PHYSICAL REVIEW A VOLUME 1, NUMBER 2 FEBRUARY 1970

Interrelation of Static, Magnetic, and Dynamic Turbulent Pressures of a Conducting Fluid in a Magnetic Field

H. E. Wilhelm
Colorado State University, Fort Collins, Colorado 80521
(Received 23 May 1969)

The relation between the mean-square fluctuations of magneto-static and dynamic pressures is derived for a turbulent fluid of ideal conductivity in an exterior magnetic field.

In turbulent fluids, stochastic magnetic fields are generated by self-excitation, if the kinematic viscosity is larger than the magnetic viscosity.[1-8] Another possibility is the induction of a random magnetic field by velocity fluctuations transverse to an exterior magnetic field. The latter phenomenon is evidently not dependent on the self-excitation process. In the following, the stationary equilibrium between magneto-static and dynamic turbulent pressures is analyzed for an ideally conducting fluid in an exterior magnetic field.

Within the frame of the magnetohydrodynamic approximation, the equation of motion of a unit fluid element in a magnetic field is[9]

$$\rho\left(\frac{\partial}{\partial t}\vec{V}+\vec{V}\cdot\nabla\vec{V}\right)=-\nabla\left(P+\frac{\vec{B}\cdot\vec{B}}{2\mu}\right)+\frac{1}{\mu}\vec{B}\cdot\nabla\vec{B}\ , \qquad (1)$$

where $\nabla\cdot\vec{V}=0$. (2)

The motion of a conducting fluid across an exterior magnetic field induces an interior magnetic field such that the resulting total magnetic field moves with the fluid ("frozen-in field"). The corresponding induction equation is[9]

$$\frac{\partial}{\partial t}\vec{B}=\nabla\times(\vec{V}\times\vec{B})\ , \qquad (3)$$

where $\nabla\cdot\vec{B}=0$. (4)

A component of any physical variable $q(\vec{r}, t)$ is composed of its spatial average $\langle q(\vec{r}, t)\rangle$ and its nonlinear fluctuation $\tilde{q}(\vec{r}, t)$:

$$q(\vec{r}, t)=\langle q(\vec{r}, t)\rangle+\tilde{q}(\vec{r}, t)\ ,$$
$$\langle q(\vec{r}, t)\rangle=\lim_{R\to\infty}\frac{1}{R}\iiint_R q(\vec{r}+\vec{r}^*, t)d\vec{r}^*\ . \qquad (5)$$

The fluid considered is assumed to be at rest, stationary, and homogeneous in the mean, and exposed to a constant exterior magnetic field. Accordingly,

$$\langle\vec{V}\rangle=0,\quad \langle\vec{B}\rangle=\vec{B}_0,\quad \langle P\rangle=P_0\ ,$$
$$\langle\rho\rangle\equiv\rho,\quad \langle\mu\rangle\equiv\mu. \qquad (6)$$

Let the exterior magnetic field be applied in the y direction and its intensity be large compared to the amplitude of the stochastic magnetic field (e.g., $\vec{B}_0$ intensities of the order of 1 Tesla):

$$\vec{B}_0=(0, B_0, 0),\quad |\vec{B}_0|\gg|\tilde{B}(\vec{r}, t)|\ . \qquad (7)$$

The turbulent fluid motion is nonhomogeneous and, in general, also nonisotropic. An appropriate collective of such turbulent fluid motions $\tilde{V}=\{\tilde{V}_x, \tilde{V}_y, \tilde{V}_z\}$ is represented by[10]

$$\tilde{V}_x=\hat{V}_x\cos\alpha x\sin\beta y\sin\gamma z\ ,$$
$$\tilde{V}_y=\hat{V}_y\sin\alpha x\cos\beta y\sin\gamma z\ , \qquad (8)$$
$$\tilde{V}_z=\hat{V}_z\sin\alpha x\sin\beta y\cos\gamma z\ .$$

The wave vector of the turbulent velocity field is $\vec{k}=\{\alpha, \beta, \gamma\}$, Eq. (8). Thus, the space occupied by the fluid is partitioned in elementary turbulence cells of side lengths

$$l_x=2\pi/\alpha,\quad l_y=2\pi/\beta,\quad l_z=2\pi/\gamma\ .$$

Because of the condition of incompressibility [Eq. (2)], the amplitudes $\hat{V}_x$, $\hat{V}_y$, and $\hat{V}_z$ of the velocity fluctuations are related by

$$\alpha\hat{V}_x+\beta\hat{V}_y+\gamma\hat{V}_z=0\ . \qquad (9)$$

In order to eliminate the time derivative, Eq. (1) is substituted into Eq. (2). So one finds upon consideration of Eqs. (6) and (7),

$$-\nabla^2(\tilde{P}+\vec{B}_0\cdot\tilde{\vec{B}}/\mu)=\nabla\cdot(\rho\tilde{\vec{V}}\cdot\nabla\tilde{\vec{V}}-\vec{B}_0\cdot\nabla\tilde{\vec{B}}/\mu)\ . \qquad (10)$$

The corresponding manipulation of Eqs. (3) and (4) leads to no contribution because $\nabla \cdot \vec{B} = 0$. Since

$$\nabla \cdot (\tilde{\vec{V}} \cdot \nabla \tilde{\vec{V}}) = \left(\frac{\partial \tilde{V}_x}{\partial x}\right)^2 + \left(\frac{\partial \tilde{V}_y}{\partial y}\right)^2 + \left(\frac{\partial \tilde{V}_z}{\partial z}\right)^2 + 2\left(\frac{\partial \tilde{V}_y}{\partial x}\frac{\partial \tilde{V}_x}{\partial y} + \frac{\partial \tilde{V}_z}{\partial y}\frac{\partial \tilde{V}_y}{\partial z} + \frac{\partial \tilde{V}_x}{\partial z}\frac{\partial \tilde{V}_z}{\partial x}\right)$$

and $\nabla \cdot (\vec{B}_0 \cdot \nabla \tilde{\vec{B}}) = \vec{B}_0 \cdot \nabla(\nabla \cdot \tilde{\vec{B}}) = 0$,

Eq. (10) yields, upon consideration of Eq. (9) and after some rearrangements,

$$-\nabla^2(\tilde{P} + \vec{B}_o \cdot \tilde{\vec{B}}/\mu) = \tfrac{1}{2}\rho(\alpha \hat{V}_x)^2(\cos 2\beta y \cos 2\gamma z - \cos 2\alpha x) + \tfrac{1}{2}\rho(\beta \hat{V}_y)^2(\cos 2\gamma z \cos 2\alpha x - \cos 2\beta y)$$
$$+ \tfrac{1}{2}\rho(\gamma \hat{V}_z)^2(\cos 2\alpha x \cos 2\beta y - \cos 2\gamma z) \quad . \tag{11}$$

Equation (11) is of the form of the Poisson equation and has the formal solution

$$\tilde{P} + \frac{\vec{B}_0 \cdot \tilde{\vec{B}}}{\mu} = \frac{1}{8\pi}\rho(\alpha \hat{V}_x)^2 \iiint_{-\infty}^{+\infty} \frac{\cos 2\beta y^* \cos 2\gamma z^* - \cos 2\alpha x^*}{[(x-x^*)^2 + (y-y^*)^2 + (z-z^*)^2]^{1/2}} dx^* dy^* dz^*$$
$$+ \frac{1}{8\pi}\rho(\beta \hat{V}_y)^2 \iiint_{-\infty}^{+\infty} \frac{\cos 2\gamma z^* \cos 2\alpha x^* - \cos 2\beta y^*}{[(x-x^*)^2 + (y-y^*)^2 + (z-z^*)^2]^{1/2}} dx^* dy^* dz^*$$
$$+ \frac{1}{8\pi}\rho(\gamma \hat{V}_z)^2 \iiint_{-\infty}^{+\infty} \frac{\cos 2\alpha x^* \cos 2\beta y^* - \cos 2\gamma z^*}{[(x-x^*)^2 + (y-y^*)^2 + (z-z^*)^2]^{1/2}} dx^* dy^* dz^* \quad , \tag{12}$$

hence
$$\tilde{P} + \frac{\vec{B}_0 \cdot \tilde{\vec{B}}}{\mu} = \tfrac{1}{8}\rho \hat{V}_x^{\,2}\left(\frac{\alpha^2}{\beta^2+\gamma^2}\cos 2\beta y \cos 2\gamma z - \cos 2\alpha x\right) + \tfrac{1}{8}\rho \hat{V}_y^{\,2}\left(\frac{\beta^2}{\gamma^2+\alpha^2}\cos 2\gamma z \cos 2\alpha x - \cos 2\beta y\right)$$
$$+ \tfrac{1}{8}\rho \hat{V}_z^{\,2}\left(\frac{\gamma^2}{\alpha^2+\beta^2}\cos 2\alpha x \cos 2\beta y - \cos 2\gamma z\right). \tag{13}$$

Squaring of Eq. (13) and subsequent averaging over an elementary turbulence cell results in the following fundamental relation ($\langle \tilde{V}_i^{\,2} \rangle = \tfrac{1}{8}\hat{V}_i^2$, $i = x, y, z$):

$$\langle (\tilde{P} + \vec{B}_0 \cdot \tilde{\vec{B}}/\mu)^2 \rangle = \kappa^2 \cdot \tfrac{1}{4}\rho^2 \langle \tilde{\vec{V}}^2 \rangle^2 \tag{14}$$

for the magneto-static turbulent pressure, $[\langle (\tilde{P} + \vec{B}_0 \cdot \tilde{\vec{B}}/\mu)^2 \rangle]^{1/2}$, and dynamic turbulent pressure $\tfrac{1}{2}\rho\langle \tilde{\vec{V}}^2 \rangle$, where

$$\kappa^2 = \left\{\hat{V}_x^{\,4}\left[2 + \left(\frac{\alpha^2}{\beta^2+\gamma^2}\right)^2\right] + \hat{V}_y^{\,4}\left[2 + \left(\frac{\beta^2}{\gamma^2+\alpha^2}\right)^2\right] + \hat{V}_z^{\,4}\left[2 + \left(\frac{\gamma^2}{\alpha^2+\beta^2}\right)^2\right]\right\} \Big/ (\hat{V}_x^{\,2} + \hat{V}_y^{\,2} + \hat{V}_z^{\,2})^2 \quad . \tag{15}$$

In the same approximation ($|\vec{B}_0| \gg |\tilde{\vec{B}}|$), one finds from $\tilde{\vec{E}} = -\tilde{\vec{V}} \times \vec{B}_0$ [Eq. (3)] the following relation:

$$\langle \tfrac{1}{2}\epsilon \tilde{\vec{E}}_\perp^{\,2} \rangle = \delta \cdot \tfrac{1}{2}\rho \langle \tilde{\vec{V}}_\perp^{\,2} \rangle \tag{16}$$

for the electric transverse turbulent pressure $\langle \tfrac{1}{2}\epsilon \tilde{\vec{E}}_\perp^{\,2} \rangle$, and dynamic transverse turbulent pressure $\langle \tfrac{1}{2}\rho \tilde{\vec{V}}_\perp^{\,2} \rangle$, where

$$\delta = B_0^{\,2}/2\mu / \tfrac{1}{2}\rho c^2 \quad . \tag{17}$$

The function $\kappa = \kappa(\alpha, \beta, \gamma)$ [Eq. (15)] depends only weakly on the form of the perturbation, i.e., on the wave-vector components α, β, and γ. It can be shown[10] that for any combination α, β, and γ

$$1 \leqslant \kappa \leqslant \sqrt{2} \; . \tag{18}$$

In particular, when the turbulence is confined to cubical partitions (free turbulence in unbounded space), Eqs. (9) and (15) give

$$\kappa = \tfrac{3}{2}\sqrt{2} \, , \quad \alpha = \beta = \gamma \; . \tag{19}$$

According to Eqs. (1)–(4), the initial shape of the turbulent velocity fluctuations [Eq. (8)] is not conserved as time progresses. Nevertheless, the basic result obtained in Eq. (14) evidently is valid for any time under conditions of stationary turbulence. The reason for this is that κ is practically independent of the spatial structure of the turbulent fluctuations [Eq. (15)]. Thus, the general conclusion can be drawn that a balance between the magneto-static and dynamic turbulent pressures prevails in a conducting fluid to which a strong magnetic field is applied [Eq. (14)]. The associated electric turbulent pressure is small compared to the dynamic turbulent pressure [Eq. (16)], since $\delta \ll 1$ [Eq. (17)].

In a nonconducting fluid, the turbulent velocity fluctuations do not interact with the magnetic field. In this case, Eq. (14) reduces to an equilibrium relation between static and dynamic turbulent pressures, which is in agreement with experimental observations on ordinary turbulent fluids,[10]

$$\langle \tilde{P}^2 \rangle^{1/2} = \tfrac{1}{2}\kappa\rho\langle \tilde{\vec{V}}^2 \rangle \, . \tag{20}$$

The previous theories on self-excited stochastic magnetic fields in conducting fluids assume a balance between magnetic and dynamic turbulent pressures.[1–3] This assumption is obviously not quite correct, since it implies vanishing of the dynamic turbulent pressure in the limit of a nonconducting fluid.[11]

The investigation was sponsored by the U. S. Office of Naval Research under the auspices of J. A. Satkowski.

---

[1]G. K. Batchelor, Proc. Roy. Soc. (London) A201, 405 (1950).

[2]L. Biermann and A. Schlueter, Phys. Rev. 82, 863 (1951).

[3]E. Fermi, Phys. Rev. 75, 1169 (1949).

[4]S. I. Syrovatskii, Usp. Fiz. Nauk 12, 247 (1957).

[5]K. Moffatt, J. Fluid Mech. 11, 625 (1961).

[6]L. S. Marochnik, Zh. Tekhn. Fiz. 34, 183 (1964) [English transl.: Soviet Phys. – Tech. Phys. 9, 145 (1964)].

[7]R. H. Kraichman and S. Nagarajan, Phys. Fluids 10, 859 (1967).

[8]A. P. Kazantsev, Zh. Eksperim. i Teor. Fiz. 53, 1806 (1968) [English transl.: Soviet Phys. – JETP 26, 1031 (1968)].

[9]L. Spitzer, Jr., Physics of Fully Ionized Gases (Willy-Interscience, Inc., New York, 1966).

[10]G. I. Taylor, Proc. Cambridge Phil. Soc. 32, 380 (1936).

[11]For further criticism of the hypothesis on the equilibrium between magnetic and dynamic turbulent pressures, see, e.g., A. Unsoeld, Physics of Stellar Atmospheres (Springer-Verlag, Berlin, 1968), p. 590.