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Statistical Analysis of Randomly Modulated Laser Light*

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A modified Michelson interferometer in which one of the mirrors is driven by a piezoelectric crystal connected to a Gaussian random-noise generator has been used to phase modulate a light beam from a Spectra Physics 119 model laser. The recombination of the modulated and the unmodulated beam reflected by a stationary prism generates a randomly varying interference pattern whose intensity distribution has been analyzed by means of a standard photon-counting technique. The experimental data agree with the theoretical semiclassical predictions in the range of linearity of the piezoelectric crystal. Deviations from the theory have been detected and correlated with the hysteresis of the crystal for large root-mean-square voltages applied to it.

I. INTRODUCTION

Photon-counting experiments have been widely discussed in recent literature as a means of analyzing the statistical properties of various electromagnetic fields.[1-27]

It has been observed, in particular,[28,29] that phase modulation of a light beam cannot be detected by a standard photoelectron detection measurement, owing to the fact that the photodetection process is only sensitive to the absolute magnitude of the electric field.

It is clear, on the other hand, that if an interference pattern is formed by superimposing a reference beam and a phase-modulated beam from the same source, and if a detector is placed in the interference region, information on the phase modulation can, in principle, be decoded from the photoelectron statistics.

An experiment has been performed where a laser beam passing through a Michelson interferometer is reflected along one arm by a randomly vibrating mirror and then recombined with the unmodulated beam. A counting apparatus suitable for the measurement of photoelectron probability distributions has been built to detect and analyze the intensity of the randomly varying interference pattern.

A semiclassical model of phase modulation due to Glauber[28] is proposed in Sec. II to predict the probability distribution of photon numbers. In Sec. III, a detailed description is offered of the experimental system, followed in Sec. IV by the analysis of the experimental results and the comparison with the theoretical description. Excellent agreement has been found between the exact solution of the model and the experimental results up to a certain degree of phase modulation. Some disagreement between theory and experiment for a higher degree of phase modulation is explained by a careful analysis of the piezoelectric driving device used to phase modulate one beam. A number of control experiments, performed to check the reliability of the operation, are also discussed.

II. THEORETICAL MODEL

In this section, we propose a semiclassical calculation based on the experiment shown schematically in Fig. 1.

With reference to Fig. 1, consider a laser source operating in a single longitudinal mode of the cavity and generating a radiation field which we assume to be in a coherent state. The laser beam is divided into two parts, one of which is randomly phase modulated with respect to the other. This is accomplished by mounting the mirror M on a piezoelectric crystal which is driven by an electrical Gaussian noise source. When the crystal is driven in its linear operating region, the resulting time-of-flight difference provides a phase difference between the two beams which is Gaussianly distributed.[30] The recombination of the two beams creates an interference pattern which is detected by a photomultiplier connected to the photoelectron counting system.

For a mathematical model of our experiment, we imagine the phase modulation to be an energy conserving process which causes random frequency shifts in the original beam. Ignoring the quantum aspects of the modulation device, we postulate the energy-conserving interaction Hamiltonian

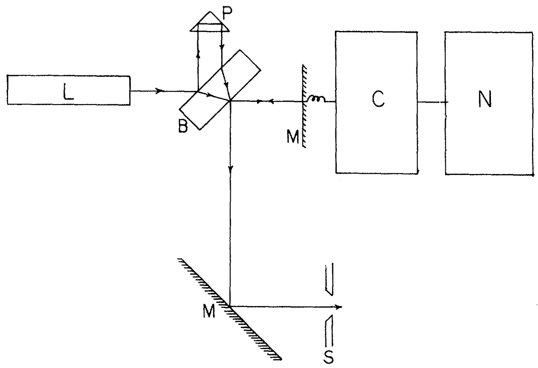

FIG. 1. Block diagram of the modified Michelson interferometer and of the modulation device used in the experiment. The letters refer to the following components: L, laser; P, prism; B, beam splitter; M, mirror; C, crystal driver; N, Gaussian noise generator; S, slit.

$$\mathcal{H}_I = \hbar f(t) a^\dagger a \;, \tag{1}$$

where a and $a^\dagger$ are, respectively, the annihilation and creation operators of the field mode and $f(t)$ is a random C-number function whose stochastic properties are assumed to be Gaussian.

More specifically,

$$\begin{aligned} &\langle f(t_1)\cdots f(t_{2n+1})\rangle\!\rangle = 0, \\ &\langle f(t_1)\cdots f(t_{2n})\rangle\!\rangle = \sum_{\substack{\text{all}\\ \text{permutations}}} \times \langle f(t_i) f(t_l)\rangle\!\rangle \langle f(t_l) f(t_m)\rangle\!\rangle \,. \end{aligned} \tag{2}$$

Relating $f(t)$ to our experiment, we note that the phase difference Φ between the two beams can be written

$$\Phi = \int_0^t f(t')\,dt'. \tag{3}$$

Making use of Eq. (2), we find that the average phase difference is zero and that the variance in the phase is given by

$$\begin{aligned} \sigma^2 &\equiv \langle (\Phi - \langle \Phi \rangle)^2 \rangle \\ &= \int_0^t dt_1 \int_0^t dt_2 \langle f(t_1) f(t_2)\rangle\!\rangle \,. \end{aligned} \tag{4}$$

Continuing with the calculation, our program is to derive an expression for the density matrix of the modulated beam and to linearly superimpose it on the unmodulated beam. The density matrix of the mixed fields will yield the probability distribution for the photon occupation number as seen by the detection device.

The formal solution of the Liouville equation in the interaction representation

$$i\hbar \frac{\partial}{\partial t} \rho = [\mathcal{H}_I, \rho] \tag{5}$$

is given by the well-known expression

$$\begin{aligned} \rho(t) = \sum_{M=0}^{\infty} \left(\frac{1}{i\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ \times [\mathcal{H}_I(t_1), [\mathcal{H}_I(t_2), \ldots,] \, [\mathcal{H}(t_n), \rho] \cdots] \,. \end{aligned} \tag{6}$$

Inserting the Hamiltonian (1) into the above equation, the density operator of the modulated beam becomes

$$\begin{aligned} \rho(t) = \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} f(t_1) \cdots f(t_n) \\ \times [a^\dagger a, \ldots, [a^\dagger a, \rho(0)] \cdots]. \end{aligned} \tag{7}$$

Since the detected properties of the modulated beam are ensemble averages over the stochastic process $f(t)$, we define the average density matrix $R(t)$

$$R(t) = \{\rho(t)\}_{\text{av over } f}, \quad R(0) = \rho(0)\,. \tag{8}$$

Utilizing Eq. (7), the averaged density matrix $R(t)$ is given by

$$\begin{aligned} R(t) = \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \\ \times \langle f(t_1) \cdots f(t_n)\rangle\!\rangle [a^\dagger a, \ldots, [a^\dagger a, R(0)] \cdots]\,. \end{aligned} \tag{9}$$

The Gaussian assumptions summarized by Eq. (2) and a standard change in the variables of integration lead to

$$R(t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\tfrac{1}{2}\sigma^2)^n$$

$$\times\,[a^\dagger a,\ldots,[a^\dagger a,R(0)]\cdots]_{2n}, \tag{10}$$

where σ^2 is the variance in phase given by Eq. (4) and $[a^\dagger a,\ldots,[a^\dagger a,R(0)]\ldots]_{2n}$ represents the $2n$th-order commutator $\{\text{e.g.}[\quad]_2\equiv[a^\dagger a,[a^\dagger a,R(0)]]\}$. It is worth noting that for the purpose of carrying out future calculations the commutator $a^\dagger a,[a^\dagger a,\ldots,[a^\dagger a,R(0)]\ldots]_{2n}$ can be put in the convenient form

$$[a^\dagger a,\ldots[a^\dagger a,R(0)]\cdots]_{2n}$$

$$=\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k(a^\dagger a)^{2n-k}R(0)(a^\dagger a)^k \tag{11}$$

so that Eq. (8) finally becomes

$$R(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{(-1)^n}{n!}$$

$$\times\left(\frac{\sigma^2}{2}\right)^n\binom{2n}{k}(-1)^k(a^\dagger a)^{2n-k}R(0)(a^\dagger a)^k. \tag{12}$$

Using Eq. (12) the expectation value of the annihilation operator $a(t)$,

$$\langle a(t)\rangle=\mathrm{Tr}\{R(t)\,a\,e^{-i\omega t}\},$$

is easily calculated to be

$$\begin{aligned}\langle a(t)\rangle&=\exp[-\tfrac{1}{2}\sigma^2(t)]\,\mathrm{Tr}\{R(0)\,a\,e^{-i\omega t}\},\\&=\exp[-\tfrac{1}{2}\sigma^2(t)]\langle a(0)\rangle\,e^{-i\omega t}.\end{aligned} \tag{13}$$

As expected, the modulation process does not affect the average intensity of the input beam since

$$\begin{aligned}\langle a^\dagger a(t)\rangle&=\mathrm{Tr}\{R(t)\,a^\dagger a\}\\&=\mathrm{Tr}\{R(0)\,a^\dagger a\}=\langle a^\dagger a(0)\rangle.\end{aligned} \tag{14}$$

Hence, a photon-counting experiment performed directly on the phase-modulated beam would not record any effect of the modulation. (Data supporting this statement are given in Sec. IV.) However, a mixing of the phase-modulated beam with the reference beam will show amplitude modulation on the mixed beam which can be detected in a photon-count distribution experiment.

In order to calculate the theoretical value for the photon distribution function of the mixed beam, it is convenient to introduce the well-known P representation for the mixed density operator $\rho_M(t)$[31]

$$\rho_M(t)=\int P_M(\alpha,t)\,|\alpha\rangle\langle\alpha|\,d^2\alpha, \tag{15}$$

where the P function $P_M(\alpha,t)$ for the combined beams, is

$$P_M(\alpha,t)=\int P_1(\alpha-\alpha',t)\,P_2(\alpha')\,d^2\alpha'. \tag{16}$$

The functions P_1 and P_2 are, respectively, the P distributions for modulated and unmodulated beams. Substitution of Eq. (16) into (15) yields

$$\rho_M(t)=\int P_2(\alpha')$$

$$\times\left[\int d^2\beta\,P_1(\beta,t)\,|\beta+\alpha'\rangle\langle\beta+\alpha'|\right]d^2\alpha', \tag{17}$$

where we have inverted the order of integration and introduced a variable β for $\alpha-\alpha'$.

Explicit calculation of Eq. (17) is made tractable by the introduction of the coherent state displacement operator defined by

$$\begin{aligned}D(\alpha)&=\exp(\alpha a^+-\alpha^* a),\\D(\alpha)\,|\xi\rangle&=|\xi+\alpha\rangle.\end{aligned} \tag{18}$$

With this transformation, Eq. (17) becomes

$$\rho_M(t)=\int P_2(\alpha')\,D(\alpha')\,R(t)\,D^\dagger(\alpha')\,d^2\alpha', \tag{19}$$

where we recognize that

$$R(t)=\int P_1(\beta_1 t)\,|\beta\rangle\langle\beta|\,d^2\beta. \tag{20}$$

With the specification that the initial state of both modulated and reference beam is a coherent state of amplitude $\tfrac{1}{2}\alpha_0$, we have

$$R(t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\sigma^2(t)}{2}\right)^n$$

$$\times\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k(a^\dagger a)^{2n-k}\,|\tfrac{1}{2}\alpha_0\rangle\langle\tfrac{1}{2}\alpha_0|\,(a^\dagger a)^k, \tag{21}$$

and $P_2(\alpha') = \delta^2(\alpha' - \frac{1}{2}\alpha_0)$. (22)

The evaluation of the photon probability distribution $p_n(t) = \langle n | \rho_m(t) | n \rangle$ is now only a matter of a few algebraic manipulations which are reported in Appendix A. The result of the calculation is

$$p_n(t) = \frac{(\frac{1}{4}|\alpha_0|^2)^n}{n!} \exp(-\tfrac{1}{2}|\alpha_0|^2) \sum_{r,s}^{\infty} \exp[-(r-s)^2 \tfrac{1}{2}\sigma^2(t)]\, L_s^{n-s}(\tfrac{1}{4}|\alpha_0|^2) L_r^{n-r}(\tfrac{1}{4}|\alpha_0|^2) , \tag{23}$$

where L_s^{n-s} is the associated Laguerre polynomial. Equation (23) can be readily checked in the limit of no modulation. For $\sigma^2(t)$ equal to zero Eq. (23) becomes

$$p_n(t) = \frac{(\frac{1}{4}|\alpha_0|^2)^n}{n!} \exp(-\tfrac{1}{2}|\alpha_0|^2) \times \sum_{r,s}^{\infty} L_s^{n-s}(\tfrac{1}{4}|\alpha_0|^2) L_r^{n-r}(\tfrac{1}{4}|\alpha_0|^2). \tag{24}$$

Making use twice of the identity

$$\sum_{n=0}^{\infty} L_n^{\alpha-n}(x) z^n = \exp(-xz)(1+z)^{\alpha} ,$$

Eq. (24) becomes

$$p_n(t) = \frac{|\alpha_0|^{2n}}{n!} \exp(-|\alpha_0|^2) ,$$

which is just the Poisson distribution for the combination of the two beams constructively interfering with each other. Equation (23) represents the central result of our analysis. Some typical computer generated distributions are shown in Figs. 2 and 3. The comparison of the theory with the experimental data is presented in Sec. IV.

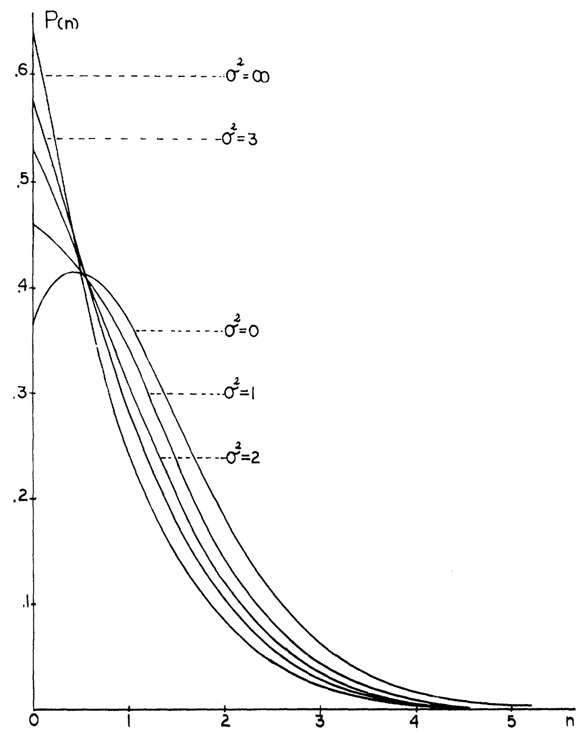

FIG. 2. Theoretical photon-counting distributions for superimposed phase-modulated and unmodulated laser beams. The mean number of photons entering the interferometer during the integration time is 1. The variable parameter is the mean-square deviation of the phase measured in radians.

III. DESCRIPTION OF EXPERIMENTAL SYSTEM

A. Optical System

The apparatus (see Fig. 1) used to generate the phase modulation and to combine the phase modulated with the unmodulated light is basically a Michelson interferometer. The beam splitter is a piece of clear glass and the stationary mirror has been replaced with a prism. With this arrangement, we have been able to superimpose two beams of the same intensity thus obtaining very nearly 100% modulation. The movable mirror was replaced with a mirror coated for maximum reflectivity at 6328 Å mounted on a piezoelectric crystal (modulator unit from a Spectra Physics Model 119 laser).

The light source is a Spectra Physics 119 helium-neon laser. Periodic checks of its stability were made by testing the unmodulated laser beam to verify that the photon counting distribution remained Poisson to within the experimental accuracy.

B. Photoelectron Counting System

The photon counting system is shown schematically in Fig. 4. The photomultiplier is an XP 1021 Amperex unit with an S-11 type photocathode which has a quantum yield of 1% at 6328 Å. The output is of the 50 Ω matched type with a rise time of 2 nsec. When the photomultiplier is run at 1800 V, a single emitted photoelectron produces a pulse with an average height of 1 mV. The pulse is then amplified using a LeCroy No. 134 amplifier with a rise time of 2 nsec and a gain of 100. The resulting 100 mV pulse is sufficient to drive the LeCroy No. 161 dual discriminator (dual discrimination is used to insure better pulse uni-

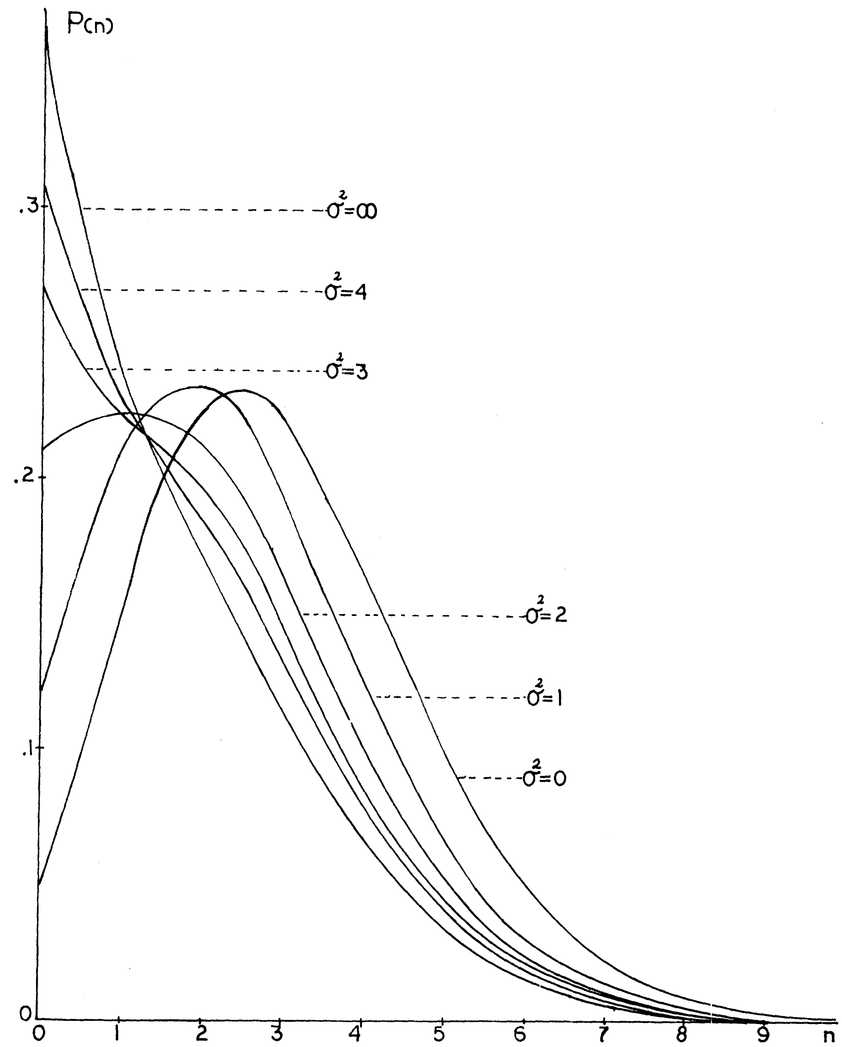

FIG. 3. Theoretical photon-counting distributions for superimposed phase-modulated and unmodulated laser beams. The mean number of photons entering the interferometer during the integration time is 3. The variable parameter is the mean-square deviation of the phase measured in radians.

formity). A fast gate on the discriminator is used to shut off incoming pulses during the time the integrator is discharging. The integrator itself was built in house and is capable of integration times from 1 μsec to 1 msec with sampling rates ranging from 10 000/sec to 1/sec. The total dead time of the system (time needed to resolve two incoming photoelectrons) is less than 10 nsec.

The output pulse of the integrator has a height which is proportional to the number of photoelectrons recorded in the particular integration period, and is recorded on a gammascope multichannel analyzer. The information stored in the analyzer is periodically dumped onto computer cards so that experiments of very long duration can be easily handled. This is necessary, for instance, to determine a reliable count distribution of the noise photoelectrons.

The integration times used in our experiment were selected on the basis of dead time considerations and the requirement that the integration time be shorter than the coherence time of the intensity fluctuations of the combined beams. The times used were 50 μsec for small modulations and 1 μsec for large modulations.

IV. ANALYSIS OF EXPERIMENTAL RESULTS

In this section we present the results of the experiments described in Sec. II. After measuring the statistical distribution of photoelectrons for different degrees of phase modulation, the signal information was separated from the noise (see Appendix C) by performing a separate counting experiment for the photomultiplier dark noise.

The signal mean number of photoelectrons $\bar{n}$ and its second moment can now be determined from the corrected experimental distribution. A comparison with the theory is then performed by evaluating the parameters $|\alpha_0|^2$ and $\sigma^2(t)$ appearing in the theoretical probability distribution Eq. (23) by means of the expressions:

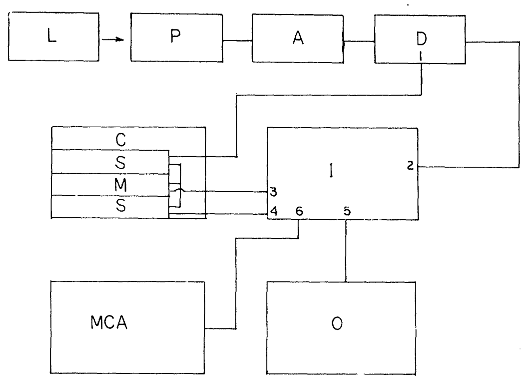

FIG. 4. Block diagram of the photoelectron counting system used in the experiment. The letters refer to the following components: L, laser; P, photomultiplier; A, amplifier; D, discriminator; C, clock; S, slave clock; M, master clock; I, integrator; O, oscilloscope; MCA, multichannel analyzer. The numbers refer to the following connections: 1, discriminator fast gate; 2, signal input to integrator; 3, integration start input; 4, output command; 5, oscilloscope output; 6, integrator output.

$$|\alpha_0|^2 = \frac{2\bar{n}^3 + \bar{n}^2[2(\overline{n^2} - \bar{n}^2 - \bar{n})]^{1/2}}{3\bar{n}^2 + \bar{n} - \overline{n^2}}, \tag{25}$$

$$\exp[-\tfrac{1}{2}\sigma^2(t)] = \frac{2(3\bar{n}^2 + \bar{n} - \overline{n^2}}{2\bar{n}^2 + \bar{n}[2(\overline{n^2} - \bar{n}^2 - \bar{n})]^{1/2}} - 1$$

which are derived in Appendix B.

Table I shows both theoretical and experimental results for a small average number of photoelectrons. For this situation, the combination of a small mean number of photoelectrons and a long integration time results in a very small dead time. The agreement in this case is excellent. Table II shows some distributions with higher average number. These were taken with the short integration time and again the fit is very good.

Table III shows a distribution with very large modulation. It can be seen that the experimental results do not fit the theoretical predications with the same degree of accuracy as in the preceding case. We attribute this deviation to hysteresis in the driving crystal. Figure 5 displays the photocurrent of a photomultiplier which detects the light from the interferometer when the crystal is driven by a periodic triangular voltage. For a linear crystal this curve should be a sine squared function. It is seen that for large driving voltages the crystal exhibits considerable hysteresis. The position of the mirror is then no longer Gaussianly distributed and, as expected, the experiment no longer fits the theory. As a further check of the theory, we performed an experiment with the ref-

TABLE I. Experimental and theoretical photoelectron distributions for mixed phase-modulated and unmodulated beams for a small mean number of photons.

n	$P(n)$ experimental	$P(n)$ theoretical
0	0.494 ±0.003	0.494
1	0.348 ±0.001	0.347
2	0.122 7 ±0.000 7	0.123 0
3	0.029 7 ±0.000 3	0.029 5
4	0.005 3 ±0.000 1	0.005 3
5	0.000 77 ±0.000 05	0.000 77
6	0.000 11 ±0.000 02	0.000 09
7	0.000 006 ±0.000 005	0.000 01
8	0.000 003 ±0.000 003	0.000 001
	$\lvert\alpha_0\rvert^2 = 0.764$ $\sigma^2 = 0.312$	

B

n	$P(n)$ experimental	$P(n)$ theoretical
0	0.589 ±0.002	0.590
1	0.294 ±0.001	0.292
2	0.091 7 ±0.000 7	0.092 6
3	0.021 0 ±0.000 3	0.021 1
4	0.003 9 ±0.000 1	0.003 7
5	0.000 55 ±0.000 05	0.000 55
6	0.000 11 ±0.000 02	0.000 07
7	0.000 005 ±0.000 005	0.000 007
	$\lvert\alpha_0\rvert^2 = 0.804$ $\sigma^2 = 1.87$	

TABLE II. Experimental and theoretical photoelectron distributions for mixed phase-modulated and unmodulated beams for increasing mean number of photons.

	A			B	
n	$P(n)$ experimental	$P(n)$ theoretical	n	$P(n)$ experimental	$P(n)$ theoretical
0	0.276 ±0.003	0.274	0	0.400 ±0.004	0.404
1	0.316 ±0.003	0.318	1	0.338 ±0.007	0.331
2	0.226 ±0.002	0.226	2	0.170 ±0.004	0.174
3	0.116 ±0.002	0.115	3	0.065 ±0.001	0.066
4	0.046 4 ±0.000 7	0.045 8	4	0.019 5 ±0.000 5	0.019 6
5	0.015 1 ±0.000 4	0.014 9	5	0.005 1 ±0.000 3	0.004 8
6	0.003 9 ±0.000 2	0.004 1	6	0.001 2 ±0.000 2	0.001 0
7	0.000 95 ±0.000 06	0.000 99	7	0.000 21 ±0.000 05	0.000 18
8	0.000 15 ±0.000 03	0.000 21	8	0.000 04 ±0.000 01	0.000 03
9	0.000 03 ±0.000 01	0.000 04	9	0.000 007 ±0.000 005	0.000 004
10	0.000 015 ±0.000 009	0.000 007			
	$\|\alpha_0\|^2=1.80$ $\sigma^2=1.15$			$\|\alpha_0\|^2=1.35$ $\sigma^2=1.55$	
	C			D	
n	$P(n)$ experimental	$P(n)$ theoretical	n	$P(n)$ experimental	$P(n)$ theoretical
0	0.365 ±0.003	0.366	0	0.474 ±0.003	0.476
1	0.311 ±0.002	0.307	1	0.304 ±0.005	0.299
2	0.189 ±0.002	0.191	2	0.145 ±0.003	0.148
3	0.089 1 ±0.000 7	0.089 3	3	0.055 3 ±0.000 8	0.055 3
4	0.032 3 ±0.000 4	0.033 0	4	0.016 3 ±0.000 4	0.016 4
5	0.010 8 ±0.000 3	0.010 1	5	0.004 3 ±0.000 3	0.004 0
6	0.002 7 ±0.000 1	0.002 6	6	0.000 89 ±0.000 05	0.000 84
7	0.000 56 ±0.000 05	0.000 59	7	0.000 15 ±0.000 03	0.000 15
8	0.000 08 ±0.000 02	0.000 11	8	0.000 022 ±0.000 01	0.000 024
9	0.000 02 ±0.000 01	0.000 02	9	0.000 004 ±0.000 004	0.000 004
	$\|\alpha_0\|^2=1.71$ $\sigma^2=2.08$			$\|\alpha_0\|^2=1.38$ $\sigma^2=2.84$	

erence beam blocked. The result of this experiment given in Table IV verifies that the statistics of the photoelectrons are not altered by a simple phase modulation of the beam.

V. CONCLUSIONS

The feasibility of analyzing the effects of phase modulation on a coherent light beam by means of photon counting techniques has been demonstrated for Gaussian phase modulation.

The comparison of the experimental and theoretical results seems to indicate that, in our case, the semiclassical description of the phase modulation discussed in Sec. II is adequate.

An interesting result, in our opinion, is the experimental verification of the often quoted state-

TABLE III. Experimental and theoretical photoelectron distributions for large phase modulation.

n	$P(n)$ experimental	$P(n)$ theoretical
0	0.372 ±0.003	0.377
1	0.274 ±0.003	0.264
2	0.179 ±0.002	0.183
3	0.101 ±0.001	0.103
4	0.047 4 ±0.000 8	0.047 1
5	0.019 0 ±0.000 5	0.018 1
6	0.005 9 ±0.000 2	0.006 0
7	0.001 8 ±0.000 1	0.001 7
8	0.000 47 ±0.000 05	0.000 44
9	0.000 09 ±0.000 02	0.000 10
10	0.000 02 ±0.000 01	0.000 02
11	0.000 004 ±0.000 004	0.000 004
	$\|\alpha_0\|^2=2.21$ $\sigma^2=3.78$	

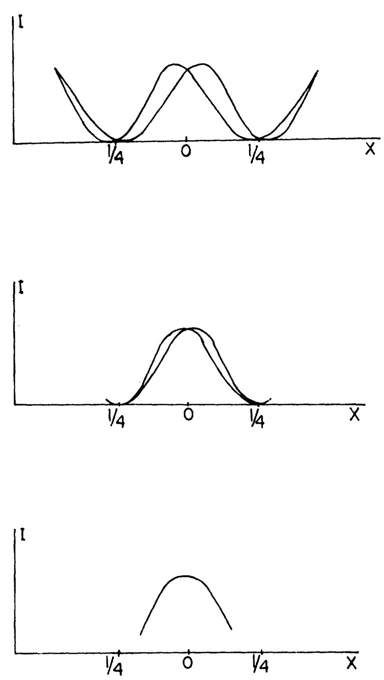

FIG. 5. Crystal hysteresis curves taken from an oscilloscope photograph. The abscissa is the applied potential difference necessary to move the crystal 6328 Å in the region of linearity. The ordinate is proportional to the intensity of the light at the output of the interferometer. The three graphs depict the crystal driven by a triangular wave at 1 kHz with approximately enough potential to move the crystal $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$ of a wavelength, respectively.

ment that simple phase modulation of a light beam does not alter the statistical distribution of photoelectrons. It seems difficult to anticipate whether or not our method could be successfully used to deal with unknown phase modulation processes; that is, to discover properties of the modulation system from the experimental knowledge of the photoelectron statistics. The difficulty is of the same nature as the one encountered in light scattering experiments where from some experimental properties of the scattered radiation one tries to characterize the scattering mechanism.

Further studies of this problem are currently under way.

ACKNOWLEDGMENTS

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APPENDIX A

In this Appendix, we derive the expression for the photon occupation number given in Eq. (23).

In Sec. II, it has been shown that the density operator describing the mixed beams can be expressed in the following form:

TABLE IV. Photoelectron distribution for phase-modulated beam alone and large phase modulation.

n	$P(n)$ experimental	$P(n)$ theoretical
0	0.038 9 ±0.000 7	0.039 5
1	0.127 3 ±0.001	0.127 6
2	0.205 9 ±0.001	0.206 2
3	0.223 0 ±0.001	0.222 1
4	0.180 7 ±0.003	0.179 5
5	0.116 1 ±0.000 9	0.116 1
6	0.062 1 ±0.000 9	0.062 5
7	0.028 4 ±0.001	0.028 9
8	0.011 75 ±0.000 2	0.011 67
9	0.004 04 ±0.000 7	0.004 19
10	0.001 36 ±0.000 03	0.001 35
11	0.000 394 ±0.000 03	0.000 398
12	0.000 114 ±0.000 02	0.000 107
13	0.000 030 ±0.000 01	0.000 027
14	0.000 007 ±0.000 002	0.000 006
	$\bar{n}=3.23$	

$$\rho_M(t) = \int P_2(\alpha')D(\alpha')R(t)D^\dagger(\alpha')\,d^2\alpha', \quad \text{(A1)}$$

where
$$P_2(\alpha') = \delta^2(\alpha' - \tfrac{1}{2}\alpha_0),\qquad D(\alpha') = \exp(\alpha' a^\dagger - \alpha'^* a), \quad \text{(A2)}$$

and
$$R(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}[\tfrac{1}{2}\sigma^2(t)]^n \sum_{k=0}^{2n} \binom{2n}{k}(-1)^k \times (a^\dagger a)^{2n-k}|\tfrac{1}{2}\alpha_0\rangle\langle\tfrac{1}{2}\alpha_0|(a^\dagger a)^k. \quad \text{(A3)}$$

Using Eqs. (A2) and (A3) in (A1) and carrying out the integration over α', the probability distribution for the photon occupation number becomes

$$P_m(t) = \langle m|D(\tfrac{1}{2}\alpha_0)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}[\tfrac{1}{2}\sigma^2(t)]^n \times \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(a^\dagger a)^{2n-k}|\tfrac{1}{2}\alpha_0\rangle \times \langle\tfrac{1}{2}\alpha_0|(a^\dagger a)^k D^\dagger(\tfrac{1}{2}\alpha_0)|m\rangle. \quad \text{(A4)}$$

By expanding the coherent states in terms of number states

$$|\tfrac{1}{2}\alpha_0\rangle = \exp(-\tfrac{1}{8}|\alpha_0|^2)\sum_{r=0}^{\infty}\frac{(\frac{1}{2}\alpha_0)^2}{(r!)^{1/2}}|r\rangle,$$

the relevant matrix elements $\langle m|D(\frac{1}{2}\alpha_0)(a^\dagger a)^{2n-k}|r\rangle$ and $\langle s|(a^\dagger a)^k D^\dagger(\frac{1}{2}\alpha_0)|m\rangle$ are easily evaluated

$$\langle m|D(\tfrac{1}{2}\alpha_0)(a^\dagger a)^{2n-k}|r\rangle = r^{2n-k}(-\tfrac{1}{2}\alpha_0^*)^{r-m}\left(\frac{m!}{r!}\right)^{1/2} L_m^{r-m}(\tfrac{1}{4}|\alpha_0|^2)e^{-|\alpha_0|^2/8}, \quad \text{(A5)}$$

and
$$\langle s|(a^\dagger a)^k D^\dagger(\tfrac{1}{2}\alpha_0)|m\rangle = s^k(\tfrac{1}{2}\alpha_0^*)^{m-s}\left(\frac{s!}{m!}\right)^{1/2} L_s^{m-s}(\tfrac{1}{4}|\alpha_0|^2)e^{-|\alpha_0|^2/8}. \quad \text{(A6)}$$

Using Eqs. (A5) and (A6) in Eq. (A4) and performing a few algebraic manipulations, we have

$$p_m(t) = \frac{(\frac{1}{4}|\alpha_0|^2)^m}{m!}\exp(-\tfrac{1}{2}|\alpha_0|^2)\sum_{r,s}^{\infty}\exp[-(r-s)^2\tfrac{1}{2}\sigma^2(t)]L_s^{m-s}(\tfrac{1}{4}|\alpha_0|^2)L_r^{m-r}(\tfrac{1}{4}|\alpha_0|^2), \quad \text{(A7)}$$

where, in order to achieve a more symmetrical form, we have used the identity[32]

$$L_l^{n-l}(x) = (-x)^{l-n}\frac{n!}{l!}L_n^{l-n}(x). \quad \text{(A8)}$$

APPENDIX B

In this Appendix, we derive the relationships given in Eq. (25). We begin by making a useful preliminary calculation of the function

$$G_{s,p} \equiv \mathrm{Tr}\{a^{\dagger s}a^p R(t)\}. \quad \text{(B1)}$$

Performing the trace in the number representation, Eq. (B1) can be written

$$G_{s,p} = \sum_{n,m}\langle n|a^{\dagger s}a^p|m\rangle\langle m|R(t)|n\rangle, \quad \text{(B2)}$$

where the projection of the identity operator in the number representation has been inserted between a^p and $R(t)$. Making use of Eq. (12), it is easily verified that

$$\langle m|R(t)|n\rangle = \exp[-(n-m)^2\tfrac{1}{2}\sigma^2(t)]\langle m|R(0)|n\rangle. \quad \text{(B3)}$$

Recognizing that

$$\langle n|a^{\dagger s}a^p|m\rangle = \langle n|a^{\dagger s}a^p|m\rangle\delta_{m-p,n-s}, \quad \text{(B4)}$$

and making use of Eq. (B3), we can write Eq. (B2) as

$$G_{s,p} = \exp[-(s-p)^2\tfrac{1}{2}\sigma^2(t)]\mathrm{Tr}\{a^{\dagger s}a^p R(0)\}. \quad \text{(B5)}$$

In our experiment, we assume

$$R(0) = |\tfrac{1}{2}\alpha_0\rangle\langle\tfrac{1}{2}\alpha_0|, \quad \text{(B6)}$$

and (B5) becomes

$$G_{s,p} = \exp[-(s-p)^2\tfrac{1}{2}\sigma^2(t)](\tfrac{1}{2}\alpha_0^*)^s(\tfrac{1}{2}\alpha_0)^p. \quad \text{(B7)}$$

In order to derive the relationships given in Eq. (25), we make use of Eqs. (A1) and (A2) to write $\rho_M(t)$ as

$$\rho_M(t) = D(\tfrac{1}{2}\alpha_0) R(t) D^\dagger(\tfrac{1}{2}\alpha_0) \ . \tag{B8}$$

Calculating the mean number of photons in the mixed beam, we have

$$\bar{n} = \mathrm{Tr}\{a^\dagger a D(\tfrac{1}{2}\alpha_0) R(t) D^\dagger(\tfrac{1}{2}\alpha_0)\}$$

$$= \mathrm{Tr}\{D^\dagger(\tfrac{1}{2}\alpha_0) a^\dagger a D(\tfrac{1}{2}\alpha_0) R(t)\} \ . \tag{B9}$$

The following properties of the displacement operator $D(\frac{1}{2}\alpha_0)$,

$$[a, D(\tfrac{1}{2}\alpha_0)] = \tfrac{1}{2}\alpha_0 D(\tfrac{1}{2}\alpha_0) \ ,$$

$$D^\dagger(\tfrac{1}{2}\alpha_0) D(\tfrac{1}{2}\alpha_0) = 1 \ , \tag{B10}$$

make it possible to write Eq. (B9) in the form

$$\bar{n} = \mathrm{Tr}\{(a^\dagger + \tfrac{1}{2}\alpha_0^*)(a + \tfrac{1}{2}\alpha_0) R(t)\} \ . \tag{B11}$$

At this point, we can use Eq. (B7) in Eq. (B11) to arrive at the result

$$\bar{n} = \tfrac{1}{2}|\alpha_0|^2 (1 + e^{-\sigma^2/2}) \ . \tag{B12}$$

In order to calculate the mean square of the photon occupation number, we make use of Eq. (B8)

$$\overline{n^2} = \mathrm{Tr}\{(a^\dagger a)^2 D(\tfrac{1}{2}\alpha_0) R(t) D^\dagger(\tfrac{1}{2}\alpha_0)\}$$

$$= \mathrm{Tr}\{D^\dagger(\tfrac{1}{2}\alpha_0) a^\dagger a a^\dagger a D(\tfrac{1}{2}\alpha_0)\} \ . \tag{B13}$$

Using the relationships in Eq. (B10) and the commutation properties of a and $a^\dagger$, Eq. (B13) takes the form

$$\overline{n^2} = \mathrm{Tr}\{[(a^\dagger + \tfrac{1}{2}\alpha_0^*)(a + \tfrac{1}{2}\alpha_0) + (a^\dagger + \tfrac{1}{2}\alpha_0^*)^2 (a + \tfrac{1}{2}\alpha_0)^2] R(t)\} \ . \tag{B14}$$

Finally, Eq. (B7) can be used to evaluate the various traces in Eq. (B14), and we have

$$\overline{n^2} = \tfrac{1}{2}|\alpha_0|^2 (1 + e^{-\sigma^2/2}) + \tfrac{1}{8}|\alpha_0|^4 (3 + 4e^{-\sigma^2/2} + e^{-2\sigma^2}) \ . \tag{B15}$$

Equations (B12) and (B15) are easily inverted to yield Eq. (25).

APPENDIX C

An important correction that must be performed on the experimental distributions before attempting a comparison with the theoretical results is the separation of the signal from the noise.

The noise consists of dark current photoelectron pulses whose voltage height is large enough to clear the dual discrimination process.

A separate experiment on the statistical distribution of dark current photoelectrons has been performed for every modulation experiment and a distribution function $W(n)$ obtained.

It is assumed that $W(n)$ is independent of the probability distribution of the signal[33] $p(n)$ so that the experimental distribution will have the form

$$p_{\mathrm{exp}}(n) = \sum_{m=0}^{n} p(m) W(n-m) \ , \tag{C1}$$

where the distribution $W(n)$ and $p_{\mathrm{exp}}(n)$ are assumed to be experimentally known. From Eq. (C1) we can derive at once

$$p(0) = p_{\mathrm{exp}}(0)/W(0) \ ,$$

and from the knowledge of $p(0)$ we can calculate

$$p(1) = [p_{\mathrm{exp}}(1) - p(0)W(1)]/W(0) \ . \tag{C2}$$

In general, any value $p(n)$ can be generated from the knowledge of $p_{\mathrm{exp}}(n)$, $W(n)$, and the previous values of $p(n)$ up to $p(n-1)$. A computer program has been set up for this purpose.

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[1]F. T. Arecchi, Phys. Rev. Letters 15, 912 (1965).

[2]F. T. Arecchi, A. Berne, and P. Burlamacchi, Phys. Rev. Letters 16, 32 (1966).

[3]F. T. Arecchi, A. Berne, A. Sona, and P. Burlamacchi, IEEE J. Quant. Elect. 2, 341 (1966).

[4]F. T. Arecchi, A. Berne, and A. Sona, Phys. Rev. Letters 17, 260 (1966).

[5]F. T. Arecchi, E. Gatti, and A. Sona, Phys. Letters 20, 27 (1966).

[6]F. T. Arecchi, E. Gatti, and A. Sona, Rev. Sci. Instr. 37, 942 (1966).

[7]F. T. Arecchi, V. Degiorgio, and B. Querzola, Phys. Rev. Letters 19, 1168 (1967).

[8]F. T. Arecchi, M. Giglio, and V. Tartari, Phys. Rev. 163, 186 (1967).

[9]F. T. Arecchi, G. S. Rodari, and A. Sona, Phys. Letters 25A, 59 (1967).
[10]J. A. Bellisio, C. Freed, and H. A. Haus, Appl. Phys. Letters 4, 5 (1964).
[11]R. F. Chang, V. Korenman, and R. W. Detenbeck, Phys. Letters 26A, 417 (1968).
[12]C. Freed and H. A. Haus, Phys. Rev. Letters 15, 943 (1966).
[13]C. Freed and H. A. Haus, IEEE J. Quant. Elect. 2, 190 (1966).
[14]S. Fray, F. A. Johnson, R. Jones, T. P. McLean, and E. R. Pike, Phys. Rev. 153, 357 (1967).
[15]E. Jakeman and E. R. Pike, J. Phys. A 1, 128 (1968).
[16]E. Jakeman, C. J. Oliver, and E. R. Pike, J. Phys. A1, 406 (1968).
[17]E. Jakeman and E. R. Pike, J. Phys. A 1, 690 (1968).
[18]E. Jakeman and E. R. Pike, J. Phys. A 2, 115 (1969).
[19]E. Jakeman and E. R. Pike, J. Phys. A 2, 411 (1969).
[20]I. A. Johnson, R. Jones, T. P. McLean, and E. R. Pike, Phys. Rev. Letters 16, 589 (1966).
[21]I. A. Johnson, R. Jones, T. P. McLean, and E. R. Pike, Optica Acta 14, 35 (1967).
[22]L. Mandel and E. Wolf, Rev. Mod. Phys. 37, 231 (1965).
[23]T. P. McLean and E. Pike, Phys. Letters 15, 318 (1965).
[24]C. J. Oliver and E. R. Pike, Brit. J. Appl. Phys. D1, 1457 (1968).
[25]D. B. Scarl, Phys. Rev. Letters 17, 663 (1966).
[26]A. W. Smith and J. A. Armstrong, Phys. Rev. Letters 16, 1169 (1966).
[27]A. W. Smith and J. A. Armstrong, Phys. Letters 19, 650 (1966).
[28]R. J. Glauber, in Quantum Optics and Electronics, edited by C. DeWitt (Gordon and Breach, Science Publishers, Inc., New York, 1965).
[29]L. Mandel, in Progress in Optics, edited by E. Wolf (North-Holland Publishing Co., Amsterdam, 1963), Vol. II.

[30]The long phase coherence time of the laser source makes it unnecessary to adjust the average path difference with a delay line.

[31]Single mode description can be used here since in performing our experiment we count for a time which is short compared to the intensity fluctuations of the combined beams and the collecting aperture of the photomultiplier is much smaller than a single interference fringe.

[32]This relationship was obtained from B. R. Mollow, Ph. D. dissertation, Harvard University, (1966), Appendix A, p. A-2.

[33]Actually great care must be exercised in selecting a photomultiplier whose dark current pulses are all independent of each other, since correlated pulses, such as those produced by back-scattered soft x rays, would invalidate the derivation. We have made a careful verification of the independence of the noise pulses. One of us (JDK) is very grateful to Dr. I. A. Johnson for pointing this out to him.