \tag{2.3a}$$

$$H_b = -\nabla_1^2 - \frac{4}{r_1} - \frac{1}{M}\nabla_\mu^2 - \frac{4}{r_\mu} + \frac{2}{|\vec{r}_1 - \vec{r}_\mu|} , \tag{2.3b}$$

is considered. The effect of the finite value of $\xi$ is estimated in Sec. II D by using perturbation theory.

### B. Method of Calculation and Numerical Results

An approximate wave function for a circular orbit may be readily found by a variational method, since a trial wave function can be easily constructed which is orthogonal to the wave functions for all lower-lying states. The variational wave function, which it is convenient to denote by $\Psi_c^{\,v}$, is chosen to be the product of two normalized hydrogenic functions which are centered on the $\alpha$ particle. One of these hydrogenic functions, which is denoted by $u_A(\vec{r}_1)$, and which is characterized by the effective nuclear charge $z$, describes an electron in a $1s$ state. The other function, which is denoted by $\phi_I(\vec{r}_\mu)$, and which is characterized by the effective nuclear charge $Z$, describes a meson of reduced mass $M$ in an orbit with principal quantum number $n$ and orbital angular momentum $(n-1)$. The two effective charges are the variational parameters. The values of $z$ and $Z$ are found by minimizing the energy $E_c^{\,v}$, which is defined by

$$E_c^{\,v} = \int \Psi_c^{\,v*} H_b \Psi_c^{\,v} \, d\tau_1 \, d\tau_\mu , \tag{2.4}$$

and which can be shown to be given in terms of $z$ and $Z$ by

$$E_c^{\,v} = -z(4-z) - (MZ/n^2)(2-Z) - 2MzZ[MZ/(MZ+nz)]^{2n} \times [(n^2 z)^{-1} + (MZ+nz)^{-1}] . \tag{2.5}$$

Numerical results are given for $\alpha\pi^- e^-$, $\alpha K^- e^-$, and $\alpha \bar{p} e^-$ atoms. The values of $z$ and $Z$ are shown in Fig. 1 as functions of $n$. The values of $E_c^{\,v}$ are shown in Fig. 2. The mean values of $r_1$ and $r_\mu$, which are denoted by $\bar{r}_1$ and $\bar{r}_\mu$, are given in terms of the effective charges by

$$\bar{r}_1 = 3/(2z) , \tag{2.6a}$$

$$\bar{r}_\mu = n(n+\tfrac{1}{2})/(MZ) . \tag{2.6b}$$

The rms velocities of the electron and the meson

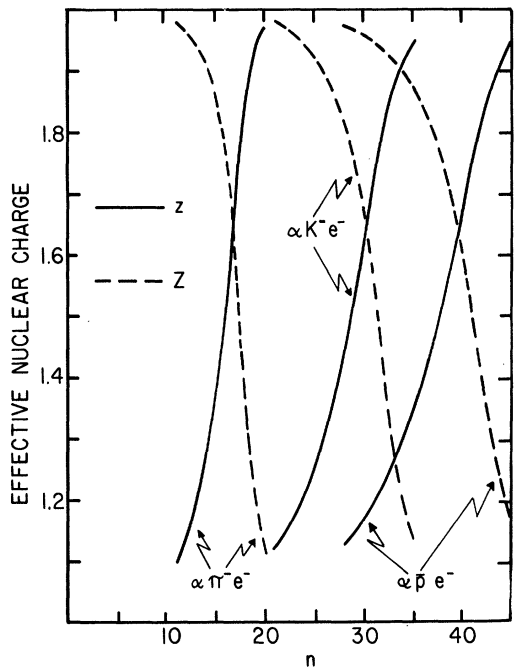


FIG. 1. Effective nuclear charges $z$ and $Z$ for the electron and the meson as functions of the meson principal quantum number $n$.

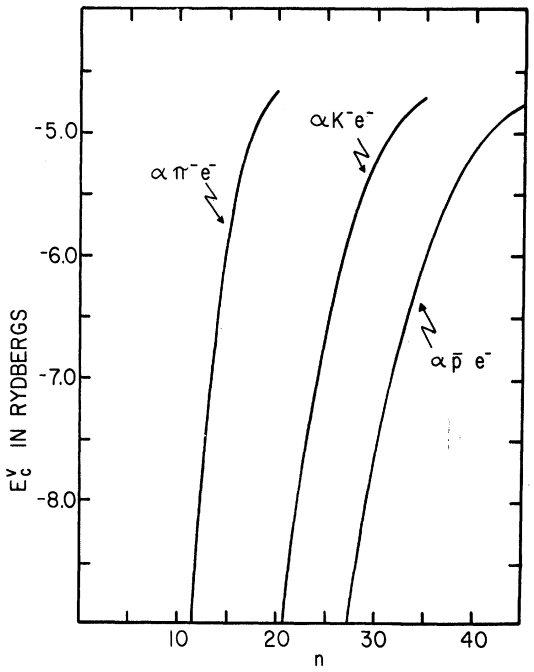


FIG. 2. Variational binding energy $E_c^v$ of the mesonic atom as a function of the meson principal quantum number $n$.

are denoted by $\bar{v}_1$ and $\bar{v}_\mu$, and they are given by

$$\bar{v}_1 = z\alpha c \,, \tag{2.7a}$$

$$\bar{v}_\mu = (Z/n)\alpha c \,, \tag{2.7b}$$

where $\alpha$ is the fine structure constant, and $c$ is the velocity of light. Unless the principal quantum number of the meson is very small, the electron moves much more rapidly than the meson.

### C. Multipolarities of Lowest Energetically Allowed Auger Transitions

Condo argued that the Auger transitions of mesons in circular orbits might have to be of a relatively high multipole order, thereby causing these states to have long lifetimes. Although the Auger rates are computed elsewhere,[1] it is convenient at this point to determine the multipolarities. The lowest possible multipole order is determined by energy conservation if the meson is initially in a circular orbit. It is equal to the lowest value of $L$ for which

$$E_a + 4M/(n-L)^2 > 0 \,, \tag{2.8}$$

where $E_a$ is the binding energy of the initial state of the mesonic atom, as specified by Eq. (2.1), and where $n$ is the principal quantum number of the meson in the initial state.

Since $E_a$ cannot be calculated exactly, the lowest possible multipole order is estimated with an approximate method. Table I shows the range of possible values of $n$ for which a particular value of $L$ is found to be the lowest which satisfies the inequality

$$E_c^v + 4M/(n-L)^2 > 0 \,. \tag{2.9}$$

TABLE I. The values of the principal quantum number $n$ for which $L$ is found to be a lower limit to the multipolarity of the lowest energetically allowed Auger transition from a circular orbit.

| Atom | $L=1$ | $L=2$ | $L=3$ | $L=4$ | $L=5$ |
|---|---|---|---|---|---|
| $\alpha\pi^-e^-$ | $n\leq 12$ | $13\leq n\leq 15$ | $16\leq n\leq 17$ | $n=18$ | $19\leq n\leq 20$ |
| $\alpha K^-e^-$ | | $21\leq n\leq 23$ | $24\leq n\leq 26$ | $27\leq n\leq 29$ | $n=30$ |
| $\alpha\bar{p}e^-$ | | | $29\leq n\leq 32$ | $33\leq n\leq 35$ | $36\leq n\leq 37$ |

It is only necessary to show that the actual values of the lowest allowed multipole order are not lower than those indicated in Table I. It is necessary, though not sufficient, that the inequality

$$E_a > E_c^{v} \tag{2.10}$$

be satisfied in any instance in which the multipole order indicated in Table I is not actually the lowest one possible. However, since $E_c^{v}$ is an approximate value of $E_b$, and since $E_a$ differs from $E_b$ only in that $E_a$ takes into account the finite value of $\xi$, it is more convenient to determine whether or not the equivalent inequality

$$(E_a - E_b) > (E_c^{v} - E_b) \tag{2.11}$$

is satisfied. The energy difference $(E_c^{v} - E_b)$, which is necessarily positive, but which is rather difficult to calculate, is estimated for a few cases of special interest in Sec. III. The energy difference $(E_a - E_b)$ is estimated in Sec. IID. Although $(E_a - E_b)$ is found to be positive also, the results of these calculations indicate that it is never large enough for the preceding inequality to be satisfied.

### D. Energy Difference $(E_a - E_b)$

According to first-order perturbation theory, the energy difference $(E_a - E_b)$ is given by

$$E_{ab} = \int \Psi_b^{*} H_{ab} \Psi_b \, d\tau_1 \, d\tau_\mu \,, \tag{2.12}$$

where $H_{ab}$ is defined by

$$H_{ab} = H_a - H_b \,. \tag{2.13}$$

Insofar as $\Psi_b$ is represented adequately by $\Psi_c^{v}$, this energy difference is given approximately by

$$E_{ab} \simeq \int \Psi_c^{v*} H_{ab} \Psi_c^{v} \, d\tau_1 \, d\tau_\mu \,. \tag{2.14}$$

The numerical values of this approximate expression for $(E_a - E_b)$ are found to be between $2\times10^{-4}$ and $6\times10^{-4}$ Ry for the circular orbits which are being considered in the present paper.

## III. MORE ACCURATE CALCULATION USING BORN-OPPENHEIMER APPROXIMATION

### A. Introduction

It seems reasonable to assume that, apart from the calculation which is described in Sec. IID, the investigation of the adequacy of $\Psi_c^{v}$ and $E_c^{v}$ as approximations to $\Psi_a$ and $E_a$ may be carried out by determining how satisfactorily they approximate $\Psi_b$ and $E_b$. In particular, it is desirable to have an estimate of the degree to which the motion of the electron is correlated with the position of the much more massive and more slowly moving meson, and to have an estimate of the energy difference $(E_c^{v} - E_b)$. Also, in order to be able to calculate the Stark transition rate, it is necessary to obtain a reliable estimate of the energy difference between two states of the mesonic atom which have the same principal quantum number $n$, but which have orbital angular momenta $(n-1)$ and $(n-2)$.

Because $u_A$ is a $1s$ state, the function $\Psi_c^{v}$ is also the solution which is obtained if the same variational method which is described in Sec. II is applied to the eigenvalue problem

$$H_c \Psi_c(\vec{r}_1, \vec{r}_\mu) = E_c \Psi_c(\vec{r}_1, \vec{r}_\mu) \,, \tag{3.1a}$$

$$H_c = -\nabla_1^2 - \frac{4}{r_1} - \frac{1}{M}\nabla_\mu^2 - \frac{4}{r_\mu} + \begin{pmatrix} 2/r_\mu, & r_1 < r_\mu \\ 2/r_1, & r_1 > r_\mu \end{pmatrix}. \tag{3.1b}$$

It is, therefore, convenient to divide the investigation of the adequacy of $\Psi_c^{v}$ and $E_c^{v}$ as approximations to $\Psi_b$ and $E_b$ into two parts. One part, which is given below, is concerned primarily with determining how accurately $\Psi_c^{v}$ and $E_c^{v}$ approximate $\Psi_c$ and $E_c$. The other part, which is given in Ref. 7, is concerned with taking into account the interaction

$$H_{bc} = H_b - H_c \,. \tag{3.2}$$

Since the Hamiltonian $H_c$ takes into account only the monopole interaction of the electron with the meson, the angular momentum operators $(\vec{r}_1 \times \vec{p}_1)$ and $(\vec{r}_\mu \times \vec{p}_\mu)$ for these two particles both commute with $H_c$. Only eigenstates of $H_c$ in which the electron is in a $1s$ state are being considered in the present paper. Therefore, in each of these eigenstates the correlation of the position of the electron with that of the meson depends on the radial coordinates $r_1$ and $r_\mu$, but not on the angular coordinates $\hat{r}_1$ and $\hat{r}_\mu$.

### B. Description of Born-Oppenheimer Approximation and Justification for its Use

A fairly accurate solution to Eq. (3.1) probably can be found by employing the Born-Oppenheimer approximation. If it is assumed that the motion of the electron depends on the position of the meson, but not on its velocity, the solution to Eq. (3.1) is given by

$$\Psi_c{}^d(\vec{r}_1, \vec{r}_\mu) = \phi_c{}^d(\vec{r}_\mu) u_{de}(\vec{r}_1, r_\mu) , \tag{3.3}$$

where the meson wave function $\phi_c{}^d(\vec{r}_\mu)$ is defined by the eigenvalue equation

$$H_c{}^d \phi_c{}^d(\vec{r}_\mu) = E_c{}^d \phi_c{}^d(\vec{r}_\mu) , \tag{3.4a}$$

$$H_c{}^d = -\frac{1}{M}\nabla_\mu{}^2 - \frac{4}{r_\mu} + E_{de}(r_\mu) . \tag{3.4b}$$

The quantity $E_{de}(r_\mu)$ in the expression for $H_c{}^d$ is defined by the eigenvalue equation

$$H_{de} u_{de}(\vec{r}_1, r_\mu) = E_{de}(r_\mu) u_{de}(\vec{r}_1, r_\mu) , \tag{3.5a}$$

$$H_{de} = -\nabla_1{}^2 - \frac{4}{r_1} + \begin{pmatrix} 2/r_\mu , & r_1 < r_\mu \\ 2/r_1 , & r_1 > r_\mu \end{pmatrix} . \tag{3.5b}$$

The eigenvalue $E_c{}^d$ in Eq. (3.4) is the approximate value of $E_c$. Unlike the variational method which is described in Sec. II, this method may be readily applied to orbits which are not circular.

The use of the foregoing relations to obtain approximate solutions to Eq. (3.1) is justified if the electron moves much more rapidly than the meson. Since the correlation of the positions of these two particles with respect to each other in an eigenstate of $H_c$ depends only on their radial coordinates $r_1$ and $r_\mu$, the velocities which should be cited in establishing the validity of approximating $\Psi_c$ and $E_c$ with $\Psi_c{}^d$ and $E_c{}^d$ are the rms radial velocities, which are found by computing the rms radial momenta.[8] It is easily shown that the rms radial velocity $\bar{v}_{\mu, r}$ of a meson which has principal quantum number $n$ and orbital angular momentum $l$, and which is described by a hydrogenic wave function, is given by

$$\bar{v}_{\mu, r} = \bar{v}_\mu \left( \frac{(2l+1)(n-l-1) + (l+1)}{n(2l+1)} \right)^{1/2} , \tag{3.6}$$

where $\bar{v}_\mu$ is its rms velocity. The velocity $\bar{v}_\mu$ is given by Eq. (2.7b) for a hydrogenic state with effective nuclear charge $Z$ and principal quantum number $n$. If the meson is in a circular orbit, Eq. (3.6) takes the form

$$\bar{v}_{\mu, r} = (2n-1)^{-1/2} \bar{v}_\mu , \quad l = n-1 . \tag{3.7}$$

If the electron is also in a hydrogenic state, its rms radial velocity $\bar{v}_{1, r}$ is given by an expression which is similar to Eq. (3.6). And if the electron is in an $s$ state, as it is in every instance which is considered in this paper, this expression takes the form

$$\bar{v}_{1, r} = \bar{v}_1 , \quad l = 0 \tag{3.8}$$

where $\bar{v}_1$ is the rms velocity. Equation (2.7a) gives an explicit expression for $\bar{v}_1$ in terms of the effective nuclear charge $z$ which is experienced by a 1$s$ electron. The relative values of $\bar{v}_1$ and $\bar{v}_\mu$, which are determined by the variational calculation given in Sec. II, indicate that $\bar{v}_{1, r}$ is two orders of magnitude larger than $\bar{v}_{\mu, r}$ for most of the circular orbits which are being considered in the present paper. It is easily verified that the value of the ratio $\bar{v}_{1, r}/\bar{v}_{\mu, r}$ for a circular orbit differs, in most instances, by less than a factor of 2 from its value for an almost circular orbit with $l = (n-2)$. It is assumed that the preceding arguments constitute sufficient justification for using the Born-Oppenheimer approximation to solve Eq. (3.1) for orbits with $l = (n-1)$ or $l = (n-2)$.

### C. Additional Approximation and Method of Calculation

So as to avoid an excessive amount of numerical work, the solution to Eq. (3.5) is approximated with a variational wave function. This variational wave function, which is denoted by $u_{dev}$, is chosen to be a normalized 1$s$ hydrogenic function. It is characterized by an effective nuclear charge $z_v$. This effective charge is the variational parameter. The value of $z_v$, which depends on $r_\mu$, is found by minimizing the expectation value of the operator $H_{de}$. This expectation value, which is also a function of $r_\mu$, is denoted by $E_{dev}$, and it is given in terms of $z_v$ and $r_\mu$ by

$$E_{dev}(r_\mu) = -z_v(4 - z_v) + 2r_\mu{}^{-1}[1 - (1 + z_v r_\mu)e^{-2z_v r_\mu}] . \tag{3.9}$$

It can be shown that, for a given value of $r_\mu$, the value of $z_v$ which minimizes the expectation value of $H_{de}$ satisfies the equation

$$(2 + 4z_v r_\mu)e^{-2z_v r_\mu} + 2z_v - 4 = 0 . \tag{3.10}$$

The solution to Eq. (3.10) may be readily obtained by numerical integration of the differential equation

$$\frac{dz_v}{dr_\mu} = \frac{4z_v{}^2 r_\mu e^{-2z_v r_\mu}}{1 - 4z_v r_\mu{}^2 e^{-2z_v r_\mu}} , \tag{3.11}$$

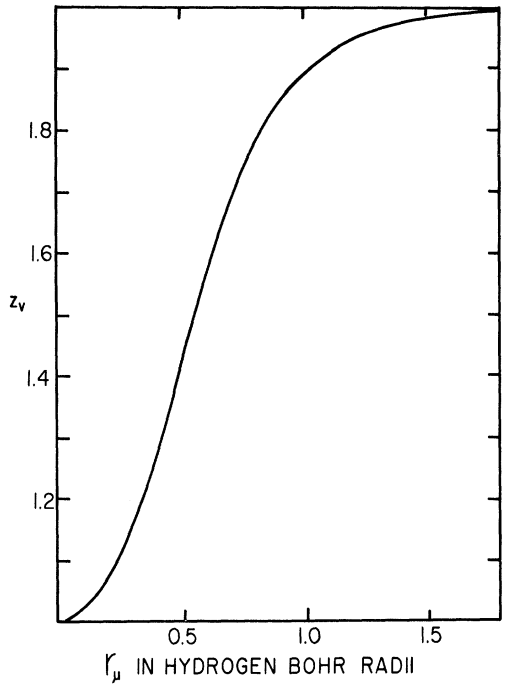


FIG. 3. Effective nuclear charge $z_v$ for the electron as a function of the meson radial coordinate $r_\mu$.

subject to the boundary condition $z_v = 1$ at $r_\mu = 0$. Values of $z_v$ are shown in Fig. 3.

The wave function $\Psi_c$ is now assumed to be given approximately by

$$\Psi_c^{dv}(\vec{r}_1, \vec{r}_\mu) = \phi_c^{dv}(\vec{r}_\mu) u_{dev}(\vec{r}_1, r_\mu) , \qquad (3.12)$$

where $\phi_c^{dv}$ is a solution to the equation

$$H_c^{dv} \phi_c^{dv}(\vec{r}_\mu) = E_c^{dv} \phi_c^{dv}(\vec{r}_\mu) , \qquad (3.13a)$$

$$H_c^{dv} = -\frac{1}{M}\nabla_\mu^2 - \frac{4}{r_\mu} + E_{dev}(r_\mu) . \qquad (3.13b)$$

The eigenvalue $E_c$ is assumed to be given rather accurately by $E_c^{dv}$. It is convenient to express the meson wave function $\phi_c^{dv}$ in the form

$$\phi_c^{dv}(\vec{r}_\mu) = (1/r_\mu)\chi_c^{dv}(r_\mu) Y_{lm}(\hat{r}_\mu) . \qquad (3.14)$$

The radial wave function $\chi_c^{dv}$ satisfies the differential equation

$$\left(-\frac{1}{M}\frac{d^2}{dr_\mu^2} + \frac{1}{M}\frac{l(l+1)}{r_\mu^2} - \frac{4}{r_\mu} + E_{dev}(r_\mu)\right)\chi_c^{dv}(r_\mu)$$

$$= E_c^{dv}\chi_c^{dv}(r_\mu) . \qquad (3.15)$$

This equation can be integrated numerically, and the allowed values of $E_c^{dv}$ can be determined if the behavior of $\chi_c^{dv}$ and its first derivative are known for both very large and very small values of $r_\mu$.

It is easily shown that near the origin the effective charge $z_v$ is given quite accurately by

$$z_v \simeq 1 + 2r_\mu^2 - \tfrac{8}{3}r_\mu^3 + 10 r_\mu^4, \quad r_\mu \ll 1. \qquad (3.16)$$

To the same order of approximation, the differential equation for $\chi_c^{dv}$ is given by

$$\left(-\frac{1}{M}\frac{d^2}{dr_\mu^2} + \frac{1}{M}\frac{l(l+1)}{r_\mu^2} - \frac{4}{r_\mu} - 1 - \tfrac{4}{3}r_\mu^2 + \tfrac{4}{3}r_\mu^3 - \tfrac{24}{5}r_\mu^4\right)\chi_c^{dv}(r_\mu)$$

$$\simeq E_c^{dv}\chi_c^{dv}(r_\mu), \quad r_\mu \ll 1 . \qquad (3.17)$$

A straightforward calculation then shows that, to the same order of approximation, the function $\chi_c^{dv}$ is given near the origin, to within a multiplicative constant, by

$$\chi_c^{dv}(r_\mu) \propto r_\mu^{l+1} e^{-2Mr_\mu/n'} \times \sum_{\nu=0}^{6} C_\nu r_\mu^\nu, \quad r_\mu \ll 1 , \qquad (3.18)$$

where $n' = 2[-M/(1 + E_c^{dv})]^{1/2}$ , (3.19)

$$C_0 = 1 , \qquad (3.20a)$$

$$C_1 = -2M[(l+1)n']^{-1}(n' - l - 1)C_0 , \qquad (3.20b)$$

$$C_2 = -2M[(2l+3)n']^{-1}(n' - l - 2)C_1 , \qquad (3.20c)$$

$$C_3 = -2M[(3l+6)n']^{-1}(n' - l - 3)C_2 , \qquad (3.20d)$$

$$C_4 = -2M[(4l+10)n']^{-1} \times [\tfrac{1}{3}n'C_0 + (n' - l - 4)C_3], \qquad (3.20e)$$

$$C_5 = -2M[(5l+15)n']^{-1}[-\tfrac{1}{3}n'C_0 + \tfrac{1}{3}n'C_1 + (n' - l - 5)C_4], \qquad (3.20f)$$

$$C_6 = -2M[(6l+21)n']^{-1}[\tfrac{6}{5}n'C_0 - \tfrac{1}{3}n'C_1$$

$$+\tfrac{1}{3}n'C_2+(n'-l-6)C_5]. \qquad (3.20g)$$

The preceding expression for $\chi_c^{dv}$ is carried to such a high order of approximation only as a matter of convenience: Since the wave function varies quite rapidly with $r_\mu$ near the origin, it is simply less troublesome to start the outward integration at a relatively large value of $r_\mu$ in order not to exceed the normal range of the computer.

The inward numerical integration of Eq. (3.15) is started at a point beyond which $E_{dev}$ is given to a high degree of accuracy by

$$E_{dev}(r_\mu)\simeq -4+2r_\mu^{-1}, \quad r_\mu \gg 1 . \qquad (3.21)$$

For these large values of $r_\mu$, the differential equation for $\chi_c^{dv}$ describes motion in a pure Coulomb field. The starting value of $\chi_c^{dv}$ is, therefore, proportional to a confluent hypergeometric function. This function and its first derivative with respect to $r_\mu$ are readily computed using asymptotic series.

### D. Numerical Results and Discussion

Numerical calculations were performed for only a few cases because a fairly large amount of computer time was required for the determination of each value of $E_c^{dv}$. Only the $\alpha\pi^-e^-$ and $\alpha K^-e^-$ atoms were considered. Of the two, the $\alpha\pi^-e^-$ atom was investigated more extensively because the somewhat less rapid variation of its radial meson wave function near the origin required less computer time. The values of $E_c^{dv}$ and $(E_c^{dv}-E_c^{v})$ for several circular orbits are given in Table II. For a given value of $n$, the difference $\epsilon_c^{dv}$ between the values of $E_c^{dv}$ for a circular and a nearly circular orbit is defined by

$$\epsilon_c^{dv}=E_{c,n-2}^{dv}-E_{c,n-1}^{dv}, \qquad (3.22)$$

where the dependence on $l$ of the eigenvalue of Eq. (3.15) is indicated by denoting it by $E_{c,l}^{dv}$, rather than by $E_c^{dv}$. Table II also gives values of $\epsilon_c^{dv}$.

Table II also gives the values, for several circular orbits, of $\bar{r}_\mu^{dv}$, which is defined by

$$\bar{r}_\mu^{dv}=\int_0^\infty[\chi_c^{dv}(r_\mu)]^2 r_\mu\, dr_\mu , \qquad (3.23)$$

and which is the mean radius of the meson orbit if the meson wave function is $\phi_c^{dv}(\vec{r}_\mu)$, as specified by Eqs. (3.13) and (3.14). These seven values of $\bar{r}_\mu^{dv}$ are equal, to within 1.5%, to the corresponding values of the mean radius $\bar{r}_\mu$, which are computed using the hydrogenic wave functions determined by the variational calculation given in Sec. II. Finally, Table II gives some values of $\Delta\bar{r}_\mu^{dv}$, which is the difference, for a given value of $n$, between the values of $\bar{r}_\mu^{dv}$ for a circular and a nearly circular orbit, and which is defined by

$$\Delta\bar{r}_\mu^{dv}=\bar{r}_{\mu,n-2}^{dv}-\bar{r}_{\mu,n-1}^{dv}, \qquad (3.24)$$

where the dependence on the orbital angular momentum $l$ is indicated by denoting the mean radius by $\bar{r}_{\mu,l}^{dv}$ rather than by $\bar{r}_\mu^{dv}$. These values of $\Delta\bar{r}_\mu^{dv}$ are used in Sec. V to estimate the values of the correction to $\epsilon_c^{dv}$ which is due to the interaction $H_{bc}$, and they are also referred to in Sec. IV, where an investigation is made of the adequacy of an approximation which is employed elsewhere in this paper and also, in some instances, in Refs. 2 and 7.

The relatively small values of the energy difference $(E_c^{dv}-E_c^{v})$ given in Table II indicate that the variational energy $E_c^{v}$ is probably a very good approximation to $E_c$. Although the magnitude of the energy correction $E_{bc}=(E_b-E_c)$, which is estimated for a few circular orbits of the $\alpha K^-e^-$ atom in Ref. 7, is larger than that of $(E_c^{dv}-E_c^{v})$, it, too, does not lower the binding energy by a substantial amount. For example, taking both of these two corrections into account lowers

TABLE II. A summary of the results of the calculation which takes into account only the monopole contribution to the interaction of the electron with the meson and which employs the Born-Oppenheimer approximation. The unit of energy is the rydberg, and the unit of distance is the hydrogen Bohr radius.

| atom | $n$ | $E_c^{dv}$ | $(E_c^{dv}-E_c^{v})$ | $\epsilon_c^{dv}$ | $\bar{r}_\mu^{dv}$ | $\Delta\bar{r}_\mu^{dv}$ |
|---|---|---|---|---|---|---|
| $\alpha\pi^-e^-$ | 18 | −4.9238 | −0.0111 | −0.0635 | 0.920 | 0.022 |
| | 17 | −5.1491 | −0.0167 | −0.0744 | 0.715 | 0.036 |
| | 16 | −5.4865 | −0.0157 | −0.0669 | 0.575 | 0.045 |
| | 15 | −5.9541 | −0.0116 | −0.0546 | 0.478 | 0.038 |
| | 14 | −6.5730 | −0.0077 | | 0.404 | |
| $\alpha K^-e^-$ | 29 | −5.4267 | −0.0091 | −0.0415 | 0.575 | 0.027 |
| | 28 | | | | | |
| | 27 | −5.9420 | −0.0064 | −0.0323 | 0.468 | 0.022 |

the estimates of the binding energies of the circular orbits of the $\alpha K^- e^-$ atom with $n=27$ and 29 from $E_c{}^v = -5.94$ and $-5.42$ Ry, respectively, to $E_b \simeq -6.00$ and $-5.50$ Ry. It is, therefore, concluded that using the variational energy $E_c{}^v$ causes the kinetic energy of the Auger electron to be overestimated by less than about 0.1 Ry. The magnitudes of these two corrections are found to indicate that in only one instance, the $\alpha K^- e^-$ atom with $n=29$, is the value of $L$ given in Table I lower than the multipolarity of the lowest energetically allowed Auger transition.

Figures 4 and 5 show the meson radial wave functions $\chi_c{}^{dv}$ for a circular orbit of the $\alpha\pi^- e^-$ atom with $n=18$ and a circular orbit of the $\alpha K^- e^-$ atom with $n=27$. For the purpose of comparison, the corresponding hydrogenic radial wave function $\chi_c{}^v$, which is characterized by the effective nuclear charge $Z$ determined in Sec. II, is shown in both instances. These particular states of the two atoms have rather different relative values of the mean radii $\bar{r}_1$ and $\bar{r}_\mu$ of the electron and meson orbits. As computed using the variational wave functions which are determined in Sec. II, the $\alpha\pi^- e^-$ atom with $n=18$ has $\bar{r}_1 = 0.815$ and $\bar{r}_\mu = 0.923$, whereas the $\alpha K^- e^-$ atom with $n=27$ has $\bar{r}_1 = 1.103$ and $\bar{r}_\mu = 0.465$. Figures 4 and 5 also show $\chi_f$, which is the similarly defined radial wave function for the meson after an Auger electron with orbital angular momentum $L$, as specified by Table I, has been ejected and the meson has been left in a hydrogenic circular orbit with principal quantum number $(n-L)$ and nuclear charge $Z_f = 2$. In both instances, and particularly in the case of the $\alpha K^- e^-$ atom with $n=27$, the variational wave function $\chi_c{}^v$ appears, in some respects, to be an acceptable approximation to $\chi_c{}^{dv}$. However, it should be noted that although the function $\chi_c{}^v$ is, in the case of the $\alpha K^- e^-$ atom with $n=27$, a fairly good approximation to $\chi_c{}^{dv}$ in the region where it overlaps to the greatest extent with $\chi_f$, the same is not true of the $\alpha\pi^- e^-$ atom with $n=18$. The accuracy of $\chi_c{}^v$ for relatively small values of $r_\mu$ increases as the ratio $\bar{r}_\mu/\bar{r}_1$ decreases.

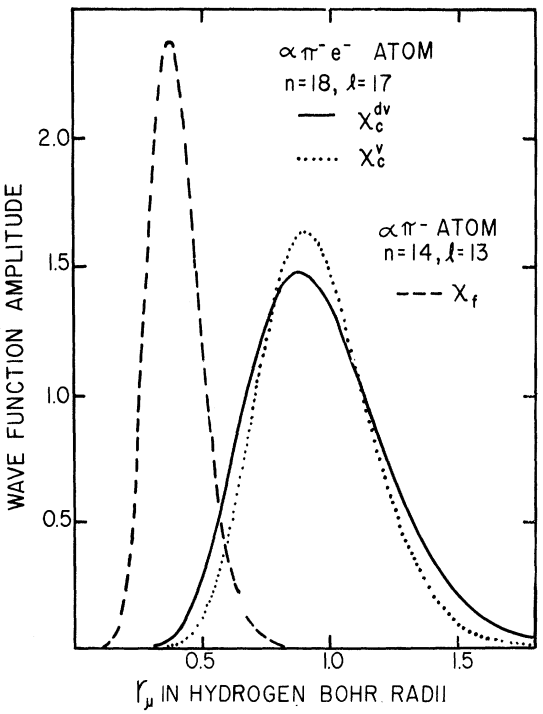


FIG. 4. Radial meson wave functions. The functions $\chi_c{}^{dv}$ and $\chi_c{}^v$ are for a circular orbit of the $\alpha\pi^- e^-$ atom with principal quantum number $n=18$. The function $\chi_f$ is for the circular orbit of the $\alpha\pi^-$ atom with $n=14$ in which the meson is left after the electron has been ejected in an $E4$ Auger transition.

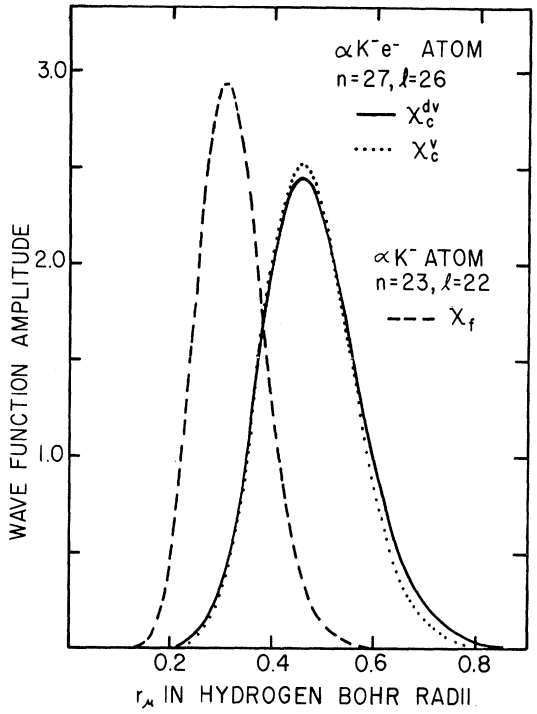


FIG. 5. Radial meson wave functions. The functions $\chi_c{}^{dv}$ and $\chi_c{}^v$ are for a circular orbit of the $\alpha K^- e^-$ atom with principal quantum number $n=27$. The function $\chi_f$ is for the circular orbit of the $\alpha K^-$ atom with $n=23$ in which the meson is left after the electron has been ejected in an $E4$ Auger transition.

## IV. SEMICLASSICAL APPROXIMATION

### A. Introduction

In this paper, and also in Refs. 2 and 7, the meson is frequently treated as a classical particle for the sake of mathematical simplicity. In these in-

stances it is assumed to be a point charge which is situated a distance $\bar{r}_\mu$ from the $\alpha$ particle. In this section, the use of this procedure is justified by demonstrating that it would not appreciably alter the results which are obtained by the variational calculation given in Sec. II. In addition, it is shown that it is possible to use this procedure in many instances to obtain rough estimates of the quantities $\epsilon_c{}^{dv}$ and $\Delta\bar{r}_\mu{}^{dv}$ which are defined by Eqs. (3.22) and (3.24).

### B. Approximate Values of Binding Energies and Effective Nuclear Charges

The expression for the variational energy $E_c{}^v$, which is given explicitly in terms of the effective charges $z$ and $Z$ by Eq. (2.5), can be left in the somewhat more general form

$$E_c{}^v = -z(4-z) - (MZ/n^2)(4-Z) + 2\int |\phi_I(\vec{r}_\mu)|^2 r_\mu{}^{-1} \times [1-(1+zr_\mu)\exp(-2zr_\mu)]d\tau_\mu . \quad (4.1)$$

The last term in this expression is the energy which is due to the interaction of the electron with the meson. If the meson is now regarded as an almost classical particle which moves in a circular orbit, and if the radius of this orbit is assumed to be approximately equal to $\bar{r}_\mu$, the probability density $|\phi_I(\vec{r}_\mu)|^2$ in Eq. (4.1) may then be replaced with a $\delta$ function which is centered on some point with $r_\mu = \bar{r}_\mu$, and $E_v{}^c$ is given approximately by

$$E_c{}^v \simeq -z(4-z) - (MZ/n^2)(4-Z) + 2\bar{r}_\mu{}^{-1}[1-(1+z\bar{r}_\mu)e^{-2z\bar{r}_\mu}]. \quad (4.2)$$

If the orbit of the meson has a well-defined radius $\bar{r}_\mu$, it seems reasonable to expect that the value of $z$ determined by the variational calculation in Sec. II should be given fairly accurately by

$$z \simeq z_v(\bar{r}_\mu) , \quad (4.3)$$

where $z_v(r_\mu)$ is the function of $r_\mu$ which is defined by Eq. (3.10), and where $\bar{r}_\mu$ is given in terms of $n$ and $Z$ by Eq. (2.6b). This approximate equality is found in every instance to be accurate to within about 1%. A semiclassical expression for the binding energy of the mesonic atom is now obtained by replacing $z$ with $z_v(\bar{r}_\mu)$ in the approximate expression for $E_c{}^v$. It is readily verified that this semiclassical energy, which is denoted by $E_c{}^{sc}$, is given by

$$E_c{}^{sc} = -(MZ/n^2)(4-Z) + E_{dev}(\bar{r}_\mu) , \quad (4.4)$$

where $E_{dev}$ is the function defined by Eq. (3.9). It is found in every instance that the values of $E_c{}^{sc}$ obtained by using those values of $Z$ shown in Fig. 1 differ at most by only a few tenths of a percent from the values of $E_c{}^v$ shown in Fig. 2. Finally, it should be remarked that the effective nuclear charges $z$ and $Z$ determined for the $\alpha K^- e^-$ and $\alpha\bar{p}e^-$ atoms by the variational calculation in Sec. II are found to be such that the values of $Z$ are equal, to within about 1%, to the values of the quantity

$$Z_{sc} = 2 - 4z^3 \int_0^{\bar{r}_\mu} r_1{}^2 e^{-2zr_1} dr_1 , \quad (4.5)$$

which is the difference between the charge of the $\alpha$ particle and the probability of finding the electron at a point with $r_1 < \bar{r}_\mu$ if the electron is described by a $1s$ hydrogenic wave function with an effective nuclear charge $z$. The values of $Z$ and $Z_{sc}$ for the $\alpha\pi^- e^-$ atom are found to differ at most by about 3%. Figure 6 shows the values of $Z_{sc}$ as

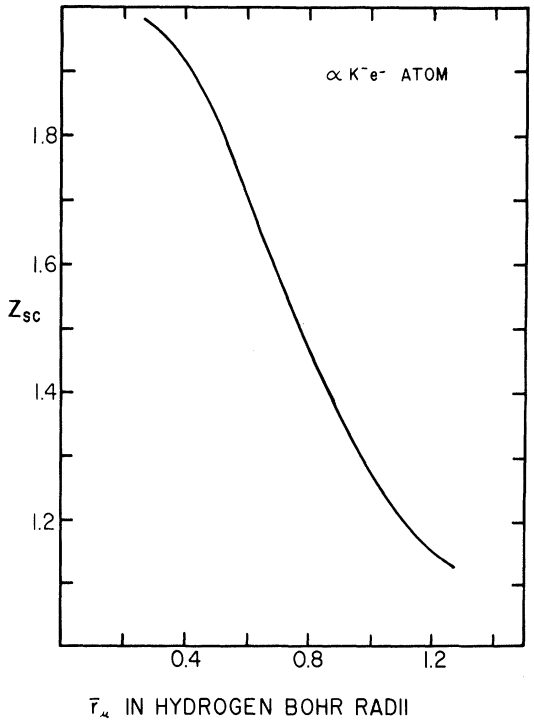


FIG. 6. The semiclassical estimate $Z_{sc}$ of the effective nuclear charge for the $K^-$ meson in a circular orbit of the $\alpha K^- e^-$ atom with mean meson orbital radius $\bar{r}_\mu$.

a function of $\bar{r}_\mu$ for the circular orbits of the $\alpha K^- e^-$ atom. The variation of $Z_{sc}$ with $\bar{r}_\mu$ for circular orbits is nearly the same for both the $\alpha\pi^- e^-$ and $\alpha\bar{p}e^-$ atoms as it is for the $\alpha K^- e^-$ atom.

### C. Rough Estimates of $\Delta\bar{r}_\mu{}^{dv}$ and $\epsilon_c{}^{dv}$

Insofar as it is possible to employ hydrogenic wave functions to describe states of the mesonic atom which differ from the ones considered above only in that, for a given value of the principal quantum number $n$, the meson orbital angular momentum $l$ is relatively large but not equal to $(n-1)$, the binding energy should still be given fairly accurately by Eq. (4.4), where, as before, $\bar{r}_\mu$ and $Z$ denote the mean radius and effective nuclear charge for the meson orbit. The values of both $\bar{r}_\mu$ and $Z$ for such states depend on $l$. That the value of $\bar{r}_\mu$ depends on $l$ is readily seen from the fact that the mean radius of a hydrogenic orbit is given in terms of $n$, $l$, and $Z$ by[9]

$$\bar{r}_\mu = [3n^2 - l(l+1)]/(2MZ)\,. \tag{4.6}$$

The effective charge $Z$ depends on $l$ because the degree to which the electron shields the meson from the field of the $\alpha$ particle depends on $\bar{r}_\mu$.

If $n$ and $Z$ are fixed, the value of $\bar{r}_\mu$ increases with decreasing $l$. For a given value of $n$, there is, in actual fact, a tendency in many instances for this increase in $\bar{r}_\mu$ with decreasing $l$ to be enhanced by a decrease in the effective charge $Z$, since the electron frequently becomes, on the average, more tightly bound to the nucleus. The mechanism which is responsible for this tendency can be discussed most conveniently by regarding the meson as a classical point charge which moves very slowly in an orbit which has a more or less elliptical shape. The wave function for the much more rapidly moving electron is then a function of the instantaneous position of the meson. At any given time the meson may be thought of as experiencing an instantaneous effective nuclear charge. This instantaneous effective charge is some function of $r_\mu$ which is surely quite similar to the function $Z_{sc}$. If this instantaneous effective nuclear charge decreases linearly with increasing $r_\mu$, or if its second derivative with respect to $r_\mu$ is negative, its average value for an elliptical orbit is surely smaller than its value for a circular orbit with the same principal quantum number: and the increase in the value of $\bar{r}_\mu$ with decreasing $l$ is, therefore, necessarily larger than it would be if the effective charge $Z$ did not vary with $l$. Of course, this conclusion does not necessarily hold in the case of an elliptical meson orbit with values of $r_\mu$ for which the second derivative of the instantaneous effective nuclear charge is positive. The behavior of the function $Z_{sc}$ shown in Fig. 6 indicates that elliptical orbits with $\bar{r}_\mu \gtrsim 0.7$ have such values of $r_\mu$. In these instances, the increase in $\bar{r}_\mu$ with decreasing $l$ may possibly be inhibited to a certain extent.

In what follows, it is convenient to indicate the dependence of $E_c{}^{sc}$ on $l$, for a given value on $n$, by denoting it by $E_{c,l}{}^{sc}$. The semiclassical expression for $\epsilon_c{}^{dv}$ is given by

$$\epsilon_c{}^{sc} = E_{c,n-2}{}^{sc} - E_{c,n-1}{}^{sc}. \tag{4.7}$$

It is also convenient to indicate the dependence of $Z$ and $\bar{r}_\mu$ on $l$, for a given value of $n$, by denoting them by $Z_l$ and $\bar{r}_{\mu,l}$. If $\Delta Z^{sc}$ and $\Delta\bar{r}_\mu{}^{sc}$ are defined by

$$\Delta Z^{sc} = Z_{n-2} - Z_{n-1}\,, \tag{4.8}$$

$$\Delta\bar{r}_\mu{}^{sc} = \bar{r}_{\mu,n-2} - \bar{r}_{\mu,n-1}\,, \tag{4.9}$$

the expression for $\epsilon_c{}^{sc}$ may be put in the form

$$\epsilon_c{}^{sc} = -(M\Delta Z^{sc}/n^2)(4 - 2Z_{n-1} - \Delta Z^{sc}) + E_{dev}(\bar{r}_{\mu,n-1} + \Delta\bar{r}_\mu{}^{sc}) - E_{dev}(\bar{r}_{\mu,n-1}). \tag{4.10}$$

Since $Z_{n-1}$ and $\bar{r}_{\mu,n-1}$ are determined in Sec. II, and since it can be shown from Eqs. (4.6), (4.8), and (4.9) that for hydrogenic states $\Delta Z^{sc}$ and $\Delta\bar{r}_\mu{}^{sc}$ must be related by

$$\Delta\bar{r}_\mu{}^{sc} = [2(n-1)Z_{n-1} - n(2n+1)\Delta Z^{sc}] \times [2MZ_{n-1}(Z_{n-1} + \Delta Z^{sc})]^{-1}\,, \tag{4.11}$$

it only remains to devise some other, independent, way of relating $\Delta Z^{sc}$ to $\Delta\bar{r}_\mu{}^{sc}$ in order to be able to estimate $\epsilon_c{}^{sc}$.

The effective nuclear charges $z$ and $Z_{n-1}$ determined for circular orbits in Sec. II, and also the quantity $Z_{sc}$ defined by Eq. (4.5), can, for a given type of atom, be represented by smoothly varying functions of a continuously varying argument $\bar{r}_{\mu,n-1}$. In order to simplify the calculation, it is now assumed that, for a given type of atom, $Z_{n-2}$ has the same dependence on $\bar{r}_{\mu,n-2}$ that $Z_{n-1}$ has on $\bar{r}_{\mu,n-1}$. A discussion which is given earlier in this part of the present section indicates that this assumption is probably valid for those values of $\bar{r}_{\mu,n-2} = \bar{r}_\mu$ for which the function $Z_{sc}(\bar{r}_\mu)$ varies almost linearly with $\bar{r}_\mu$. It is, therefore, tentatively assumed that $\Delta Z^{sc}$

and $\Delta\bar{r}_\mu{}^{sc}$ are given in terms of each other with sufficient accuracy by

$$\Delta Z^{sc} = \frac{dZ_{sc}}{d\bar{r}_\mu}\,\Delta\bar{r}_\mu{}^{sc}\,, \tag{4.12}$$

where the derivative of $Z_{sc}(\bar{r}_\mu)$ is evaluated at $\bar{r}_\mu = \bar{r}_{\mu,n-1}$. It is easily shown from the definition of $Z_{sc}$ that

$$\frac{dZ_{sc}}{d\bar{r}_\mu} = -4z^3\bar{r}_\mu{}^2\left(1+z^{-1}\bar{r}_\mu\frac{dz}{d\bar{r}_\mu}\right)e^{-2z\bar{r}_\mu}\,. \tag{4.13}$$

Since for a given value of $r_\mu = \bar{r}_{\mu,n-1}$ the effective nuclear charge $z$ is found to be approximated rather accurately by the function $z_v$, as defined in terms of $r_\mu$ by Eq. (3.10), it is assumed, by analogy to Eq. (3.11), that the derivative of $z$ which appears in Eq. (4.13) is given by

$$\frac{dz}{d\bar{r}_\mu} = \frac{4z^2\bar{r}_\mu e^{-2z\bar{r}_\mu}}{1-4z\bar{r}_\mu{}^2 e^{-2z\bar{r}_\mu}}\,. \tag{4.14}$$

Values of $\Delta\bar{r}_\mu{}^{sc}$, which are obtained by employing Eqs. (4.11)–(4.14), are shown in Fig. 7. For the purpose of comparison, the values of $\Delta\bar{r}_\mu{}^{dv}$ which are computed in Sec. III and which are given in Table II are also shown in Fig. 7. Figure 7 also shows values of the quantity

$$\Delta\bar{r}_\mu{}^{0} = (n-1)/(MZ_{n-1})\,, \tag{4.15}$$

which is the expression for $\Delta\bar{r}_\mu{}^{sc}$ if $\Delta Z^{sc}$ is set equal to zero. The relative values of $\Delta\bar{r}_\mu{}^{dv}$ and $\Delta\bar{r}_\mu{}^{0}$ tend to support the qualitative discussion given earlier in this part of the present section concerning the relative values of the mean radii of circular and nearly circular meson orbits. Although the values of $\Delta\bar{r}_\mu{}^{sc}$ are obviously unrealistic for levels with relatively large values of $r_{\mu,n-1}$, the fairly good agreement between the values of $\Delta\bar{r}_\mu{}^{sc}$ and $\Delta\bar{r}_\mu{}^{dv}$ for levels of the $\alpha\pi^-e^-$ and $\alpha K^-e^-$ atoms with $\bar{r}_{\mu,n-1} \simeq 0.5$ is an additional indication that treating the meson as a classical particle is probably a valid approximation in many instances. The large discrepancy between the values of $\Delta\bar{r}_\mu{}^{sc}$ and $\Delta\bar{r}_\mu{}^{dv}$ for the levels of the $\alpha\pi^-e^-$ atom with $n = 17$ and 18 is probably a consequence of Eq. (4.12) being an inadequate approximation, since in these instances the function $Z_{sc}$ does not vary linearly with the radial position of the meson.

Figure 8 shows values of the energy difference

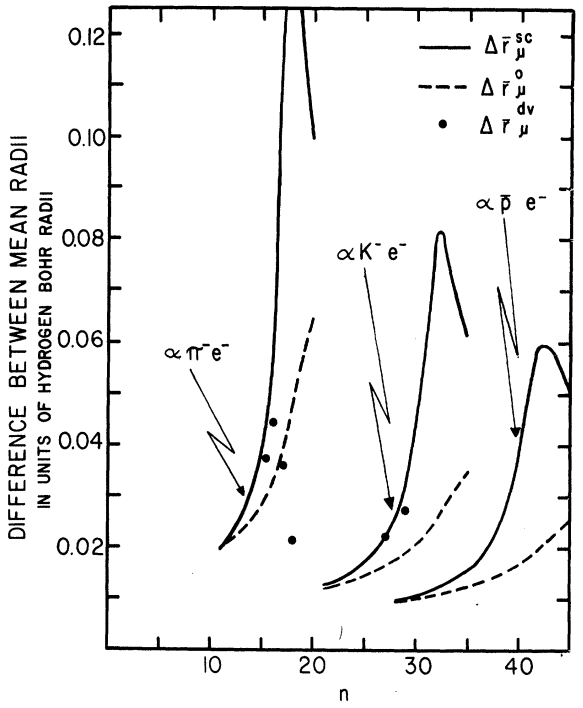


FIG. 7. Estimates of the difference between the mean meson orbital radii for two states of a mesonic atom with the same principal quantum number $n$, but with orbital angular momenta $(n-2)$ and $(n-1)$.

$\epsilon_c{}^{sc}$. For the purpose of comparison, Fig. 8 also shows the values of $\epsilon_c{}^{dv}$ which are calculated in Sec. III and which are given in Table II. In view of the rapid variation of this energy difference with the principal quantum number, and in view of the assumptions which are made in this part of the present section, the agreement between the values of $\epsilon_c{}^{dv}$ and $\epsilon_c{}^{sc}$ can probably be regarded as satisfactory, at least for the purpose of partially justifying treating the meson as a classical particle. The discrepancy between the values of $\epsilon_c{}^{sc}$ and $\epsilon_c{}^{dv}$ for those levels of the $\alpha\pi^-e^-$ and $\alpha K^-e^-$ atoms for which $\Delta\bar{r}_\mu{}^{sc}$ is found to be a fairly good approximation to $\Delta\bar{r}_\mu{}^{dv}$ is, to some extent, probably due to $E_{dev}(\bar{r}_{\mu,n-2})$ not being quite as accurate an approximation to the expectation value of $H_{de}$ for the orbit with $l = (n-2)$ as $E_{dev}(\bar{r}_{\mu,n-1})$ is for the circular orbit.

## V. ENERGY DIFFERENCE $\epsilon$

### A. Introduction

The energy difference $\epsilon$ between two states of the mesonic atom with the same principal quantum number $n$, but with orbital angular momentum $(n-1)$ in one state and $(n-2)$ in the other, may be defined as

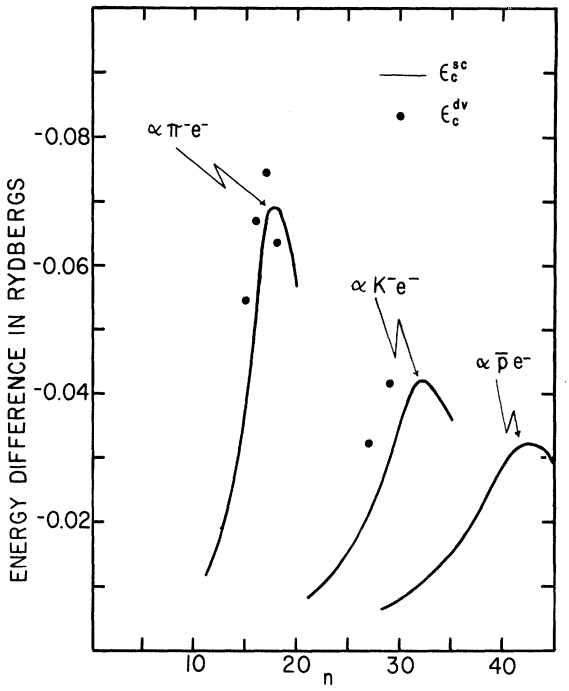


FIG. 8. Estimates of the difference between the binding energies of two states of a mesonic atom with the same principal quantum number $n$, but with orbital angular momenta $(n-2)$ and $(n-1)$. These estimates take into account only the monopole contribution to the interaction between the electron and the meson.

$$\epsilon = E_{a,n-2} - E_{a,n-1}\,, \tag{5.1}$$

where the dependence of an eigenvalue of the exact Hamiltonian $H_a$ on the meson orbital angular momentum $l$ is indicated by now denoting it by $E_{a,l}$ rather than by $E_a$. The energy difference $\epsilon_c^{dv}$, which is defined by Eq. (3.22), is only an approximate value of $\epsilon$. This is partly because $E_c{}^{dv}$, which is denoted by $E_{c,l}{}^{dv}$ in Eq. (3.22) and which is an eigenvalue of $H_c{}^{dv}$, is not precisely an eigenvalue of $H_c$; but probably it is chiefly because the approximate Hamiltonian $H_c$ does not take into account the dipole and higher multipole interactions of the electron with the meson. These interactions are given by

$$H_{bc} = H_b - H_c\,, \tag{5.2}$$

where $H_b$ is the form taken by $H_a$ if $\xi$ is set equal to zero. The very small values of the energy correction $E_{ab}$ indicate that taking into account the energy operator

$$H_{ab} = H_a - H_b \tag{5.3}$$

would not significantly affect an estimate of $\epsilon$.

It is now assumed that no appreciable error is introduced into a computation of $\epsilon$ by ignoring that correction which is due to $E_c{}^{dv}$ not being precisely an eigenvalue of $H_c$, since the arguments presented in Sec. III make it seem not unlikely that, for a given value of $n$, the value of $(E_c - E_c{}^{dv})$ changes by only a negligible amount if the value of $l$ is changed from $(n-1)$ to $(n-2)$. It therefore only remains to estimate values of the contribution to $\epsilon$ which is due to $H_{bc}$. This contribution is denoted by $\epsilon_{bc}$.

### B. Estimates for Some Levels of the $\alpha K^- e^-$ Atom

The values of $\epsilon_{bc}$ are now estimated for the levels of the $\alpha K^- e^-$ atom with $n = 27$ and 29. These are two of the three levels which are also studied in Ref. 2, where some estimates are made of the effects of the interaction of this mesonic atom with a nearby helium atom. The following estimates of these two values of $\epsilon_{bc}$ are based on some numerical results which are obtained in Ref. 7. These numerical results are approximate values of $E_{bc}$, which is that correction to the binding energy for a circular orbit which arises from taking into account the interaction $H_{bc}$.

Since the meson is moving relatively slowly in a rather well-defined orbit, the approximate expressions for $E_{bc}$ are given as functions of the mean radius $\bar{r}_\mu$ of the meson orbit, as determined in Sec. II of the present paper. Since the values of $\bar{r}_\mu$ and $\bar{r}_\mu{}^{dv}$ are shown in Sec. III to be nearly the same for a given circular orbit, it seems reasonable to suppose that a fairly reliable estimate of $\epsilon_{bc}$, which is, in a sense, based on the eigenfunctions of the approximate Hamiltonian $H_c{}^{dv}$, is given by

$$\epsilon_{bc}{}^{dv} = \frac{dE_{bc}}{d\bar{r}_\mu}\,\Delta\bar{r}_\mu{}^{dv}\,, \tag{5.4}$$

provided $\Delta\bar{r}_\mu{}^{dv}$ is not too large. The accuracy of the estimates of $E_{bc}$ probably does not warrant a more elaborate attempt to calculate $\epsilon_{bc}$. The estimates of $E_{bc}$ for the $\alpha K^- e^-$ atom, which are given in Tables II and III of Ref. 7, indicate that assuming the derivative of $E_{bc}$ to be given approximately by

$$\frac{dE_{bc}}{d\bar{r}_\mu} = -0.20, \quad \alpha K^- e^- \text{ atom} \quad n = 27, \tag{5.5a}$$

$$\frac{dE_{bc}}{d\bar{r}_\mu} = -0.14, \quad \alpha K^- e^- \text{ atom} \quad n = 29, \tag{5.5b}$$

probably does not cause an inaccuracy of more

TABLE III. A summary of the results of a calculation of the energy difference $\epsilon$ for two levels of the $\alpha K^- e^-$ atom, as obtained by first taking into account only the monopole interaction of the electron with the meson and employing the Born-Oppenheimer approximation, and then estimating roughly the effect of the dipole and higher multipole interactions. The unit of energy is the rydberg.

| $n$ | $\epsilon_c^{dv}$ | $\epsilon_{bc}^{dv}$ | $\epsilon^{dv}$ |
|---|---|---|---|
| 29 | −0.0415 | −0.0038 | −0.0453 |
| 28 | | | |
| 27 | −0.0323 | −0.0044 | −0.0367 |

than 20% in the evaluation of the expression for $\epsilon_{bc}^{dv}$. Table III gives the values of $\epsilon_{bc}^{dv}$ which are obtained by employing Eqs. (5.4)–(5.5b). For the purpose of comparison, the values of $\epsilon_c^{dv}$ which are given for these two levels of the $\alpha K^- e^-$ atom in Table II are given again in Table III. Table III also gives values of the quantity $\epsilon^{dv}$, which is defined by

$$\epsilon^{dv} = \epsilon_c^{dv} + \epsilon_{bc}^{dv} , \tag{5.6}$$

and which is assumed to be a fairly accurate approximation to $\epsilon$. It should be noted that in these two instances a relatively large inaccuracy in the estimate of $\epsilon_{bc}$ results in a much smaller relative inaccuracy in the estimate of $\epsilon$.

### C. Estimates for Some Levels of the $\alpha\bar{p}e^-$ Atom

Because an experiment has been proposed to determine whether or not Stark transitions are suppressed in antiprotonic atoms formed in liquid helium,[1] and because a Born-Oppenheimer calculation requires a substantial amount of computer time, it is appropriate to attempt to estimate the values of $\epsilon$ for a few levels of the $\alpha\bar{p}e^-$ atom in a way which does not require any additional numerical solutions to Eq. (3.15). A semiclassical method is first employed to find rough values of the energy difference $\epsilon$ for the levels of the $\alpha\bar{p}e^-$ atom with $n = 36$, 37, and 38. For the purpose of comparison, this method is also applied to the levels of the $\alpha K^- e^-$ atom with $n = 27$, 28, and 29. The results which are obtained by using this semiclassical method form, to a limited extent, a basis for some of the assumptions which are made at the end of this section in what is probably a more accurate estimate of values of $\epsilon$ for the $\alpha\bar{p}e^-$ atom.

The semiclassical estimate of $\epsilon$ is denoted by $\epsilon^{sc}$. By analogy to Eq. (5.6), it is defined by

$$\epsilon^{sc} = \epsilon_c^{sc} + \epsilon_{bc}^{sc} . \tag{5.7}$$

By analogy to Eq. (5.4), the term $\epsilon_{bc}^{sc}$ is defined by

$$\epsilon_{bc}^{sc} = \frac{dE_{bc}}{d\bar{r}_\mu} \Delta\bar{r}_\mu^{sc} . \tag{5.8}$$

The derivative of $E_{bc}$ which appears in Eq. (5.8) is the same quantity which appears in Eq. (5.4).

Values of the derivative of $E_{bc}$ are now estimated in much the same way as they are in the preceding part of this section. The estimates of $E_{bc}$, which are made in Ref. 7 by employing a semiclassical approximation, depend only on the mean radius $\bar{r}_\mu$ of the meson orbit, and not on the meson mass. Each of the circular orbits of the $\alpha\bar{p}e^-$ atom being considered here has a value of $\bar{r}_\mu$ which is not very different from that for one of the circular orbits of the $\alpha K^- e^-$ atom with $n = 27$, 28, or 29. It is now assumed that for these few states of the two atoms the derivative of $E_{bc}$ varies linearly with $\bar{r}_\mu$ in such a way that Eqs. (5.5a) and (5.5b) are satisfied.

Table IV gives values of $\Delta\bar{r}_\mu^{sc}$, $\epsilon_c^{sc}$, $\epsilon_{bc}^{sc}$, and $\epsilon^{sc}$. Values of the approximate binding energy

TABLE IV. A summary of the results of a calculation of the energy difference $\epsilon$ for some levels of the $\alpha K^- e^-$ and $\alpha\bar{p}e^-$ atoms, as obtained by first taking into account only the monopole interaction of the electron with the meson and employing the semiclassical approximation, and then estimating roughly the effect of the dipole and higher multipole interactions. The unit of energy is the rydberg, and the unit of distance is the hydrogen Bohr radius.

| atom | $n$ | $E_c^v$ | $\bar{r}_\mu$ | $\Delta\bar{r}_\mu^{sc}$ | $\epsilon_{bc}^{sc}$ | $\epsilon_{bc}^{sc}$ | $\epsilon^{sc}$ |
|---|---|---|---|---|---|---|---|
| $\alpha K^- e^-$ | 29 | −5.4176 | 0.569 | 0.0322 | −0.0296 | −0.0045 | −0.0341 |
| | 28 | −5.6551 | 0.514 | 0.0257 | −0.0249 | −0.0044 | −0.0293 |
| | 27 | −5.9356 | 0.465 | 0.0214 | −0.0210 | −0.0043 | −0.0253 |
| $\alpha\bar{p}e^-$ | 38 | −5.4175 | 0.567 | 0.0244 | −0.0225 | −0.0034 | −0.0259 |
| | 37 | −5.5955 | 0.523 | 0.0204 | −0.0198 | −0.0034 | −0.0232 |
| | 36 | −5.7982 | 0.484 | 0.0176 | −0.0172 | −0.0033 | −0.0205 |

$E_c{}^v$ and the mean meson orbital radius $\bar{r}_\mu$ for the circular orbits, as determined in Sec. II, are also given. Although each of the three levels of the $\alpha\bar{p}e^-$ atom which are listed in Table IV has a circular orbit with values of $\bar{r}_\mu$ and $E_c{}^v$ which are not very different from those for a circular orbit of one of the three levels listed for the $\alpha K^- e^-$ atom, the values of $\epsilon^{sc}$ do not agree quite so closely. This difference is due to the difference between the values of $\Delta\bar{r}_\mu{}^{sc}$. For each of these three particular pairs of more or less comparable levels of the $\alpha\bar{p}e^-$ and $\alpha K^- e^-$ atoms, the ratio of the values of $\epsilon^{sc}$ is very nearly equal to the ratio of the values of $\Delta\bar{r}_\mu{}^{sc}$.

Although a comparison of the results given for the $\alpha K^- e^-$ atom in Table IV with those given in Table III shows that the semiclassical method does not result in an especially accurate estimate of $\epsilon^{dv}$, there is reason to believe that the relative values of $\epsilon^{sc}$ for each of the particular pairs of levels of the $\alpha K^- e^-$ and $\alpha\bar{p}e^-$ atoms listed in Table IV are probably similar to the relative values of $\epsilon^{dv}$. If the values of $\epsilon_c{}^{sc}$ and $\epsilon_c{}^{dv}$ shown for the $\alpha\pi^- e^-$ and $\alpha K^- e^-$ atoms in Fig. 8 can be regarded as providing a reliable indication, a computation of $\epsilon_c{}^{dv}$ for a level of the $\alpha\bar{p}e^-$ atom with $n = 36$, 37, or 38 would result in a value with a magnitude which is somewhat larger than that of $\epsilon_c{}^{sc}$, but which is still somewhat smaller than the magnitude of $\epsilon_c{}^{dv}$ for an $\alpha K^- e^-$ atom of nearly the same size and binding energy. And if the values of $\epsilon_c{}^{dv}$ and $\Delta\bar{r}_\mu{}^{dv}$ given in Table II for the levels of the $\alpha\pi^- e^-$ atom with $n = 15$ and 16 and the levels of the $\alpha K^- e^-$ atom with $n = 27$ and 29 can be regarded as providing a reliable indication, the ratio of the value of $\epsilon_c{}^{dv}$ for an $\alpha\bar{p}e^-$ atom with $n = 36$, 37, or 38 to that for an $\alpha K^- e^-$ atom of nearly the same size and binding energy is likely to be very nearly equal to the ratio of the values of $\Delta\bar{r}_\mu{}^{dv}$ for these two atoms. The values of $\Delta\bar{r}_\mu{}^{sc}$ and $\Delta\bar{r}_\mu{}^{dv}$ given for the $\alpha\pi^- e^-$ and $\alpha K^- e^-$ atoms in Fig. 7, and also the discussion given in Sec. IV C, indicate that the values of $r_\mu$ and $Z$ for an $\alpha\bar{p}e^-$ atom with $n = 36$, 37, or 38 are probably such that $\Delta\bar{r}_\mu{}^{sc}$ is a fair approximation to $\Delta\bar{r}_\mu{}^{dv}$. It therefore seems likely that the magnitudes of $\epsilon^{dv}$ for $\alpha\bar{p}e^-$ atoms with $n = 36$, 37, and 38 are roughly 80% as large as the magnitudes of $\epsilon^{dv}$ for $\alpha K^- e^-$ atoms with $n = 27$, 28, and 29, respectively.

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