which is the exact classical Josephson equation. Perhaps the most convenient way to interpret Eqs.(31a) and (31b) is to write

$$\kappa \dot{J}_{\phi\,\mathrm{crossed}} \equiv -\sum_i \Delta J_{\phi i} \frac{\Delta\kappa_i}{\Delta t} .$$

$$= \text{rate at which vorticity is flowing across the streamlines of } \vec{v}_\phi . \quad (32)$$

We then use our original interpretations, but say that when $\vec{v}_\phi$ is not uniform over short sections of vortex core, we must use Eq. (32) for calculating the rate at which vortex lines (or vorticity) is crossing the potential current.

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Electron-Impact Excitation of Be^+, Mg^+, and Ca^+ Ions

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The classical binary-encounter model for the ionization of ions by impact of electrons given by Thomas and Garcia has been extended to calculate the excitation cross section of ions. The electron-impact excitation cross sections for the $3s$-$3p$ and the $3s$-$3d$ transitions in Mg^+, the $2s$-$2p$ transition in Be^+, and the $4s$-$4p$ transition in Ca^+ have been calculated. The results are compared with calculations based on the close-coupling and the Coulomb-Born approximations. Our results agree better with close-coupling calculations than with calculations based on the Coulomb-Born approximation.

INTRODUCTION

In recent years, considerable effort has been devoted to the study of the electron-impact excitation of atoms.[1] Very few attempts, however, have been made to calculate the excitation cross section of positive ions because of the difficulty of including the Coulomb field which acts upon the incident electron throughout its trajectory and distorts the linear path. The quantum-mechanical calculations using the Coulomb-Born and the close-coupling approximations have been made for a few ions.[2–7] The classical binary-encounter model,[8] which provides a simple method of estimating the ionization and the excitation cross sections of atoms,[9,10] has not yet been used to calculate the excitation cross section of ions.

Recently, Thomas and Garcia[11] have discussed a model solution of the problem of the ionization of ions within the framework of the binary-encounter approximation in which they have taken into consideration the residual field of the ion. Here, we have extended the same approach to calculate the electron-impact excitation cross section of the positive ions Be^+, Mg^+, and Ca^+. The transitions studied are: $Be^+(2s\text{-}2p)$, $Mg^+(3s\text{-}sp$ and $3s\text{-}3d)$, and $Ca^+(4s\text{-}4p)$.

THEORY

In the binary-encounter model, the significant interaction is the energy exchange between the incident charged particle of velocity $\vec{v}_1$ and an atomic electron in the ith shell of velocity $\vec{v}_{2i}$. The excitation cross section from the ground state to a state n of any atom due to an incident electron of kinetic energy E_1, under classical impulse approximation, is given by

$$\sigma_{\text{exc}} = \sum_i n_i \int_{U_n}^{U_{n+1}} \sigma_{\Delta E}^{\text{eff}}(\vec{v}_1, \vec{v}_{2i})\, d(\Delta E) , \quad \text{if } E_1 > U_{n+1} \qquad \text{(1a)}$$

$$= \sum_i n_i \int_{U_n}^{E_1} \sigma_{\Delta E}^{\text{eff}}(\vec{v}_1, \vec{v}_{2i})\, d(\Delta E) , \quad \text{if } U_n < E_1 < U_{n+1} \qquad \text{(1b)}$$

where U_n and U_{n+1} are the relative energies of the states n and $n+1$, $\sigma_{\Delta E}^{\text{eff}}$ is the cross section for the exchange of energy ΔE, and n_i is the number of electrons in the shell whose energy is U_i. Equation (1) is to be averaged over the velocity distribution of the bound electrons.

In the case of ions of effective charge Z', Thomas and Garcia assumed that the binary collision took place at a distance ξ from the nucleus which resulted in an energy transfer ΔE. If ionization energy is U, then for ionization $\Delta E \geqslant U$, whereas for excitation $U_n \leqslant \Delta E \leqslant U_{n+1}$. The kinetic energy at the collision radius ξ is $E_1' = E_1 + Z'/\xi \geqslant E_1$, so that the total cross section for the energy exchange collision is

$$\sigma'_{\text{exc}}(E_1', U_n) = \left\langle \int_{U_n}^{U_{n+1}} \sigma_{\Delta E}^{\text{eff}}(\vec{v}_1', \vec{v}_2)\, d(\Delta E) \right\rangle, \quad \text{if } E_1' > U_{n+1} \qquad \text{(2a)}$$

$$= \left\langle \int_{U_n}^{E_1'} \sigma_{\Delta E}^{\text{eff}}(\vec{v}_1', \vec{v}_2)\, d(\Delta E) \right\rangle, \quad \text{if } U_n < E_1' < U_{n+1} . \qquad \text{(2b)}$$

Here $\langle\ \rangle$ denotes the average over the speed distribution of the bound electron. Following an approach analogous to that of Thomas and Garcia, the expression for the total excitation cross section from the ground state to the state n of the ion is given by

$$\Sigma = \tfrac{1}{4}\Sigma' \left(1 + \left\{ 1 + \frac{2Z'\pi}{\beta_1 \Sigma'} \left[\frac{Z'}{\beta_1' - \beta_1} - \left(\frac{Z'^2}{(\beta_1' - \beta_1)^2} - \frac{\Sigma'}{\pi} \right)^{1/2} \right] \right\}^{1/2} \right)^2 . \qquad (3)$$

Here $\Sigma = U^2\sigma$, $\Sigma' = U^2\sigma'$, $\beta_1 = E_1/U$,

$$\beta_1' = E_1'/U = \beta_1 + \tfrac{3}{2}z'(z'+1)/[\tfrac{3}{4}(z'+1)^2 + \Delta^2] , \quad \text{if } \tfrac{1}{2}(z'+1) > \Delta$$

$$= \beta_1 + 3z'\Delta/\{3\Delta^2 + [\tfrac{1}{2}(z'+1)]^2\} , \quad \text{if } \tfrac{1}{2}(z'+1) < \Delta ,$$

$$R^2 = U_{n+1}/U, \quad \text{and} \quad M^2 = U_n/U .$$

For excitation, Δ is given by

$$\Delta = (\beta_1 - M^2)^{-1}[(\beta_1/M^2 - 1)^{1/2} + 1] .$$

The factor to the right of $\frac{1}{4}\Sigma'$ in Eq. (3) represents the magnification due to the curvature of the path of the incident electron in the field of the ion. When $E_1' > U_{n+1}$, Σ' is given by

$$\Sigma'(\beta_1', \beta_2, R^2) = \frac{\pi}{\beta_1'} \left[\frac{2}{3}\beta_2 \left(\frac{1}{M^4} - \frac{1}{R^4} \right) + \left(\frac{1}{M^2} - \frac{1}{R^2} \right) \right] , \quad \text{if } 0 \leqslant \beta_2 \leqslant \beta_1^1 - R^2$$

$$= \frac{\pi}{3\beta_1'} \left(\frac{2\beta_2 + 3M^2}{M^4} - \frac{3}{\beta_1' - \beta_2} - \frac{2(\beta_1' - R^2)^{3/2}}{\beta_2^{1/2} R^4} \right) , \quad \text{if } \beta_1' - R^2 \leqslant \beta_2 \leqslant \beta_1' - M^2$$

$$= \frac{2\pi}{3\beta_1' \beta_2^{1/2}} \left(\frac{(\beta_1' - M^2)^{3/2}}{M^4} - \frac{(\beta_1' - R^2)^{3/2}}{R^4} \right) , \quad \text{if } \beta_1' - M^2 \leqslant \beta_2 . \qquad \text{(4a)}$$

For the case $U_n \leqslant E_1' \leqslant U_{n+1}$, the expression for Σ' is

$$\Sigma'(\beta_1', \beta_2) = \frac{\pi}{3\beta_1'} \left(\frac{2\beta_2 + 3M^2}{M^4} - \frac{3}{\beta_1' - \beta_2} \right) , \quad \text{if } 0 \leqslant \beta_2 \leqslant \beta_1' - M^2$$

$$= \frac{2\pi}{3\beta_1' \beta_2^{1/2}} \frac{(\beta_1' - M^2)^{3/2}}{M^4} ,$$

$$\text{if } \beta_1' - M^2 \leq \beta_2 . \tag{4b}$$

Here $\beta_2 = E_2/U$, and E_2 is the kinetic energy of bound electron. If we use a hydrogenic velocity distribution function for the bound electrons.

$$f(k) = (32/\pi) k^2 / (1 + k^2)^4 , \tag{5}$$

where $k^2 = \beta_2$, we get

$$\langle \Sigma' \rangle = \int_0^\infty \Sigma'(\beta_1', k, R^2) f(k)\, dk . \tag{6}$$

With the help of Eqs. (4)–(6), if $E_1' > U_{n+1}$, we can write

$$\begin{aligned}
\Sigma'_{\text{exc}}(\beta_1', R^2) = \frac{32}{3\beta_1'} \Bigg\{ & \frac{(\beta_1' - M^2)^{1/2}}{2(C - M^2)} \left(\frac{3M^2 + 2}{8M^4} - \frac{2 + 3M^2}{3(C - M^2)^2 M^4} - \frac{8\beta_1' - 2(C - M^2) - 3(C - M^2) M^2}{12(C - M^2)^2 M^4} \right) \\
& + \frac{(\beta_1' - R^2)^{1/2}}{2R^2(C - R^2)} \quad \frac{1}{(C - R^2)^2} + \frac{2 - R^2}{4R^2(C - R^2)} - \frac{2 + 3R^2}{8R^2} \Bigg) + \frac{2 + 3M^2}{16M^4} \tan^{-1}(\beta_1' - M^2)^{1/2} \\
& - \frac{2 + 3R^2}{16R^4} \tan^{-1}(\beta_1' - R^2)^{1/2} - \frac{3\beta_1'^{1/2}}{C^4} \ln \frac{R}{M} \left(\frac{\beta_1'^{1/2} + (\beta_1' - M^2)^{1/2}}{\beta_1'^{1/2} + (\beta_1' - R^2)^{1/2}} \right) + 3\beta_1' \Bigg[\frac{\beta_1' - C}{6\beta_1' C} \left(\frac{(\beta_1' - R^2)^{1/2}}{(C - R^2)^3} \right. \\
& \left. - \frac{(\beta_1' - M^2)^{1/2}}{(C - M^2)^3} \right) + \left(\frac{5}{24} \frac{\beta_1' - C}{\beta_1' C} + \frac{1}{4C^2} \right) \left(\frac{(\beta_1' - R^2)^{1/2}}{(C - R^2)^2} - \frac{(\beta_1' - M^2)^{1/2}}{(C - M^2)^2} \right) \\
& + \left(\frac{5}{16} \frac{\beta_1' - C}{\beta_1' C} \quad \frac{3}{8C^2} + \frac{1}{2C^3} \right) \left(\frac{(\beta_1' - R^2)^{1/2}}{C - R^2} - \frac{(\beta_1' - M^2)^{1/2}}{C - M^2} \right) \\
& - \left(\frac{5}{16} \frac{\beta_1' - C}{\beta_1' C} + \frac{3}{8C^2} + \frac{1}{2C^3} + \frac{1}{C^4} \right) \left[\tan^{-1}(\beta_1' - M^2)^{1/2} - \tan^{-1}(\beta_1' - R^2)^{1/2} \right] \Bigg] \Bigg\} ,
\end{aligned} \tag{7}$$

where $C = \beta_1' + 1$.

The corresponding expression for the case when $U_n < E_1' < U_{n+1}$ is obtained by replacing R^2 by β_1' in Eq. (7).

RESULTS AND DISCUSSION

The electron-impact excitation cross sections were calculated on the basis of the above formulation for the ions Mg^+, Be^+, and Ca^+. Figures 1(a) and 1(b) are plots of the excitation cross section, σ against incident energy for the $3s$-$3p$ and $3s$-$3d$ transitions of Mg^+, and Figs. 2(a) and 2(b) display σ for the $2s$-$2p$ transition in Be^+ and the $4s$-$4p$ transition in Ca^+, respectively. The results are compared with the close-coupling calculations of Burke and Moores[2] for Ca^+ and Mg^+. The results of calculations of Bely *et al.*[5] for Be^+ and Mg^+, and Petrini[12] for Ca^+, based on Coulomb-Born approximation are also shown. No experimental data are available for the excitations of these ions. The energies of various eigenstates in Ca^+ and Mg^+ were taken from the paper of Burke and Moores. For Be^+, these energies are from the compilation by Weise, Smith, and Glennon.[13]

For the $3s$-$3p$ transition in Mg^+, we observe that our results are in good agreement with the calculation of Burke and Moores based on the close-coupling approximation with exchange. The Coulomb-Born results of Bely *et al.* give a very high value of the cross section as compared to our calculation. For the $3s$-$3d$ transition in Mg^+ and the $4s$-$4p$ transition in Ca^+, our results deviate within a factor of 2 from the close-coupling results with exchange. The Coulomb-Born results of Petrini yield a high value of the cross section at the threshold. For the $2s$-$2p$ transition in Be^+, our results agree with the Coulomb-Born results of Bely *et al.* beyond the value 12 eV of the incident electron energy, but at lower energies there is a marked disagreement between the two calculations. It can thus be seen that the classical binary-encounter model gives fair estimates of the electron-impact excitation cross sections of ions, especially at higher impact energies. The quantum-mechanical calculations give a nonzero value of the excitation cross section at the threshold, whereas our calculations yield a cross section which vanishes at threshold.

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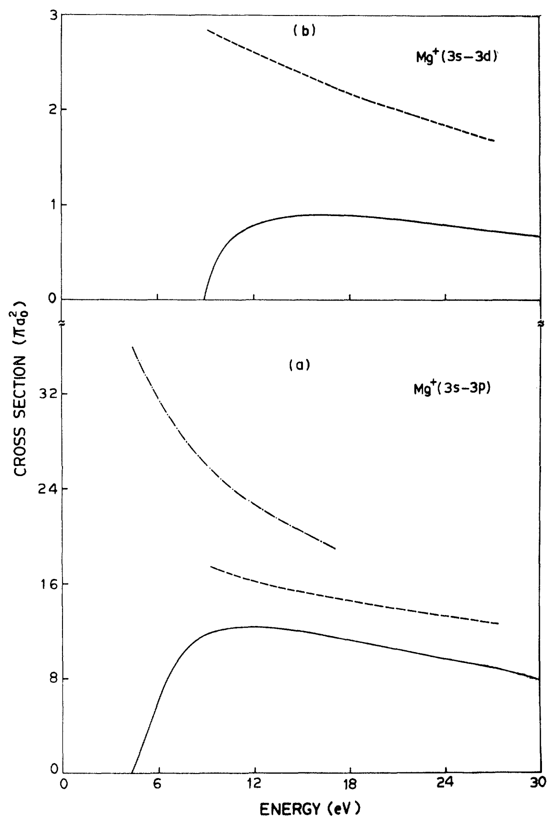

FIG. 1. (a) Electron-impact excitation cross section of Mg^+ ($3s$-$3p$) transition: dashed line, close-coupling calculations of Burke *et al.*, dot-dashed line, Coulomb-Born calculations of Bely *et al.*; solid line, present calculations. (b) Electron-impact excitation cross section of Mg^+ ($3s$-$3d$) transition: dashed line, close-coupling calculations of Burke *et al.*; solid line, present calculations.

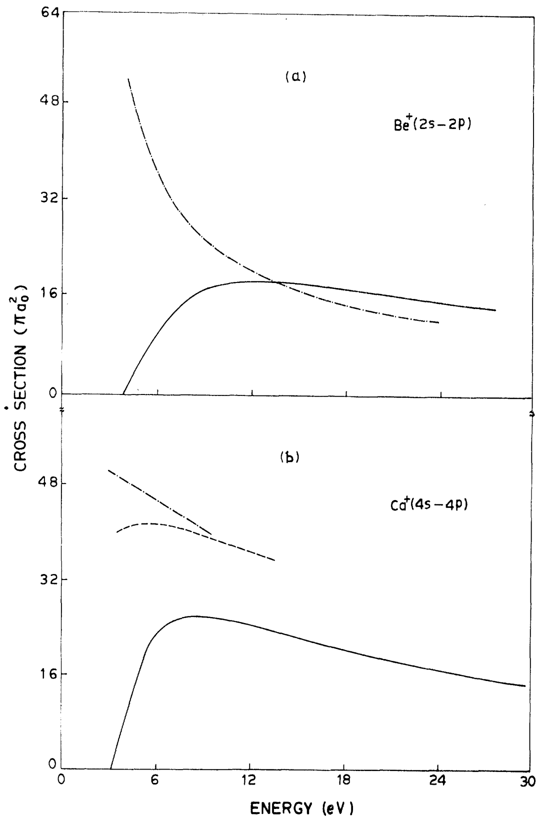

FIG. 2. (a) Electron-impact excitation cross section of Be^+ ($2s$-$2p$) transitions: dot-dashed line Coulomb-Born calculations of Bely *et al.*; solid line, present calculations. (b) Electron-impact excitation cross section of Ca^+ ($4s$-$4p$) transition: dashed line, close-coupling calculations of Burke *et al.*; dot-dashed line, Coulomb-Born calculations of Petrini; solid line, present calculations.

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