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Quantum-Mechanical Amplification and Frequency Conversion with a Trilinear Hamiltonian

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A description of the parametric amplifier and frequency converter is presented without introducing the classical (i.e., parametric) approximation for the pumping field. Constants of the motion are found which reduce the solution of the Schrödinger equation to the diagonalization of a matrix. This diagonalization is accomplished numerically, and the eigenvalues and eigenfunctions of a system with fixed energy are calculated. The time-dependent behavior of the mean number of photons in the amplified or frequency up-converted field is presented. The time evolution of the probability distributions is illustrated. The technique is extended to the problem of coherent spontaneous emission from a system of N two-level atoms interacting with the radiation field where both the atomic system and the radiation field are quantized.

1. INTRODUCTION

Recently, there have been great advances made in the construction of light amplifiers and frequency converters. These devices are based on the coupling of light waves in nonlinear dielectric crystals such as $LiNbO_3$.[1,2] A photon from an intense monochromatic laser beam, the pump, couples with a signal photon to induce emission of a photon in the idler field. In parametric application, the idler photon is emitted at the difference frequency, whereas in frequency conversion the idler photon is emitted at the sum frequency. In a particular crystal, one or the other process is suppressed by the requirements of phase matching. Frequency up-conversion of infrared wavelengths into the visible is now finding application in infrared image converters.[2]

The microwave versions of the parametric amplifier and frequency converter have been used in electrical engineering applications for some time. At optical frequencies, photons have sufficient energy to be individually detectable; thus, measurements of photon statistics are possible which provide much more information about the field than do spectroscopic measurements alone. The spontaneous emission of quanta, which is not predicted by classical theory, is an important contribution at optical frequencies. Thus, a theoretical description of the amplification and frequency conversion of light must take quantum effects into account.

Quantum-mechanical models of the parametric amplifier and frequency converter were first proposed by Louisell, Yariv, and Siegman.[3] Detailed analyses of the statistical properties of these devices have been made based on these models.[4,5] In these models, the incident laser or pump field is treated as a classical electromagnetic field of constant amplitude, and is termed the parametric approximation. The parametric approximation does not take into account the depletion of the laser field with the result that the mean number of amplified photons in the parametric approximation grows exponentially with time. An analysis of the amplifier that takes into account the depletion of the pump field has been made by Bloembergen *et al.*,[6] where all three coupled fields are treated classically. This classical analysis treats the spatial behavior of the three interacting fields, whereas our quantum-mechanical treatment will give only temporal effects. As such, our analysis is applicable to three standing waves coupled in a cavity. An example of how quantum theory may treat traveling-wave situations has been given by Tucker and Walls.[7]

We present in this paper a quantum-mechanical model for parametric amplification and frequency conversion without introducing the parametric approximation. Thus, the depletion of the laser field is automatically included in a full quantum-mechanical context. Our calculation refers to an idealized device, since no account is taken of the damping of the modes, which is present in any practical situation. A quantum-mechanical analysis of parametric oscillation including the damping of the cavity modes has been made by Graham and Haken.[8] Their solution is reached by introducing the parametric approximation below threshold and by using a quasilinearization technique above threshold.

We have also studied the problem of coherent spontaneous emission from a system of N two-level atoms interacting with the radiation field, where both the atomic system and the radiation field are quantized. The technique we employ is not limited to the above problems but is applicable to related problems in nonlinear optics, for example, Raman and Brillouin scattering.

2. BASIC HAMILTONIAN

The presence of an electromagnetic field in a dielectric causes a polarization of the medium. Following the now standard procedure,[9] we assume that the polarization $P(\vec{r},t)$ can be expanded in powers of the instantaneous electric field:

$$\vec{P}(\vec{r},t)=\vec{\chi}\cdot\vec{E}(\vec{r},t)+\vec{\chi}:\vec{E}(r,t)\vec{E}(r,t)+\cdots . \tag{1}$$

Here, the first term defines the usual linear susceptibility; the second term defines the lowest-order nonlinear susceptibility. The time-dependent Hamiltonian $H_1(t)$ describing the interaction of the electromagnetic field with the dielectric medium is

$$H_1(t)=-\int\vec{E}(\vec{r},t)\cdot\vec{P}(\vec{r},t)\,d^3\vec{r} . \tag{2}$$

The electric field operator can be expanded in terms of normal modes[10] as

$$\vec{E}(\vec{r},t)=i\hat{e}_k\sum_k\left(\frac{\hbar\omega_k}{2V\epsilon}\right)^{1/2}\times\left(a_k(t)e^{i\vec{k}\cdot\vec{r}}-a_k^+(t)e^{-i\vec{k}\cdot\vec{r}}\right). \tag{3}$$

Here $a_k(t)$ and $a_k^+(t)$ are the annihilation and creation operators for the kth mode; they obey the boson commutation rules

$$[a_k(t),a_k{}^\dagger(t)]=\delta_{kl},\qquad [a_k(t),a_l(t)]=[a_k{}^\dagger(t),a_l{}^\dagger(t)]=0 . \tag{4}$$

V is the normalization volume, ϵ_k is the dielectric constant evaluated at frequency ω_k, and $\hat{e}_k$ is a unit polarization vector with the usual polarization indices omitted for simplicity.

When Eq. (3) is substituted into Eq. (2), the result is a complex coupling of the modes of the interaction Hamiltonian H_1. This expression simplifies considerably if we assume that only three modes are coupled strongly. This situation could be realized by the requirements of phase matching for three modes coupled in a cavity. Based on this assumption, the interaction Hamiltonian becomes

$$H_1(t) = \hbar\kappa[a(t)b^\dagger(t)c^\dagger(t) + a^\dagger(t)b(t)c(t)], \qquad (5)$$

where the three modes are labeled a, b, c, and κ is the coupling constant (taken to be real for convenience). The free-field Hamiltonian $H_0(t)$ for these modes is

$$H_0(t) = \hbar\omega_a a^\dagger(t)a(t) + \hbar\omega_b b^\dagger(t)b(t) + \hbar\omega_c c^\dagger(t)c(t) . \qquad (6)$$

The total time-dependent Hamiltonian is

$$H(t) = H_1(t) + H_0(t) . \qquad (7)$$

This Hamiltonian can represent the parametric amplifier, provided we identify $a(t)$, $b(t)$, $c(t)$ as the annihilation operators of the laser, signal, and idler modes, respectively. Our Hamiltonian can also describe the process of frequency conversion in which a signal photon and a laser photon combine to form an idler photon at the sum frequency if we make the following identifications: $a(t)$ is the idler mode, $b(t)$ is the signal mode, and $c(t)$ is the pump mode.

The same Hamiltonian, in addition, can also be used to study the problem of coherent emission from a system of N two-level atoms interacting with a single mode of the radiation field. The first quantum-mechanical treatment of this problem is due to Dicke[11] who based his solution on first-order time-dependent perturbation theory. Further computations on this problem have been performed by Abate and Haken,[12] Tavis and Cummings,[13] and Bonifacio and Preparata.[14] The Hamiltonian describing this system is the sum of H_0 and H_1, where

$$H_0 = \hbar\omega c c^\dagger + \hbar\omega J_z ,$$
$$H_1 = \hbar\kappa\{cJ^+ + c^+ J^-\}, \qquad (8)$$

where c is the annihilation operator of the single mode of the electromagnetic field with frequency ω. The J operators are defined in terms of σ_z (z components of spin for each atom), σ^+, and σ^- (spin-flip operators for each atom) thus,

$$J_z = \sum\sigma_z, \quad J^+ = \sum\sigma^+, \quad J^- = \sum\sigma^- . \qquad (9)$$

However, the angular momentum operators, J^+ and J^-, can themselves be represented in terms of two operators a, b obeying the boson commutation rules, Eq. (4), such that[15]

$$J^+ = ba^\dagger, \quad J^- = b^\dagger a . \qquad (10)$$

With this substitution, the Hamiltonian in Eq. (9) and the Hamiltonian in Eq. (7) for the parametric amplifier are formally identical. We may make the identification: laser pump equivalent to the upper atomic level; signal photon equivalent to the lower atomic level; idler photon equivalent to the emitted photon.

The eigenvalues of $a^\dagger a$ and $b^\dagger b$ are the actual occupation numbers of the upper and lower levels, respectively, provided the cooperation number[11] J of the N atom system is a maximum, i.e., $J = \frac{1}{2}N$. If J is not a maximum, i.e., $0 \leq J < \frac{1}{2}N$, then the eigenvalues of $a^\dagger a$ and $b^\dagger b$ are the effective occupation numbers of the upper and lower levels, respectively.[14] We have assumed $J = \frac{1}{2}N$ in our analysis, thus the eigenvalues n_a and n_b can be identified as the actual occupation numbers. This means that the amplifier system and the two-level atom system are formally equivalent and possess identical eigenvalues and eigenfunctions. The results for one system are directly applicable to the other.

3. AMPLIFIER, SPONTANEOUS EMISSION

We shall now present an exact solution of the quantum-mechanical amplifier as described by the Hamiltonian given in Eq. (7). The resultant Heisenberg equations of motion are nonlinear operator equations. Rather than attempt a direct attack upon these equations, we employ an indirect attack and search for constants of the motion. Two constants of the motion can be found and the Schrödinger equation of the system solved.

Under conditions of perfect energy conservation, i.e., $\omega_a = \omega_b + \omega_c$, it can be shown that the Hamiltonians $H_0(t)$ and $H_1(t)$ commute with each other:

$$[H_0(t), H_1(t)] = [H(t), H_0(t)] = [H(t), H_1(t)] = 0. \qquad (11)$$

Thus, $H_0(t)$ and $H_1(t)$ are themselves two constants of the motion. Since $H_0(t)$ and $H_1(t)$ commute with each other, a representation of $H(t)$ exists in which the Hamiltonian is diagonal. Once $H(t)$ has been diagonalized, the Schrödinger equation has been solved. The advantage of this approach lies in the fact that we have reduced the intractable nonlinear problem to a tractable linear one.

The eigenstates of the free field Hamiltonian H_0 are the number states of the form $|n_a, n_b, n_c\rangle$, where

$$a^\dagger a|n_a, n_b, n_c\rangle = n_a|n_a, n_b, n_c\rangle ,$$
$$b^\dagger b|n_a, n_b, n_c\rangle = n_b|n_a, n_b, n_c\rangle , \qquad (12)$$
$$c^\dagger c|n_a, n_b, n_c\rangle = n_c|n_a, n_b, n_c\rangle .$$

If we assume that there are n_a initial laser photons and n_b initial signal photons, then the possible eigenstates of H_0 are

$$\vec{\Psi} = (|n_a, n_b, 0\rangle, |n_a - 1, n_b + 1, 1\rangle, \ldots, \times|0, n_a + n_b, n_b, \rangle)^{\dagger}, \tag{13}$$

all with eigenvalues $\hbar\omega_a(n_a) + \hbar\omega_b n_b$. The eigenstates of the interaction Hamiltonian H_1 are as yet unknown, but we will attempt to express these eigenstates as linear combinations of the eigenstates of H_0. The result of the interaction Hamiltonian operating on the eigenstates of H_0 can be written in matrix notation as

$$H_1\vec{\Psi} = \hbar\kappa\vec{A}_{n_a}\vec{\Psi}, \tag{14}$$

where $\vec{A}_{n_a}$ is an $(n_a+1)\times(n_a+1)$ symmetric matrix, all of whose elements are zero except those on the two diagonals immediately above and below the principal diagonal:

$$\vec{A}_{n_a} = \begin{bmatrix} 0 & a_1 & 0 & 0 & & & & & \\ a_1 & 0 & a_2 & 0 & & & & & \\ 0 & a_2 & 0 & a_3 & & & & & \\ 0 & 0 & a_3 & 0 & \cdot & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \cdot & \\ & & & & & & \cdot & \cdot & a_{n_a} \\ & & & & & & & a_{n_a} & 0 \end{bmatrix} \tag{15}$$

and $$a_r = [(n_a - r + 1)(n_b + r)]^{1/2}, \quad (r = 1, 2, \ldots, n_a). \tag{16}$$

This matrix was derived by induction. Since $\vec{A}_{n_a}$ is symmetric, its eigenvalues are real. In addition, $\vec{A}_{n_a}$ is a continuant matrix.[16] It is possible to prove from the theory of such matrices that the eigenvalues are symmetrically displaced about the value zero.

In order to find the eigenstates and eigenvalues of H_1 we must diagonalize the matrix A_{n_a}. Suppose the orthogonal matrix which diagonalizes A_{n_a} is U_{n_a}, then

$$\vec{U}_{n_a}\vec{A}_{n_a}\vec{U}_{n_a}^{\dagger} = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{n_a}), \tag{17}$$

where $\vec{U}_{n_a}\vec{U}_{n_a}^{\dagger} = \vec{I}$ and the elements of $\vec{U}_{n_a}$ are the eigenvectors of $\vec{A}_{n_a}$ labeled in the form u_{ij} with i denoting the particular eigenvalue of $\vec{A}_{n_a}$ and j the component. (This labeling is contrary to standard practice but is very convenient for our purposes.) Thus, the eigenvalues of H_1 are: $\lambda_0\hbar\kappa, \lambda_1\hbar\kappa, \ldots, \lambda_{n_a}\hbar\kappa$ and the eigenstates ϕ_j of H_1 are given by

$$\Phi = \vec{U}_{n_a}\vec{\Psi}, \tag{18}$$

where $$\vec{\Phi} = (\vec{\phi}_0, \vec{\phi}_1, \ldots, \phi_{n_a})^{\dagger}. \tag{19}$$

We note the inverse transform

$$\vec{\Psi} = \vec{U}_{n_a}^{-1}\vec{\Phi} = \vec{U}_{n_a}^{\dagger}\vec{\phi}, \tag{20}$$

and the orthonormality relation

$$\langle\phi_i|\phi_j\rangle = \delta_{ij}. \tag{21}$$

The eigenstates ϕ_j of H_i are also eigenstates of H_0 and are thus eigenstates of the total Hamiltonian, H. The eigenvalue equation for H is

$$H\phi_j = \hbar(n_a\omega_a + n_b\omega_b + \lambda_j\kappa)\phi_j \,. \tag{22}$$

We first consider the case of spontaneous emission in which the number of signal photons initially is zero, $n_b = 0$. In the atomic system this situation corresponds to all atoms initially excited, i.e., $m = J$ is the eigenvalue of J_z. As an example, the case $n_a = 24$ was calculated using a high-speed computer to obtain the eigenvalues and eigenfunctions of $\vec{A}_{n_a}$.

The probability $p(n, \lambda)$ that the eigenstate having eigenvalue λ contains n idler photons is given by $|u_{\lambda_n}|^2$. A plot of some typical values for $n_a = 24$ is shown in Fig. 1. We note that the eigenstate

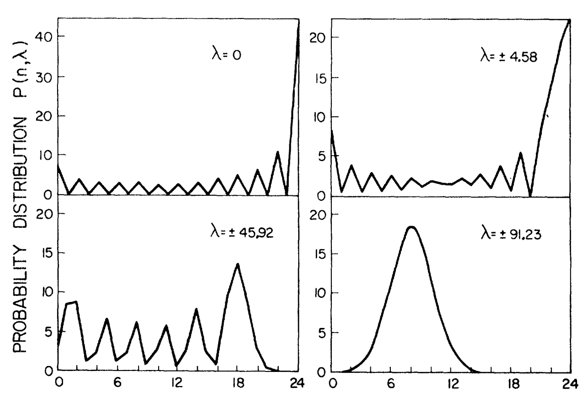

FIG. 1. Probability distribution $P(n, \lambda)$ for $n_a = 24$, $n_b = 0$ for eigenvalues: $\lambda = 0$, ± 4.58, ± 45.92, ± 91.23.

with $\lambda = 0$ has a high probability of containing the full 24 idler photons, while it has zero probability of containing an odd number of photons. The eigenstates with $\lambda = \pm\lambda_{max}$, that is, the ground state and the most highly excited state, have approximately a Poisson distribution. Calculations for other values of n_a (i.e., $n_a = 12$, 13, 23) are qualitatively similar.

The probability of having n idler photons at $\tau \equiv \kappa t$ is

$$P(n, \tau) = |\langle n_a - n, n, n | \exp(-iHt/\hbar) | n_a, 0, 0\rangle|^2 (n \le n_a). \tag{23}$$

In order to evaluate Eq. (23), we expand $|n_a, 0, 0\rangle$ into a linear combination of eigenfunctions using Eq. (20), and using the eigenvector [Eq. (22)] we obtain

$$P(n, \tau) = \left| \langle n_a - n, n, n | \sum_{l=0}^{n_a} U_{l0} e^{-i\lambda_l \tau} \phi_l \rangle \right|^2 . \tag{24}$$

This expression is further reduced when we employ Eq. (18) and the orthonormality relation, Eq. (21). The final result is

$$P(n, \tau) = \left| \sum_{l=0}^{n_a} U_{l0} U_{ln} e^{-i\lambda_l \tau} \right|^2 . \tag{25}$$

Thus, $P(n, \tau)$ has been expressed as the absolute square of a finite trigonometric series. This series is not a Fourier-type series because the λ_l are not constant multiples of each other. The series is actually an almost periodic series[17] and $P(n, \tau)$ is actually an almost periodic function. The mean number $\bar{n}_c(\tau)$ of idler photons at time τ is given by

TABLE I. Amplifier eigenvalues λ_l for $n_a = 24$.

$n_b = 0$	$n_b = 1$	$n_b = 4$	$n_b = 24$
0	0	0	0
±4.58	±5.48	±7.01	±11.81
±10.05	±11.39	±14.16	±23.64
±16.25	±17.84	±21.56	±35.51
±23.00	±24.79	±29.25	±47.43
±30.23	±32.18	±37.27	±59.43
±37.88	±39.97	±45.61	±71.52
±45.92	±48.15	±54.27	±83.72
±54.33	±56.69	±63.26	±96.03
±63.09	±65.56	±72.55	±108.47
±72.17	±74.76	±82.13	±121.04
±81.57	±84.76	±92.01	±133.76
±91.28	±94.09	±102.17	±146.64

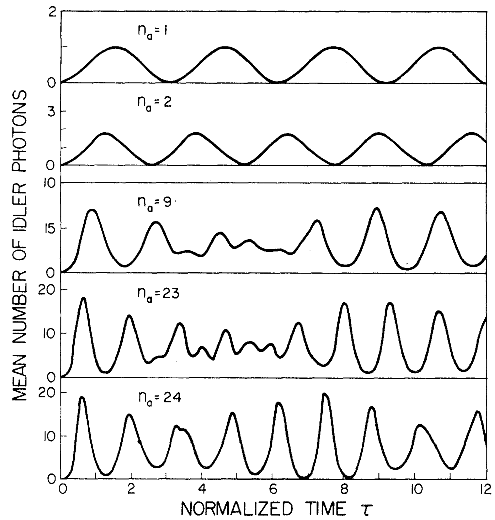

FIG. 2. Mean number of idler photons for $n_a = 1, 2, 9, 23, 24$ as a function of the normalized time.

$$\bar{n}_c(\tau) = \sum_{n=0}^{n_a} n P(n, \tau) . \tag{26}$$

Since $\bar{n}_c(\tau)$ is a sum of almost periodic functions, it is also an almost periodic function. A typical set of eigenvalues is listed in the first column of Table I; inspection of the numerical values confirms that they are not constant multiples of each other.

The mean number of idler photons as a function of time is plotted in Fig. 2 for $n_a = 1, 2, 9, 23, 24$. Number conservation ensures that the mean number of laser photons at time τ is

$$\bar{n}_a(\tau) = n_a(0) - \bar{n}_c(\tau) . \tag{27}$$

We note the oscillating (almost periodic) behavior in direct contrast to the exponentially increasing solution predicted by the parametric approximation. The fluctuations are due to the fact that the vacuum fluctuations for the spontaneous emission are amplified. The contributions from the vacuum will always have an appreciable effect for a small number of photons. If, however, a large number of photons are present, then the vacuum fluctuations, though initially the source of the field, are quickly masked. As the number of photons becomes very large the classical limit is reached and the mean number of photons behaves as an elliptic function.[14]

The probability distribution $P(n, \tau)$ is plotted in Fig. 3 at several fixed times for the case $n_a = 24$. For very short times ($\tau \le 0.05$), the probability distribution follows a power-law distribution.[14] This result may readily be seen by observing that for short times we may replace $a(t)$ by a constant

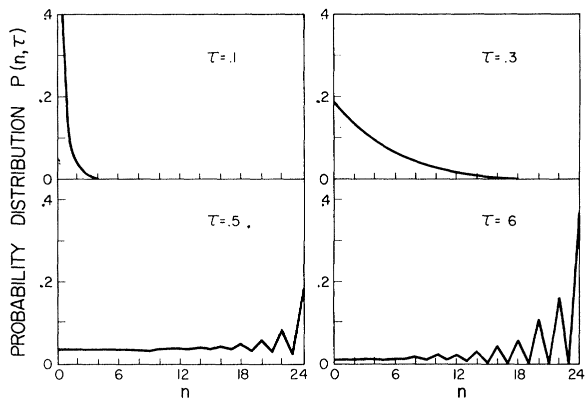

FIG. 3. Probability distribution $P(n, \tau)$ for $n_a = 24$ as a function of n.

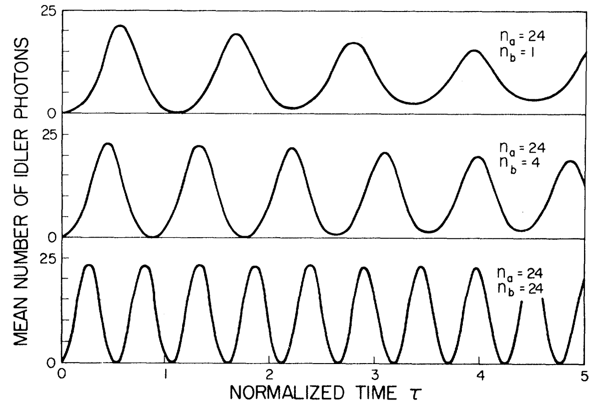

FIG. 4. Mean number of idler photons, $\bar{n}_c(\tau)$, for: (a) $n_a = 24$, $n_b = 1$; (b) $n_a = 24$, $n_b = 4$; (c) $n_a = 24$, $n_b = 24$.

classical amplitude $\alpha e^{-i\omega_a t}$. The Hamiltonian (5) then reduces to that of the parametric amplifier which leads to emission of photons in a chaotic state.[4] However, as the time increases, the distribution becomes peaked about increasingly higher values of $\bar{n}_c$ out to the first maximum of n_c at $\tau \approx 0.6$ (see Fig. 2). The probability distribution in the vicinity of $\tau = 0.6$ has zero probability for containing an odd number of photons. This phenomenon is due to the fact that the eigenstate with $\lambda = 0$ (Fig. 1) dominates near $\bar{n}_{\max}$. The photon distributions exhibit an oscillatory behavior with respect to time similar to that of the mean numbers of photons.

Due to the conservation relation $m + n = \frac{1}{2}N$, the probability distribution may also be considered as a function of m, $P(m)$. It has been advocated[11] that a means of preparing a superradiant state is to completely invert the system $m = J$ and let it decay by spontaneous emission. However, one sees from Fig. 3 that there is no peak of the probability distribution $P(m)$ about $m = 0 (n = 12)$, at least for this small number of atoms. Thus, it appears that the back reaction of the emitted chaotic photons on the atoms prevents the formation of a superradiant state by this method. (The authors are indebted to Dr. H. Thomas for this observation.) The other method suggested by Dicke for preparing a superradiant state, by starting with the atoms in the ground state $m = -J$ and pumping the system to the $m = 0$ state is discussed in Sec. 5.

4. AMPLIFIER STIMULATED EMISSION

Now consider the case where there are n_a incident laser photons and n_b signal photons (i.e., stimulated emission). The mean number of idler photons for $n_a = 24$ is shown in Fig. 4. We see a considerable smoothing of the behavior of $\bar{n}_c(\tau)$ as the number of signal photons is increased. The behavior becomes almost completely classical for $n_b > 1$.[6] Thus, the presence of a finite number of signal photons is sufficient to mask the effects of the vacuum fluctuations. The eigenvalues are listed in Table I for the various values of n_b. As n_b is increased, the eigenvalues tend toward becoming constant multiples of each other with the result that the behavior of $\bar{n}_c(\tau)$ approaches truly periodic behavior.

The special case for which $n_a = n_b$ is equivalent in the atom system to the superradiant state $m = 0$, where the total number of atoms is $n_a + n_b$ (see Fig. 4(c) for $n_a = n_b = 24$). A smooth classical-type behavior is exhibited. The probability distribution of the emitted photons, as shown in Fig. 5, closely follows a Poisson distribution for very short times (i.e., $\tau < 0.05$). This distribution is the same distribution of photons as is present in a coherent state.[18] This result may be seen by observing that for short times we may replace the operators $a(t)$ and $b(t)$ by classical amplitudes $\alpha e^{-i\omega_a t}$ and $\beta e^{-i\omega_b t}$. The Hamiltonian (5) then reduces to that

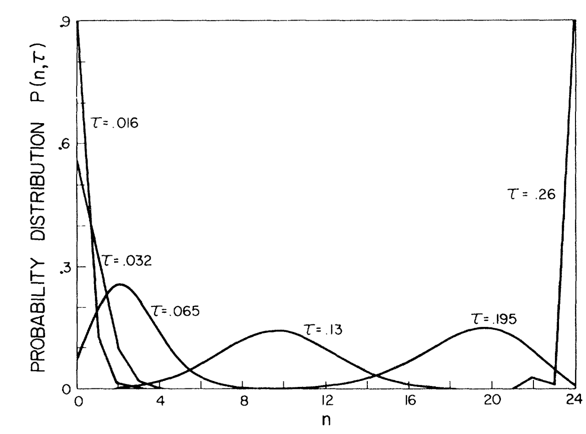

FIG. 5. Probability distribution $P(n, \tau)$ for $n_a = n_b = 24$ (superradiant state) as a function of n.

of a classically driven oscillator which radiates into a coherent state. For large times, the distributions proceed smoothly peaking at higher values of $\bar{n}$ out to the first maximum at $\tau = 0.26$.

5. FREQUENCY CONVERTER

The Hamiltonian listed in Eq. (9) can also describe the process of frequency conversion. We adhere to the same definitions for the operators and write the Hamiltonian for frequency conversion as

$$H_1(t) = \hbar\kappa[a(t)b(t)c^+(t) + a^+(t)b^+(t)c(t)] \ . \qquad (28)$$

Perfect energy conservation requires $\omega_a = \omega_c - \omega_b$. We observe that there is no spontaneous emission for the frequency converter. If the initial signal photons are not present, then the converter does not go. Hence, we shall consider the case of n_a incident laser photons and n_b signal photons where $n_a \geq n_b$. In practice, we usually encounter $n_a \gg n_b$.

The eigenstates of $H_0(t)$ are

$$\vec{\Psi} = (|n_a, n_b, 0\rangle, |n_a - 1, n_b - 1, 1\rangle, \ldots, |n_a - n_b, 0, n_b\rangle)^+ \ , \qquad (29)$$

with eigenvalues

$$\hbar(n_a\omega_a + n_b\omega_b) \ . \qquad (30)$$

The result of H_1 operating on the eigenstates of H_0 is

$$H_1\vec{\Psi} = A_{n_b}\vec{\Psi} \ , \qquad (31)$$

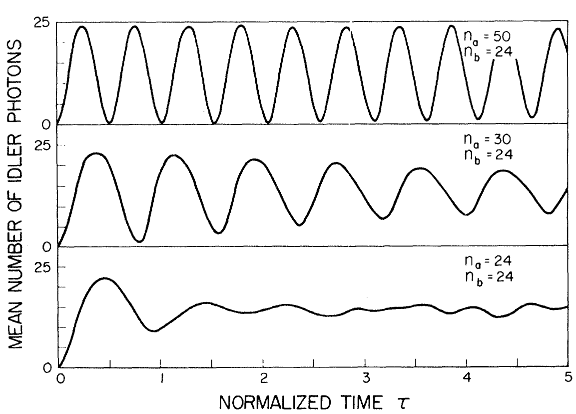

FIG. 6. Mean number of idler photons (frequency converter) for: $n_a = 50$, $n_b = 24$; $n_a = 30$, $n_b = 24$; $n_a = 30$, $n_b = 24$; $n_a = 24$, $n_b = 24$ as a function of the normalized time.

TABLE II. Frequency converter eigenvalues λ_l for $n_b = 24$.

$n_a = 50$	$n_a = 24$
0	0
± 12.16	4.58
± 24.33	10.05
± 36.55	16.25
± 48.81	23.00
± 61.14	30.23
± 73.56	37.88
± 86.07	45.92
± 98.69	54.33
± 111.43	63.09
± 124.29	72.17
± 137.29	81.57
± 150.44	91.28

where A_{n_b} is a matrix of the form Eq. (15), with

$$a_r = [(n_a - r + 1)(n_b - r + 1)r]^{1/2} \ . \qquad (32)$$

This expression was also derived by induction.

The behavior of the frequency converter is illustrated in Fig. 6 for two sets of initial conditions:

(a) $n_a = 50$, $n_b = 24$, (b) $n_a = 30$, $n_b = 24$,

(c) $n_a = 24$, $n_b = 24$.

We observe that in case (a), the frequency converter follows closely a sinusoidal behavior with essentially complete conversion of the signal taking place each cycle. In case (c), where the number of pump photons equals the number of signal photons, the nonlinear effect dominates and the behavior deviates considerably from any cyclic pattern. Case (b) is intermediate in behavior. The eigenvalues for these three cases are listed in Table II.

Thus, we can conclude that for $n_a \gg n_b$, the parametric approximation is a valid approximation for the frequency converter. We contrast this with the parametric amplifier where the laser depletion is the source of the amplification and the parametric approximation is valid only for very short times.

The probability distribution of the idler photons is shown in Fig. 7 for times up to the first maximum. We note the smooth transition from a distribution peaked about $\bar{n} = 0$ at $\tau \approx 0$ out to a distribution peaked about $\bar{n}_{max}$ at $\tau = 0.6$. The initial distribution is Poisson irrespective of the values of n_a and n_b, although the extent to which it remains Poisson in time depends on n_a and n_b.

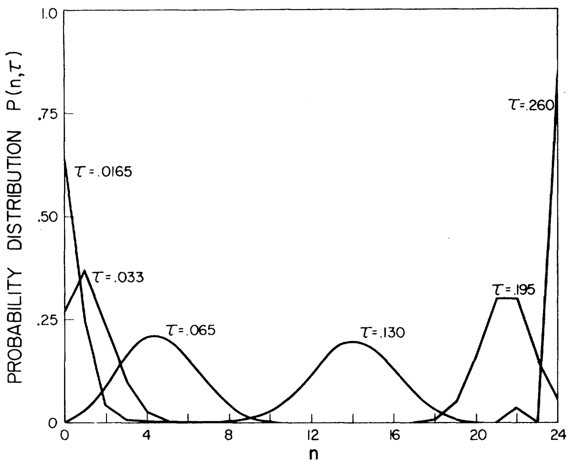

FIG. 7. Probability distribution $P(n, \tau)$ for $n_a = 50$, $n_b = 24$ as a function of n.

The process of frequency conversion corresponds to pumping an atomic system from the ground level to the excited level. This is the second method advocated by Dicke[11] for preparing a superradiant state. If one considers the probability distribution in Fig. 7 as a function of m, one observes that between $\tau = 0.065$ and $\tau = 0.130$ a state peaked about $m = 0$, $(n = 12)$ will be formed. Thus, this appears to be a suitable method for preparing a superradiant state.

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