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Generalizations of Gaussian Optical Fields

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Stationary Gaussian optical fields are also often called thermal fields because they are generated by natural or thermal sources. By using the amplitude of the field, we show that thermal fields are only a particular case of the more general class of Gaussian fields. Such nonthermal Gaussian fields can be obtained experimentally, and we calculate some properties, concerning particularly the effect of Hanbury Brown and Twiss, photocounting, and coincidence experiments. We show that the thermal fields are the less chaotic or incoherent Gaussian fields. Finally, we introduce pseudo-Gaussian fields which appear in some experiments of diffusion with laser light. They are non-Gaussian fields, but their intensity has the same properties as that of a Gaussian field.

I. INTRODUCTION

Statistical properties of optical fields have been extensively studied in recent years. In particular, we now have very good descriptions of coherence, photon coincidence, and photocounting experiments.[1–3] These descriptions can be achieved by using classical concepts, where the statistical nature of the electromagnetic field is described by means of an appropriate stochastic process. Coherence properties are thus defined by a set of coherence functions which are particular moments of the process. However, the statistical nature of the field is also described quantum-mechanically by a density matrix which allows us to introduce quantum coherence functions.

The correspondence, which is in some cases an equivalence between the classical and the quantum description, can be studied by using coherent states and the P representation. The main result of this study is that, even though the quantum description is more appropriate to a microscopic description, the two points of view are completely equivalent in the case of fields actually studied in the laboratory.[4] Theoretically, there are fields which have no classical equivalence,[5] but up to now they have not been obtained experimentally.

This is particularly the case for natural light and laser light. Ideal laser (or coherent) light is represented by a coherent state or by a nonrandom function of time. Thermal (or Gaussian or chaotic) light is generated by a natural source, and its Gaussian properties appear as a consequence of the central-limit theorem.[6] This statement is also true in the quantum description. An expression for thermal light can also be used in the case of optical fields obtained by the diffusion of laser light in a statistical medium, and some recent experiments seem to indicate that such fields are approximately thermal.[7-10]

Nevertheless, it has never been pointed out that thermal fields are only a particular case of Gaussian optical fields. We will present and study here the most general quasimonochromatic Gaussian field. For this description it will be necessary to summarize the characteristic properties of thermal fields. In particular, we will show that the analytic signal and the amplitude of such fields must possess some special properties. By changing some of these properties, we can obtain fields which are Gaussian but are no longer thermal. Evidently, the question arises whether such fields are only hypothetical or are obtainable experimentally. We will discuss this point and show that it is actually possible to obtain such electromagnetic fields. Therefore, it becomes very interesting to specify the most important statistical properties of such fields; we will present results concerning interference, intensity correlations, coincidence, and photocounting. In particular, we will show that the effect of Hanbury Brown and Twiss can be more important for these than it is for thermal fields. Finally we will introduce the concept of pseudo-Gaussian fields. In fact, there are some non-Gaussian fields whose light-intensity properties are exactly the same as those of the Gaussian field. This situation occurs particularly in diffusion experiments with laser light: The diffused field cannot be Gaussian, because the laser light is not a pure sinusoidal wave, but has phase fluctuations.[11] Nevertheless, counting and coincidence experiments give the same results as for a thermal field. Some properties of such pseudo-Gaussian fields are reviewed. In our discussion, we have chiefly used classical concepts, because the equivalence theorem can be applied to the fields studied here.

II. THERMAL FIELDS

In this section we will summarize some characteristic properties of thermal fields, which will allow us to introduce, in the following sections, Gaussian but nonthermal fields.

We are not interested in polarization problems, and therefore the field can be described by a scalar random function of time $X(t)$. Moreover, throughout the following discussion we will consider only quasimonochromatic fields. The function $X(t)$ is quasimonochromatic if its spectral representation has nonzero components only in a frequency range $\Delta\nu$, such that $(\Delta\nu/\nu_0) \ll 1$, where ν_0 is the mean frequency.

The field is *thermal* if $X(t)$ is a real, zero-mean, quasimonochromatic, stationary, and Gaussian random process, which is therefore defined by a correlation function $\Gamma_X(\tau)$.

For optical problems, and particularly for introducing the light intensity, it is convenient to use the analytic signal (a. s.) of $X(t)$ defined by

$$Z(t) = X(t) + iY(t) \quad , \tag{2.1}$$

where $Y(t)$ is the Hilbert transform of $X(t)$.

Moreover, we introduce the *amplitude* $\overline{Z}(t)$ of the field by

$$\overline{Z}(t) = \overline{X}(t) + i\overline{Y}(t) = Z(t)e^{-i\omega_0 t} \quad , \tag{2.2}$$

where ω_0 is the mean (angular) frequency of the field. Therefore the light intensity is

$$I(t) = |Z(t)|^2 = |\overline{Z}(t)|^2 \quad . \tag{2.3}$$

Now we will summarize the fundamental properties of the amplitude of a thermal field. (i) Since $X(t)$ is quasimonochromatic, $\overline{Z}(t)$ is band-limited and the mean frequency is evidently 0. (ii) Thus, $\overline{Z}(t)$ is not an a. s., because there are negative frequencies in its spectral representation. Therefore, $\overline{X}(t)$ and $\overline{Y}(t)$ are not Hilbert transforms. (iii) The random amplitude $\overline{Z}(t)$ is fully stationary to second order,[12] which means that $\langle\overline{Z}(t_1)\,\overline{Z}(t_2)\rangle$ and $\langle\overline{Z}(t_1)\overline{Z}^*(t_2)\rangle$ are only functions of $t_1 - t_2$. In fact, if $X(t)$ is stationary, we know that the a. s. $Z(t)$ satisfies[13]

$$\langle Z(t_1)Z(t_2)\rangle = 0 \quad , \tag{2.4}$$

$$\langle Z(t_1)Z^*(t_2)\rangle = \Gamma_Z(t_1 - t_2) \quad , \tag{2.5}$$

where $\Gamma_Z(\tau)$ is the first-order coherence function of the field. Therefore, with Eq. (2.2) we obtain for the amplitude $\overline{Z}(t)$,

$$\langle\overline{Z}(t_1)\overline{Z}(t_2)\rangle = 0 \tag{2.6}$$

and

$$\langle\overline{Z}(t_1)\overline{Z}^*(t_2)\rangle = \Gamma_{\overline{Z}}(t_1 - t_2) = \Gamma_{\overline{Z}}(t_1 - t_2)e^{-i\omega_0(t_1 - t_2)} \quad . \tag{2.7}$$

Thus, $\overline{Z}(t)$ is fully stationary to second order. Conversely, if Eqs. (2.6) and (2.7) hold, the field $X(t)$ is stationary. (iv) By introducing the correlation functions of $\overline{X}(t)$ and $\overline{Y}(t)$, Eq. (2.6) can be written in an equivalent form

$$\Gamma_{\overline{X}\overline{X}}(\tau) = \Gamma_{\overline{Y}\overline{Y}}(\tau) \quad , \tag{2.8}$$

$$\Gamma_{\overline{X}\overline{Y}}(\tau) = -\Gamma_{\overline{Y}\overline{X}}(\tau) \quad , \tag{2.9}$$

and therefore Eq. (2.7) becomes

$$\Gamma_{\overline{Z}}(\tau) = 2[\Gamma_{\overline{X}\overline{X}}(\tau) + i\Gamma_{\overline{Y}\overline{X}}(\tau)] \quad . \tag{2.10}$$

We may note that we have exactly the same equations for $X(t)$ and $Y(t)$, as for $\overline{X}(t)$ and $\overline{Y}(t)$ but with the important difference that $\Gamma_{XX}(\tau)$ and $\Gamma_{YX}(\tau)$ are Hilbert transforms, because $\Gamma_Z(\tau)$ is an a.s. That is not true for $\Gamma_{\overline{Z}}(\tau)$, so $\Gamma_{\overline{X}\overline{X}}(\tau)$ and $\Gamma_{\overline{Y}\overline{X}}(\tau)$ are not Hilbert transforms. (v) The amplitude $\overline{Z}(t)$ of a thermal field is a Gaussian random function. Thus, the components $\overline{X}(t)$ and $\overline{Y}(t)$ are also real and Gaussian. (vi) An immediate consequence of Eqs. (2.8) and (2.9) for $\tau = 0$ is that $\langle\overline{X}^2(t)\rangle = \langle\overline{Y}^2(t)\rangle$ and $\langle\overline{X}(t)\overline{Y}(t)\rangle = 0$, so that the instantaneous phase of $\overline{Z}(t)$ has a uniform distribution.

III. INTRODUCTION OF GAUSSIAN NONTHERMAL FIELDS

Let us suppose that the amplitude $\overline{Z}(t)$ is still fully stationary to second order, but that Eq. (2.6) does not hold. In this case the field is only *quasistationary*.[14] In fact, Eq. (2.5) is still valid, and there is always a stationary first-order coherence function, but we have

$$\langle Z(t_1)Z(t_2)\rangle = \langle\overline{Z}(t_1)\overline{Z}(t_2)\rangle e^{i\omega_0(t_1+t_2)} \quad , \tag{3.1}$$

which is not a function of $(t_1 - t_2)$, even if $\langle\overline{Z}(t_1)\,\overline{Z}(t_2)\rangle$ is stationary. Thus, $Z(t)$ is not fully stationary, but only to second order, and $X(t)$ is not stationary. Now if the amplitude $\overline{Z}(t)$ is Gaussian, we obtain a quasistationary Gaussian field which is nonthermal.

A nonstationary field can be considered as unphysical. But the expression "quasistationary" means that there are some physical properties of the field which are still stationary, and in our case one such property is the light intensity defined by Eq. (2.3). Therefore, for experiments on light intensity, the field appears as stationary.

Thus, we can define the most general quasimonochromatic Gaussian field by the correlation matrix of its amplitude $\overline{Z}(t)$

$$\underline{\Gamma}(\tau) = \begin{pmatrix} \Gamma_{\overline{X}\overline{X}}(\tau) & \Gamma_{\overline{X}\overline{Y}}(\tau) \\ \Gamma_{\overline{Y}\overline{X}}(\tau) & \Gamma_{\overline{Y}\overline{Y}}(\tau) \end{pmatrix} \quad , \tag{3.2}$$

and the only condition is $\Gamma_{\overline{X}\overline{Y}}(\tau) = \Gamma_{\overline{Y}\overline{X}}(-\tau)$, because of the stationarity of $\overline{Z}(t)$. A thermal field is a particular case defined by Eqs. (2.8) and (2.9).

In Sec. IV we will study fields with real amplitude for which $\overline{Y}(t) = 0$, and therefore $\underline{\Gamma}(\tau)$ has only one nonzero element, $\Gamma_{\overline{X}\overline{X}}(\tau)$. Before discussing properties of such fields, it is important to consider whether quasistationary Gaussian fields can be obtained physically, or are only theoretical concepts.

It is effectively possible to obtain such fields in experiments on the propagation or diffusion of light in random media. Let us consider a coherent monochromatic beam which passes through a random medium. If we neglect phase fluctuations, the incident beam is described by $e^{i\omega_0 t}$. If the random medium creates only amplitude fluctuations, the beam obtained can be written

$$Z(t) = \overline{X}(t)e^{i\omega_0 t} \tag{3.3}$$

where $\overline{X}(t)$ is, for example, the random transparency of the medium. If $\overline{X}(t)$ is real, stationary, and Gaussian, the output beam is quasistationary and Gaussian, but the light intensity $\overline{X}^2(t)$ is stationary. A more general case with complex transparency can also be considered.

At this point it is interesting to study the complex amplitude of the field obtained by the diffusion of a laser beam in a statistical medium. In fact, as previously noted, many recent experiments using counting and coincidence measurements have shown that such a field is approximately thermal.[7-10]

To discuss this problem we can use a macroscopic description of the fluctuations of the medium which are the origin of the diffusion, by introducing a refractive index $n(\vec{r}, t)$ of the medium whose fluctuations are

$$A(\vec{r}, t) = n(\vec{r}, t) - \langle n\rangle \ . \tag{3.4}$$

In many cases we can assume that $A(\vec{r}, t)$ is a real Gaussian function. The optical field diffused in a direction defined by the impulse transfer $\vec{k}$ can be obtained by a calculation similar to the Born approximation, and in the first approximation the amplitude of the diffused field is [15]

$$\overline{Z}_k(t) = \int_V A(\vec{r}, t)e^{i\vec{k}\cdot\vec{r}}\,d\vec{r} \quad , \tag{3.5}$$

where V is the scattering volume.

To see whether the scattered field is a thermal one or not, we have to study whether the relation (2.6) is satisfied. From Eq. (3.5) we obtain

$$\langle\overline{Z}_k(t)\overline{Z}_k(t-\tau)\rangle = \int_V\int_V \langle A(\vec{r}_1, t)A(\vec{r}_2, t-\tau)\rangle \times e^{i\vec{k}\cdot(\vec{r}_1+\vec{r}_2)}\,d\vec{r}_1\,d\vec{r}_2 \ . \tag{3.6}$$

The index fluctuations $A(\vec{r}, t)$ can be supposed stationary in space and time, and we suppose that the correlation function can be factorized as

$$\langle A(\vec{r}_1, t)A(\vec{r}_2, t-\tau)\rangle = f(\vec{r}_1 - \vec{r}_2)\Gamma_A(\tau) \ . \qquad (3.7)$$

After a change of variables and two integrations, we find

$$\langle \overline{Z}_k(t)\overline{Z}_k(t-\tau)\rangle = \tfrac{1}{2}\Gamma_A(\tau) f(\vec{k})\delta(k) \ , \qquad (3.8)$$

where $f(\vec{k})$ is the Fourier transform of $f(\vec{r})$.

Thus, for other directions than the incident one ($\vec{k}=0$), Eq. (3.8) is identical to Eq. (2.6), and therefore the diffused field is thermal. To understand this fact we may note that the diffusion process creates fluctuations of the amplitude and phase of the field, and therefore the phenomenon is more complex than in the case of random transparency.

IV. PROPERTIES OF GAUSSIAN FIELDS WITH REAL AMPLITUDE

In this section we will study Gaussian fields defined by Eq. (3.3), where $\overline{Z}(t)$ is real, because among the nonthermal fields they are experimentally the most feasible and theoretically the simplest to study.

A. Interference and First-Order Coherence

Even though the field $Z(t)$ defined by Eq. (3.3) is not in the strict sense stationary, we can introduce a correlation (or first-order coherence) function of the field by

$$\Gamma_Z(\tau) = \langle Z(t)Z^*(t-\tau)\rangle$$
$$= \Gamma_{\overline{X}}(\tau)e^{i\omega_0\tau} \ , \qquad (4.1)$$

where $\Gamma_{\overline{X}}(\tau)$ is the correlation function of the amplitude $\overline{X}(t)$. This function is used for the interpretation of the interference experiments. The main intensity obtained in a two-beam interferometer with a delay τ is given by

$$I_\tau = 2[\Gamma_{\overline{X}}(0) + \Gamma_{\overline{X}}(\tau)\cos\omega_0 I]$$
$$= 2I[1 + \gamma_{\overline{X}}(\tau)\cos\omega_0\tau] \ . \qquad (4.2)$$

Therefore, in interference phenomena there is no difference between quasistationary Gaussian and thermal fields. Furthermore, we can even say that for every Gaussian field we can find a thermal field which gives the same interference fringes. This field is the stationary Gaussian field whose correlation function is $\Gamma_{\overline{X}}(\tau)\cos\omega_0\tau$. Thus, to distinguish thermal and Gaussian fields, higher-order properties must be studied.

B. Intensity Correlation Experiments and Second-Order Coherence

Because $\overline{X}(t)$ is real and Gaussian, the correlation function of the light intensity $I(t)=\overline{X}^2(t)$ is given by

$$\Gamma_I(\tau) = I^2[1 + 2\gamma_{\overline{X}}^2(\tau)] = I^2[1 + 2|\gamma_Z(\tau)|^2] \ , \qquad (4.3)$$

where $\gamma_Z(\tau)$ is the complex degree of coherence of the field. For thermal fields, we have a similar expression, but without the factor 2. Therefore, if we characterize the effect of Hanbury Brown and Twiss due to the second-order incoherence of the field by the ratio $h=\Gamma_I(0)/\Gamma_I(\infty)$, we obtain $h=3$, compared with $h=2$ for a thermal field. In some sense we can say that this field is more chaotic or more incoherent in second order than a thermal field.

C. Light-Intensity Probability Distribution

By using the fact that $\overline{X}(t)$ is Gaussian, we find for the probability distribution of the random variable $I(t)$ at the time t,

$$p(i) = [\pi^{1/2}(2i_0 i)^{1/2}]^{-1} e^{-i/2i_0} \ , \qquad (4.4)$$

where $i_0 = \langle I(t)\rangle$, the mean value of the intensity. For thermal fields we have an exponential distribution; a comparison between all Gaussian fields will be given in Sec. 5 (see Fig. 1).

D. Photocount Distribution

Photocounting experiments provide an important tool for investigating the statistical properties of optical fields. These experiments give information about the point stochastic process determined by

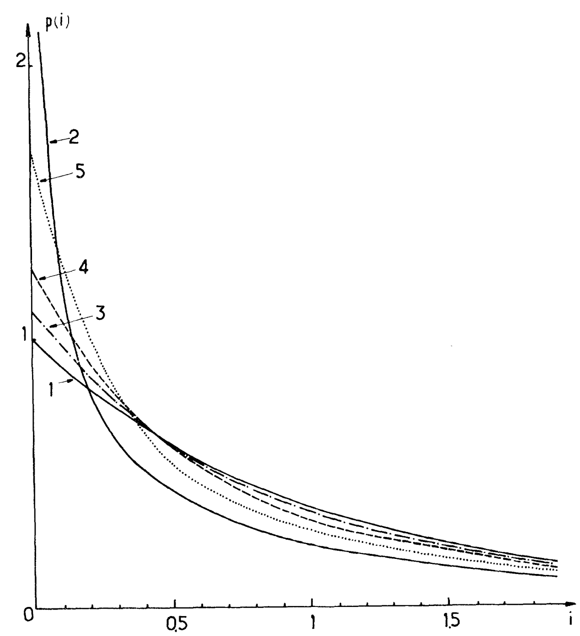

FIG. 1. Intensity probability distribution for Gaussian fields and $i_0=1$. (1) Thermal field; (2) Gaussian field with real amplitude; (3) general Gaussian fields with $\rho=0.4$; (4) with $\rho=0.6$; (5) with $\rho=0.8$.

the emission times $\{t_i\}$ of photoelectrons as measured by a detector immersed in the field.[16]

As recently pointed out,[17] there are many kinds of photocount distributions; for this discussion we will only consider the two most important ones.

An ordinary photocount experiment gives the probability $p_n(T)$ for the emission of n photoelectrons in the time interval $(t, t+T)$. Since the process of $\{t_i\}$ is a Poisson compound stochastic process, we have

$$p_n(T) = (1/n!)\langle\{\exp[-\int_t^{t+T} \alpha I(\theta)\,d\theta]\} \times [\int_t^{t+T} \alpha I(\theta)\,d\theta]^n\rangle \ . \qquad (4.5)$$

To investigate the properties of the field, it is also interesting to study the probability $q_n(T)$ for the emission of n photoelectrons in $(t, t+T)$ with the condition that there is a point at the time t (triggered photocounting distribution). The result is

$$q_n(T) = (1/n!\,i_0)\langle\{\exp[-\int_t^{t+T} I(\theta)\,d\theta]\} \times I(t)[\int_t^{t+T} I(\theta)\,d\theta]^n\rangle \ . \qquad (4.6)$$

In the following equations, for simplicity we put $\alpha = 1$. To avoid complex calculations,[18] we suppose that the sampling time T is much smaller than the coherence time τ_c of the field, so that $\int_t^{t+T} I(\theta)d\theta = I(t)T$; then from Eq. (4.4) we obtain

$$p_n(T) = \frac{\Gamma(n+\frac{1}{2})}{\pi^{1/2} n!} \frac{(2\langle N\rangle)^n}{(2\langle N\rangle + 1)^{n+1/2}} \qquad (4.7)$$

and

$$q_n(T) = 2\,\frac{\Gamma(n+\frac{3}{2})}{\pi^{1/2} n!} \frac{(2\langle N\rangle)^n}{(2\langle N\rangle + 1)^{n+3/2}} \ , \qquad (4.8)$$

where Γ is the factorial (Euler) function, and $\langle N\rangle = i_0 T$ is the mean number of photoelectrons in the time interval $(t, t+T)$.

Some elementary properties of such distributions are of interest. First a comparison between the mean value of N for the distribution p_n and q_n gives

$$\langle N\rangle_q = 3\langle N\rangle_p = 3\langle N\rangle \ , \qquad (4.9)$$

instead of $N_q = 2\langle N\rangle_p$ for a thermal field. This ratio is a measure of the bunching effect of photoelectrons, which is therefore stronger than for thermal fields.

Secondly, we can compute the sequence[9, 17]

$$F(n) = (n+1)\,p_{n+1}/p_n \ , \qquad (4.10)$$

which is an increasing sequence for a compound Poisson process. For a pure Poisson process we have $F(n) = F(0)$; for the thermal field, $F(n) = (n+1)\,F(0)$; and here we deduce from Eq. (4.7) that

$$F(n) = (2n+1)F(0) \ . \qquad (4.11)$$

Therefore, as previously, we can conclude that Eq. (4.9) or (4.11) shows that the field is more chaotic than a thermal field.

E. P-Time Joint Photocounting Distributions

If we wish to evaluate some statistical distributions which are not instantaneous but related to the time evolution of the field, we may in particular study the multitime distribution of the photocounts, $p(n_1T_1, \ldots, n_PT_P)$, which is the probability that the numbers of photoelectrons received during the sampling time intervals $(t_1, t_1+T_1) \cdots (t_P, t_P+T_P)$ are $n_1, \ldots, n_P$. For a Gaussian thermal field this was calculated for $T_i \ll \tau_c$ by Bédard,[18] and more recently by Dialetis[19] for an arbitrary value of T_i. In this section we calculate in the first approximation $(T_i \ll \tau_c)$ the P-fold joint probability distribution $p(n_1T_1, \ldots, n_PT_P)$ given by

$$p(n_1T_1, \ldots, n_PT_P) = \langle \prod_{k=1}^{P} \exp(-I_kT_k)\,[(I_kT_k)^{n_k}/n_k!\,]\rangle \ , \qquad (4.12)$$

where I_k is the light intensity during the time interval (t_k, t_k+T_k). Such a distribution can be obtained by multiple differentiation of the P-fold generating function $G(s_1, \ldots, s_P)$ defined by the relation

$$G(s_1, \ldots, s_P) = \langle \prod_{i=1}^{P} (1-s_i)^{N_i}\rangle \ , \qquad (4.13)$$

where N_i is the random number of photoelectrons detected in the time interval (t_i, t_i+T_i).

For a real-amplitude Gaussian field and when $T_i \ll \tau_c$, we have

$$G(s_1, \ldots, s_p) = \langle \prod_{i=1}^{P} \exp[-(s_iT_i\overline{X}_i^{\,2})]\rangle \ , \qquad (4.14)$$

where $\overline{X}_i = \overline{X}(t_i)$.

Moreover, the P-fold joint probability distribution for the zero-mean real amplitude $\overline{X}(t)$ is given by

$$p(\overline{X}_1, \ldots, \overline{X}_P) = \frac{(\det A)^{1/2}}{(2\pi)^{P/2}} \exp(-\tfrac{1}{2}\overline{X}^{\dagger} A\overline{X}) \ , \qquad (4.15)$$

where $\overline{X}$ is the column matrix formed by the $\overline{X}_i$,

$i=1, 2, \ldots, P$; $\overline{X}^\dagger$ is the corresponding row matrix, and A^{-1} is the symmetric covariances matrix. Using Eqs. (4.14) and (4.15), we see

$$G(s_1, \ldots, s_P) = \frac{(\det A)^{1/2}}{(2\pi)^{N/2}} \times \int \cdots \int \exp(-\tfrac{1}{2}\overline{X}^\dagger B \overline{X})\, dx_1 \cdots dx_p , \quad (4.16)$$

where B is a matrix related to A by

$$B_{ij} = A_{ij} + 2T_i s_i \delta_{ij} \quad . \quad (4.17)$$

Finally, the result can be written

$$G(s_1, \ldots, s_p) = (\det \Delta_N)^{-1/2} , \quad (4.18)$$

where Δ_N is the matrix with elements

$$(\Delta_N)_{ij} = \delta_{ij} + 2\langle X_i X_j \rangle T_i s_i \quad . \quad (4.19)$$

For $P=1$, we obtain $G(s) = (1 + 2\langle N\rangle s)^{-1/2}$ which gives the photocount distribution $p_n(T)$ of Eq. (4.7). For $P=2$, we obtain

$$G(s_1, s_2) = [1 + 2\langle N_1\rangle s_1 + 2\langle N_2\rangle s_2 + 4\langle N_1\rangle\langle N_2\rangle s_1 s_2 (1-\gamma^2)]^{-1/2} , \quad (4.20)$$

where N_1 and N_2 are the numbers of photoelectrons in the two intervals, and γ the normalized (and in this case real) degree of coherence. From Eq. (4.20) we can derive the twofold probability distribution $p(n_1, t_1, T_1;\ n_2, t_2, T_2)$; in particular, we have

$$p(n, t_1, T_1; 1, t_2, T_2) = \frac{\Gamma(n+\frac{1}{2})}{n!\,\pi^{1/2}} \left(4n\langle N_1\rangle\langle N_2\rangle(1-\gamma^2) \frac{A^{n-1}}{B^{n+1/2}} + 2(n+\tfrac{1}{2}) \frac{A^n}{B^{n+3/2}} [\langle N_2\rangle + 2\langle N_1\rangle\langle N_2\rangle(1-\gamma^2)]\right) , \quad (4.21)$$

where $A = 2\langle N_1\rangle + 4\langle N_1\rangle\langle N_2\rangle(1-\gamma^2)$

and $B = 1 + 2\langle N_1\rangle + 2\langle N_2\rangle + 4\langle N_1\rangle\langle N_2\rangle(1-\gamma^2)$.

In these expressions, $\langle N_1\rangle$ and $\langle N_2\rangle$ are, respectively, the mean numbers of photoelectrons registered during the time intervals (t_1, t_1+T_1) and (t_2, t_2+T_2).

When $t_1 = t_2$ and $T_2 \to 0$, we obtain the conditional probability that n photoelectrons are registered during the time interval $(t, t+T)$ on the condition that one photoelectron is registered at the time t. If we set $\gamma = 1$ and $\langle N_2\rangle = 0$ in Eqs. (4.21) we obtain

$$q_n(T) = p(n, t, T; 1, t, 0)/p(1, t, 0) , \quad (4.22)$$

which is just the same as Eq. (4.8).

F. Time-Interval Distributions

Another way to study the statistical properties of photoelectrons is to consider the probability distribution of time intervals. Let us define the lifetime as the random variable T which is the time interval between two photoelectrons, and "residual waiting time" as the time interval between an arbitrary time and the first photoelectron registered after it; and let us denote by $l(\tau)$ and $w(\tau)$ the corresponding probability distributions. In the case of a Poisson compound process we have

$$w(\tau) = \langle I(t+\tau)\exp[-\int_t^{t+\tau} I(\theta)\, d\theta]\rangle \quad (4.23)$$

$$l(\tau) = (i_0)^{-1}\langle I(t)I(t+\tau)\exp[\int_t^{t+\tau} I(\theta)\, d\theta]\rangle .$$

For very large τ_c and a real Gaussian field, we obtain

$$w(\tau) = i_0/(1+2i_0\tau)^{3/2} ,$$
$$l(\tau) = 3i_0/(1+2i_0\tau)^{5/2} . \quad (4.24)$$

Such functions are represented in Fig. 2, where they are compared with the corresponding thermal distributions given by[20]

$$w(\tau) = i_0/(1+i_0\tau)^2 ,$$
$$l(\tau) = 2i_0/(1+i_0\tau)^3 . \quad (4.25)$$

G. Generalization for Arbitrary Coherence Time

When time intervals are not much smaller than τ_c, we can evaluate various statistical distributions in which the random functional $E = \int_t^{t+T} \overline{X}^2(\theta)\, d\theta$ appears. Such calculations have been performed for thermal fields with Lorentzian spectra.[17–19] On the same assumption, we can express the generating function of photocounts $G_\gamma(s)$, for a Gaussian field with real amplitude by

$$G_\gamma(s, \tau) = \frac{e^{\Gamma\tau/2}}{\cosh z + \sinh z[(\Gamma\tau/2z) + (z/2\Gamma\tau)]^{1/2}} , \quad (4.26)$$

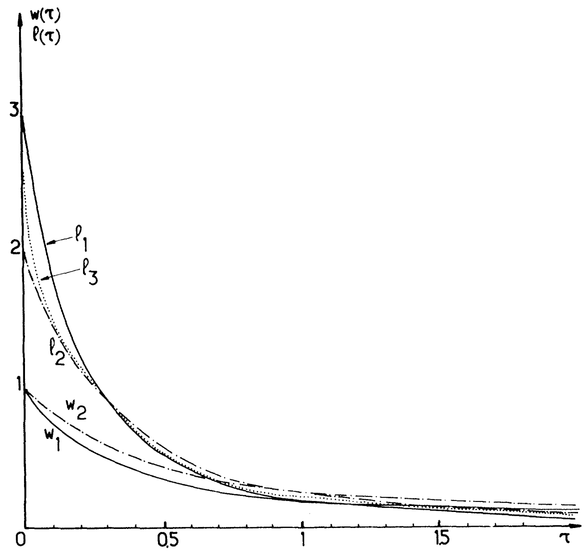

FIG. 2. Time-interval distributions: lifetime l; residual waiting time w; $i_0=1$. l_1, w_1: real Gaussian field l_2, w_2: thermal field l_3: Gaussian field with $\rho=0.5$.

$$\text{where} \quad z=\Gamma\tau(1+4\langle N\rangle s/\Gamma\tau)^{1/2} \quad . \tag{4.27}$$

In these equations, Γ is equal to τ_c^{-1}, and τ is the time interval in which the Kharünen-Loève expansion is valid. To obtain the photocount distribution $p_n(T)$, we put $\tau=T$ and differentiate repeatedly. The time interval distributions $l(\tau)$ and $w(\tau)$ are given by

$$\begin{aligned} w(\tau) &= G_\gamma(2,\tau)A(\tau) \quad , \\ l(\tau) &= G_\gamma(2,\tau)3A^2(\tau) \quad , \end{aligned} \tag{4.28}$$

where

$$A(\tau)=4i_0\Gamma\,\frac{(\Gamma+\alpha)e^{\alpha\tau}-(\Gamma-\alpha)e^{-\alpha\tau}}{(\Gamma+\alpha)^2e^{\alpha\tau}-(\Gamma-\alpha)^2e^{-\alpha\tau}} \quad , \tag{4.29}$$

where $\alpha=\Gamma(1+4I_0/\Gamma)^{1/2}$. These equations are generally valid for all Γ. For $\Gamma=0$ (τ_c infinite), $A(\tau)$ is equal to $i_0/1+2i_0\tau$, which gives Eq. (4.24).

V. GENERAL GAUSSIAN FIELD WITH COMPLEX AMPLITUDE

In this section we study the most general Gaussian field, defined by the correlation matrix of the complex amplitude $\overline{Z}(t)$ given by Eq. (3.2). Our aim is to obtain some general results and to discuss the comparison with a thermal field.

A. Intensity Correlation

When the amplitude $\overline{Z}(t)$ is complex, the intensity of the field can be written as

$$I(t)=\overline{X}^2(t)+\overline{Y}^2(t) \ , \tag{5.1}$$

where $\overline{X}(t)$ and $\overline{Y}(t)$ are Gaussian. Thus, the intensity correlation function $\Gamma_I(\tau)=\langle I(t)I(t-\tau)\rangle$ can be expressed in terms of the correlation functions of $\overline{X}(t)$ and $\overline{Y}(t)$ by

$$\begin{aligned} \Gamma_I(t) = \langle I\rangle^2 + 2[\Gamma_{\overline{X}\overline{X}}{}^2(\tau)+\Gamma_{\overline{Y}\overline{Y}}{}^2(\tau) \\ +\Gamma_{\overline{X}\overline{Y}}{}^2(\tau)+\Gamma_{\overline{Y}\overline{X}}{}^2(\tau)] \ , \end{aligned} \tag{5.2}$$

where $\langle I\rangle$ is the mean value of the intensity, defined in terms of the variances $\sigma_{\overline{X}}$ and $\sigma_{\overline{Y}}$ of $\overline{X}(t)$ and $\overline{Y}(t)$ by

$$\langle I\rangle = \sigma_{\overline{X}}{}^2\sigma_{\overline{Y}}{}^2/(\sigma_{\overline{X}}{}^2+\sigma_{\overline{Y}}{}^2)^2 \ . \tag{5.3}$$

By using the determinant of the matrix $\underline{\Gamma}(\tau)$, this expression can be written

$$\Gamma_I(\tau)=\langle I\rangle^2+2[\,|\Gamma_{\overline{Z}}(\tau)|^2-2\det\underline{\Gamma}(\tau)] \ , \tag{5.4}$$

whereas for thermal fields we have

$$\Gamma_I(\tau)=\langle I\rangle^2+|\Gamma_{\overline{Z}}(\tau)|^2 \ . \tag{5.5}$$

As previously, we can characterize the effect of Hanbury Brown and Twiss by the ratio $h=\Gamma_I(0)/\Gamma_I(\infty)$ which is

$$h=3-4(1-\rho^2)\,\sigma_{\overline{X}}{}^2\sigma_{\overline{Y}}{}^2/(\sigma_{\overline{X}}{}^2+\sigma_{\overline{Y}}{}^2)^2 \ , \tag{5.6}$$

where ρ is the correlation coefficient of the two random variables $\overline{X}(t)$ and $\overline{Y}(t)$. We can easily see that

$$2\le h\le 3 \ . \tag{5.7}$$

The maximum value is obtained if $\rho=1$ or if one of the variances is equal to zero. The first case means that $\overline{X}(t)$ and $\overline{Y}(t)$ are almost surely proportional, and therefore $\overline{Z}(t)=e^{i\alpha}\overline{X}(t)$.

The minimum value is obtained if $a=0$ and $\sigma_{\overline{X}}=\sigma_{\overline{Y}}$, which means that $\overline{X}(t)$ and $Y(t)$ have the same variances and are independent for the same t. That is evidently true for thermal fields. But as this relation is necessary only for the same t, there are nonthermal fields for which $h=2$.

B. Intensity Distribution

To simplify the notation we now suppose that $\sigma_{\overline{X}} = \sigma_{\overline{Y}} = \sigma$. The twofold probability distribution for the random variables $\overline{X}$ and $\overline{Y}$ at the time t is[21]

$$p(\overline{X}, \overline{Y}) = \frac{1}{2\pi\sigma^2(1-\rho^2)^{1/2}} \exp\left(-\frac{\overline{X}^2+\overline{Y}^2-2\rho\overline{X}\overline{Y}}{2\sigma^2(1-\rho^2)}\right). \tag{5.8}$$

By using Eq. (5.1), we obtain for the intensity probability distribution

$$p(i) = \frac{1}{2\sigma^2(1-\rho^2)^{1/2}} \exp\left(-\frac{i}{2\sigma^2(1-\rho^2)}\right) I_0\left(\frac{\rho i}{2\sigma^2(1-\rho^2)}\right) \tag{5.9}$$

When $\overline{X}$ and $\overline{Y}$ are independent random variables ($\rho = 0$), we obtain the usual exponential function corresponding to the thermal-field intensity distribution. We have plotted, in Fig. 1, $p(i)$ given by Eq. (5.9) for some values of ρ between 0 and 1, and we present in the same figure the probability distribution of the intensity for a real-amplitude Gaussian field given by Eq. (4.4). We see that the shapes of these curves are particularly different for small intensities ($i < \frac{1}{4} i_0$).

C. Photocount Distribution For $T \ll \tau_c$

The intensity distribution given by Eq. (5.9) can be directly used to obtain the photocount distribution when the sampling time T is much smaller than the coherence time. We perform the calculation by residues and obtain

$$p_n(T) = (1-p^2)^{1/2} \frac{a^n}{(1+a)^{n+1}[1-p^2/(1+a)^2]^{n+1/2}} \frac{1}{n!} \sum_{p=0}^{n} \binom{n}{p} \frac{(n+p)!}{p!} \left(-\frac{1}{2}\right)^p \left(1+\frac{1}{[1-p^2/(1+a)^2]^{1/2}}\right)^p, \tag{5.10}$$

where a is equal to $2\sigma^2 T(1-\rho^2)$.

We verify that the usual expression for a thermal field[22] is obtained from Eq. (5.10) by putting $\rho = 0$, and from Eq. (4.7) when $\rho \to 1$.

D. Time-Interval Distribution for Large τ_c

The general formulas for the lifetime and residual-waiting-time distributions $l(\tau)$ and $w(\tau)$ for very large τ_c can be evaluated from Eqs. (5.9) and (4.23). We obtain

$$w(\tau) = (1-\rho^2)^{3/2} i_0 \frac{1+i_0(1-\rho^2)\tau}{\{[1+i_0(1-\rho^2)\tau]^2-\rho^2\}^{3/2}},$$

$$l(\tau) = (1-\rho^2)^{3/2} \frac{i_0}{\{[1+i_0(1-\rho^2)\tau]^2-\rho^2\}^{3/2}} \times\left(\frac{3}{1-\rho^2[1+i_0(1-\rho^2)\tau]^{-2}}-1\right), \tag{5.11}$$

where $i_0 = 2\sigma^2 = \langle I \rangle$.

We have plotted these functions in Fig. 2 for Gaussian fields with a complex amplitude ($p = \frac{1}{2}$ and 1) and with a real amplitude. The functions $w(\tau)$ are nearly independent of ρ; in contrast, the lifetime depends strongly on the correlation coefficient ρ. Its value for $\tau = 0$ is 2, or 3 times $\omega(0)$, because of the chaotic properties of the Gaussian field.

VI. PSEUDO-GAUSSIAN FIELDS

As previously noted, thermal fields or quasistationary Gaussian fields can be generated in experiments on propagation or diffusion in random media. For simplicity in exposition, we have supposed that the incident field $e^{i\omega_0 t}$ was perfectly coherent and monochromatic. But in practice even the best stabilized monomode lasers have phase fluctuations,[23] and therefore the incident field must be written as

$$Z_i(t) = \exp i[\omega_0 t + \Phi_i(t)] , \tag{6.1}$$

where $\Phi_i(t)$ is a random process describing the phase evolution of the incident field.[24] Therefore, instead of Eq. (3.3), the diffused field is now described by

$$Z(t) = \overline{Z}(t) \exp i[\omega_0 t + \Phi_i(t)] . \tag{6.2}$$

However, as a consequence of this expression, we see that even if the amplitude $\overline{Z}(t)$ is Gaussian, the field $Z(t)$ is no longer Gaussian, because of the random process $\Phi_i(t)$. We call such a field a pseudo-Gaussian field. Indeed, for experiments on light intensity this field has exactly the same properties as the Gaussian one defined by $\overline{Z}(t) e^{i\omega_0 t}$. And all the experiments described previously are on light intensity (intensity correlations, photocounting, etc.). To show that the field is not Gaussian it would be necessary to study the statistical properties of the phase, which has not yet been done.

Another consequence of Eq. (6.2) is that the field is now stationary. Indeed, its amplitude is evidently $\bar{Z}(t)e^{i\Phi_i(t)}$ and, in general, the phase at any time is uniformly distributed, so that Eq. (2.6) holds.

Finally, in such experiments we obtain a stationary pseudo-Gaussian field. Evidently, this field can be pseudothermal, and that is probably the case in experiments on diffusion in rotating ground glass or solutions of macromolecules.[25] We can see that such a field is not thermal by comparing first- and second-order coherence experiments. If we suppose that random processes $\bar{Z}(t)$ and $\Phi_i(t)$ are independent, the first-order coherence function is given by

$$\Gamma_Z(\tau) = \Gamma_{\bar{Z}}(\tau)\,\langle e^{i\Delta\Phi(\tau)}\rangle e^{i\omega_0\tau} \quad , \qquad (6.3)$$

where $\Delta\Phi(\tau) = \Phi_i(t+\tau) - \Phi_i(t)$.

Therefore, if we carry out an interference experiment, or a measurement of optical spectrum, we obtain a result which depends on second-order properties of $\bar{Z}(t)$ and on the incident light. For example, if the random medium has only slow fluctuations, the coherence time of the field is that of the incident field. So with very slowly rotating ground glass it is not possible to obtain thermal fields with very long coherence times[26]; they cannot be smaller than that of the incident field. Nevertheless, in an intensity-correlation or photocounting experiment, the phase fluctuations disappear completely, and we obtain exactly the same results as for a true thermal field.

More complex experiments[27] on second-order coherence functions may be able to prove that the field is not Gaussian but only pseudo-Gaussian.

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