Energy-Dissipation Theorem and Detailed Josephson Equation for Ideal Incompressible Fluids

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The dissipation of energy in the flow of an ideal incompressible fluid is described by a theorem whose derivation relies upon the exact three-dimensional Magnus formula discussed in the previous paper. The theorem, which explicitly demonstrates the role of vortex motion in the process of energy dissipation, can be used to calculate the trajectories of vortices. Also derived is a detailed Josephson equation — an extension of Anderson's "new corollary in classical hydrodynamics" — which provides an exact non-time-average relation between chemical potentials and vortex motion.

In this paper, we study the dissipation of energy in the flow of an ideal incompressible fluid (e. g., superfluid helium). The fluid velocity field $\vec{v}$ is uniquely separated into a potential flow and a vortex velocity field. With a convenient choice of boundary conditions, we eliminate the energy of interaction between these fields and obtain an energy dissipation theorem and a detailed Josephson equation for ideal classical fluids.

The energy dissipation theorem provides a detailed analysis of energy transfer, via vortex motion, between the potential flow and the vortex system. The theorem can also be used to calculate the trajectories of vortex lines in the flow channel; an example is provided to illustrate such calculations.

The detailed Josephson equation, which is a generalization of both the "pressure equation" in hydrodynamics[1] and Anderson's "new corollary to classical hydrodynamics,"[2] provides a detailed relation between chemical potentials and vortex motion. The relation can be applied to both static and accelerated flows without relying upon time averages or assumptions about "quasisteady" flows.

Throughout this paper we will consider the flow of an ideal incompressible fluid through an orifice or channel connecting two long cylindrical reservoirs, as shown in Fig. 1(a). We will explicitly assume that the fluid velocity $\vec{v}$ is uniform for a finite depth at the open ends of each reservoir, having values $\vec{v}_1$ and $\vec{v}_2$ at fixed horizontal surfaces S_1 and S_2, which lie just below the free surfaces. (In practice, it should be possible to add sufficiently long tubes to an experiment so that there will be no vorticity and $\vec{v}$ will be uniform near the ends of the reservoirs.) In our discussion, V will be the volume bounded by the fixed surface S consisting of S_1 and S_2, and the fixed channel walls which we will call S_w. It is assumed that V is a singly connected region.

In the absence of any vorticity in V, there will be pure potential flow with $\vec{v} = \vec{v}_\phi \equiv \vec{\nabla}\phi$, where $\vec{v}_\phi$ is completely determined by the values of $\vec{v}_1$, $\vec{v}_2$, and the geometry of the flow channel. Because of our assumption that $\vec{v}$ is uniform at the ends of the flow
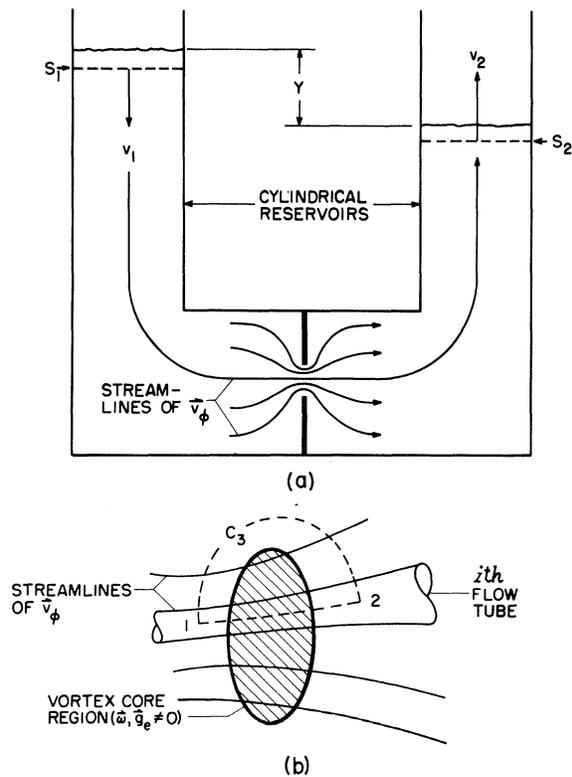

(a)

(b)

FIG. 1. (a) Flow channel between cylindrical reservoirs; (b) Core of a vortex in the flow channel.

channel, the shape of the streamlines of $\vec{v}_\phi$ within V (below S_1 and S_2) will not change as the free surfaces move. When vorticity is present, we can define a vortex velocity field $\vec{v}_v$ by the relation $\vec{v}_v \equiv \vec{v} - \vec{v}_\phi$, so that we have in effect separated the velocity field $\vec{v}$ into a potential and a vortex velocity field given by

$$\vec{v} = \vec{v}_\phi + \vec{v}_v \ . \tag{1}$$

To prove that this separation is unique, we use the theorem that a vector field is uniquely determined in a singly connected region by specifying the field's curl and divergence in the region and its normal components at the boundary. For a given vorticity $\vec{\omega} = \vec{\nabla} \times \vec{v}$ in the flow channel, we chose as conditions upon the three vectors $\vec{v}$, $\vec{v}_\phi$, and $\vec{v}_v$:

$$\vec{\nabla} \cdot \vec{v} = 0, \quad \vec{\nabla} \times \vec{v} = \vec{\omega}, \quad \vec{v} \cdot \vec{n}\big|_{S1} = \vec{v}_1, \quad \vec{v} \cdot \vec{n}\big|_{S2} = \vec{v}_2, \quad \vec{v} \cdot \vec{n}\big|_{Sw} = 0; \tag{2a}$$

$$\vec{\nabla} \circ \vec{v}_\phi = 0, \quad \vec{\nabla} \times \vec{v}_\phi = 0, \quad \vec{v}_\phi \cdot \vec{n}\big|_{S1} = \vec{v}_1, \quad \vec{v}_\phi \cdot \vec{n}\big|_{S2} = \vec{v}_2, \quad \vec{v}_\phi \circ \vec{n}\big|_{Sw} = 0; \tag{2b}$$

$$\vec{\nabla} \cdot \vec{v}_v = 0, \quad \vec{\nabla} \times \vec{v}_v = \vec{\omega}, \quad \vec{v}_v \cdot \vec{n}\big|_S = 0. \tag{2c}$$

Mathematically, Eqs. (2a)–(2c) uniquely specify $\vec{v}$, $\vec{v}_\phi$, and $\vec{v}_v$; we then prove that these vectors are related by Eq. (1), by noting that the two vectors $\vec{v}$ and $(\vec{v}_\phi + \vec{v}_v)$ have the same curl and divergence in V and the same normal components at S, and thus must be the same uniquely determined vectors.

Having separated the velocity field into a potential and vortex part by Eqs. (1) and (2), we can now separate the energy E of the fluid into a stream energy E_ϕ and a vortex energy E_v. To do this, we use Eq. (1) to write

$$E = \int_V \tfrac{1}{2}\rho v^2\, d\tau = \int_V \tfrac{1}{2}\rho v_\phi{}^2\, d\tau + \int_V \tfrac{1}{2}\rho v_v{}^2\, d\tau$$

$$+ \int_V \rho \vec{v}_\phi \cdot \vec{v}_v\, d\tau = E_\phi + E_v + E_{\text{int}} \ . \tag{3}$$

Because of the conditions $\vec{\nabla} \cdot \vec{v}_v = 0$ and $\vec{v}_v \cdot \vec{n}\big|_S = 0$, we get

$$E_{\text{int}} = \int_V \rho \vec{\nabla}\phi \cdot \vec{v}_v\, d\tau = \int_V \vec{\nabla} \circ (\rho\phi\vec{v}_v)\, d\tau$$

$$= \int_S \rho\phi\vec{v}_v \cdot \vec{n}\big|_S\, dS = 0.$$

Thus, there is no interaction energy, and we have

$$E = E_\phi + E_v \ . \tag{4}$$

To obtain the rate of change of E_ϕ and E_v, we will assume that we have an ideal incompressible fluid where (i) all vorticity $\vec{\omega}$ is confined to small but finite fluid cores of vortices; (ii) there can be external forces $\vec{g}_e$ (per unit mass) which act only on the fluid in or near the vortex cores[3] (the localized region where either $\vec{\omega}$ or $\vec{g}_e$ is nonzero will be called the "core region"); and (iii) that all other forces acting on the fluid are conservative body forces $\vec{g}_\Omega \equiv -\vec{\nabla}\Omega$, derivable from the poten-

tial Ω. The equation of motion for the fluid is

$$\dot{\vec{v}} = -\vec{\nabla}\mu + \vec{v} \times \vec{\omega} + \vec{g}_e \ , \tag{5}$$

where $\dot{\vec{v}} \equiv \partial\vec{v}/\partial t$; $\vec{\omega} \equiv \vec{\nabla} \times \vec{v}$; and the "chemical potential" μ is defined as

$$\mu = \Omega + p/\rho + \tfrac{1}{2}v^2 \ . \tag{6}$$

An equation for $\dot{\vec{v}}_\phi$ is obtained by remembering that the shape of the streamlines of $\vec{v}_\phi$ does not change. Thus, $\vec{v}_\phi(t + \delta t)$ must be proportional to $\vec{v}_\phi(t)$ and one gets

$$\dot{\vec{v}}_\phi = \alpha(t)\vec{v}_\phi = \vec{\nabla}(\alpha\phi) \ . \tag{7}$$

The spatially constant $\alpha(t)$ can be evaluated at S_1, giving $\alpha(t) = a_1/v_1$, where a_1 is the downward acceleration of the free surface above S_1. $\alpha(t)$ is a measure of over-all accelerations, such as one has in the case of U-tube oscillations; when the total current is constant, α is zero.

The equation for $\dot{\vec{v}}_v$ is obtained from Eqs. (5) and (7) with the result

$$\dot{\vec{v}}_v = \dot{\vec{v}} - \dot{\vec{v}}_\phi = -\vec{\nabla}(\mu + \alpha\phi) + \vec{v} \times \vec{\omega} + \vec{g}_e \ . \tag{8}$$

We will now use Eqs. (5), (7), and (8) to calculate

$$\dot{E} = \frac{\partial E}{\partial t}, \quad \dot{E}_\phi = \frac{\partial E_\phi}{\partial t}, \quad \text{and } \dot{E}_v = \frac{\partial E_v}{\partial t} \ .$$

Equation (5) gives

$$\dot{E} = \int_V \rho\vec{v} \cdot \dot{\vec{v}}\, d\tau$$

$$= -\int_V \rho\vec{v} \cdot \vec{\nabla}\mu\, d\tau + \int_V \rho\vec{v} \cdot (\vec{v} \times \vec{\omega} + \vec{g}_e)\, d\tau. \tag{9}$$

The first integral on the right-hand side of Eq. (9) becomes (since $\vec{\nabla} \cdot \vec{v} = 0$)

$$-\int_V \vec{\nabla} \cdot (\rho\mu\vec{v})d\tau = -\int_{S_1 S_2} \rho\mu(\vec{v} \cdot \vec{n})dS = J(\mu_1 - \mu_2),$$

where $J = -\int_{S_1} \rho\vec{v} \cdot \vec{n}\,dS = \int_{S_2} \rho\vec{v} \cdot \vec{n}\,dS$ is the total mass current in the channel. Noting that $\vec{v} \cdot (\vec{v} \times \vec{\omega}) = 0$, Eq. (9) can now be written in the form

$$\dot{E} = J(\mu_1 - \mu_2) + \int_V \rho\vec{v} \cdot \vec{g}_e\,d\tau. \tag{10}$$

The interpretation of the following results will require that the two terms on the right-hand side of Eq. (10) be formally distinct. One can show that the term involving $\vec{g}_e$ cannot be written as part of $J(\mu_1 - \mu_2)$ if there exists some path $C(1,2)$ in the fluid connecting S_1 and S_2, such that $\oint_{C(1,2)}\vec{g}_e \cdot d\vec{l} = 0$. This condition on $\vec{g}_e$ can be considered a minimum formal condition for what we have been calling a "localized" force.

Using Eq. (7) and the fact that $\vec{\nabla} \cdot \vec{v}_\phi = 0$, we get for $\dot{E}_\phi$,

$$\dot{E}_\phi = \int_V \rho\vec{v}_\phi \cdot \vec{v}_\phi\,d\tau = \int_V \rho\vec{v}_\phi \cdot \vec{\nabla}(\alpha\phi)d\tau$$

$$= \int_{S_1 S_2} \rho\alpha\phi\vec{v}_\phi \cdot \vec{n}\,dS,$$

$$\dot{E}_\phi = \alpha J(\phi_2 - \phi_1), \tag{11}$$

where our boundary condition $\vec{v}_\phi \cdot \vec{n} = \vec{v} \cdot \vec{n}$ at S_1 and S_2 allows us to equate the total current J and the total potential current.

The rate of change $\dot{E}_v$ of the energy of the vortex system is calculated using Eq. (8):

$$\dot{E}_v = \int_V \rho\vec{v}_v \cdot \vec{v}_v\,d\tau = -\int_V \rho\vec{v}_v \cdot \vec{\nabla}(\mu + \alpha\phi)d\tau$$

$$+ \int_V \rho\vec{v}_v \cdot (\vec{v} \times \vec{\omega} + \vec{g}_e)d\tau. \tag{12}$$

The first integral on the right-hand side of Eq. (12) is zero because $\vec{v}_v \cdot \vec{n}|_S = 0$. Since $\vec{v}_v \cdot (\vec{v} \times \vec{\omega}) = -\vec{v}_\phi \cdot (\vec{v} \times \vec{\omega})$, because $\vec{v} \cdot (\vec{v} \times \vec{\omega}) = 0$, and since $\vec{v}_v \cdot \vec{g}_e = \vec{v} \cdot \vec{g}_e - \vec{v}_\phi \cdot \vec{g}_e$, Eq. (12) can now be written

$$\dot{E}_v = -\rho\int_V \vec{v}_\phi \cdot (\vec{v} \times \vec{\omega} + \vec{g}_e)d\tau + \int_V \rho\vec{v} \cdot \vec{g}_e\,d\tau. \tag{13}$$

Before we summarize and interpret our results, it is necessary to interpret the term

$$\chi = -\rho\int_{\text{core region}} \vec{v}_\phi \cdot (\vec{v} \times \vec{\omega} + \vec{g}_e)d\tau, \tag{14}$$

which appears in Eq. (13).[4] We will first derive an approximate but convenient form for χ, a form that leads not only to a simple interpretation of the energy equations, but also allows us to use these equations to predict the motion of vortex lines. At the end of the paper we will evaluate

χ exactly.

In Eq. (14), we have already used the fact that $\vec{\omega}$ and $\vec{g}_e$ are zero outside the region of the vortex cores to reduce the integration volume from V to the core region. For our approximate evaluation of χ, we will assume that $\vec{v}_\phi$ is sufficiently uniform over the volume occupied by a reasonable length of vortex-line core that $\vec{v}_\phi$ can be taken outside the integral in Eq. (14). Consider a short length of line as shown in Fig. 2, assume $\vec{v}_\phi$ is uniform over this length, choose the y axis to be in the direction of $\vec{v}_\phi$, and slice the vortex by a series of xy planes a distance Δz_i apart. Then the contribution to χ from this section of vortex line can be written in the form

$$\chi = -\sum_i \rho\Delta z_i v_{\phi(y)}\int_{S_{0i}} (\vec{v} \times \vec{\omega} + \vec{g}_e)_y\,dS, \tag{15}$$

where $v_{\phi(y)}$ is $\vec{v}_\phi$, assumed to be in the y direction, and S_{0i} is the surface area, in the ith xy plane, across the core region.

From a study of the exact three-dimensional Magnus formula,[5] it was found that the x component of the velocity of the c.m. of the vortex line (the coordinate X of the c.m. of the line being defined by $\kappa X = \int_{S_0} x\omega_z\,dS$) is given by

$$\kappa V_{vx} = -\int_{S_0} (\vec{v} \times \vec{\omega} + \vec{g}_e)_y\,dS = \kappa\frac{\Delta X}{\Delta t}. \tag{16}$$

Using Eq. (16) in Eq. (15) gives

$$\chi = \sum_i \rho v_{\phi(y)}\Delta z_i \kappa V_{vxi} = \kappa\sum_i (\rho v_{\phi(y)})\frac{\Delta z_i \Delta X_i}{\Delta t}.$$
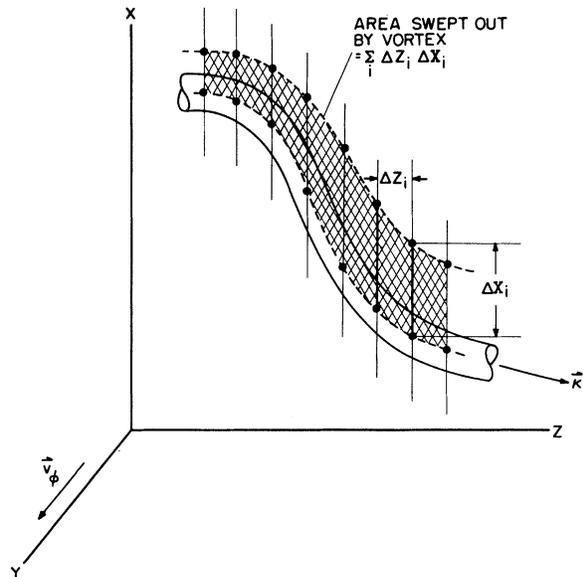

FIG. 2. Area swept out by a vortex line crossing the potential current J_ϕ.

From Fig. 2 we can see that $\sum_i \Delta z_i \Delta X_i$ is the area, in the xz plane, crossed by the vortex in a time Δt. To be more explicit, it is the area crossed by the "center-of-mass line," a line constructed by connecting the points of the c. m. of vorticity in each of the xy planes. Since $\vec{v}_\phi$ is normal to the xz plane, we have

$$\Delta J_{\phi\text{ crossed}} \equiv \rho v_{\phi(y)} \sum_i \Delta z_i \Delta X_i \qquad (17)$$

representing the potential current crossed in a time Δt, and Eq. (16) can be written

$$\chi = \kappa (\Delta J_{\phi\text{ crossed}}/\Delta t) = \kappa \dot{J}_{\phi\text{ crossed}}. \qquad (18)$$

Thus, χ is κ times the rate at which the potential current is being crossed by vortex lines. A check of numerical signs will show that $\dot{J}_{\phi\text{ crossed}}$ is positive when the vortex is moving in the direction of $\vec{v}_\phi \times \vec{k}$ (i.e., $\vec{V}_v \cdot [\vec{v}_\phi \times \vec{k}]$ is positive), and negative otherwise.

In the special case where we have distinct vortex lines, each of which cross the entire channel (each line crosses the entire current J), then

$$\kappa \dot{J}_{\phi\text{ crossed}} \to \kappa J \nu, \qquad (19)$$

where ν is the frequency at which lines cross. When the situation is more complicated and lines are crossing only part way, $\chi = \kappa \dot{J}_{\phi\text{ crossed}}$ is interpreted as the rate at which potential current is being crossed by the "center-of-mass line" of the vortices. In the most general case where $\vec{v}_\phi$ is not uniform over short sections of vortex core, we must use the results of our exact calculation, Eq. (32).

Summarizing our equations for $\dot{E}, \dot{E}_\phi$, and $\dot{E}_v$ [using Eqs. (10), (13), and (18)] we get

$$\dot{E} = J(\mu_1 - \mu_2) + \int_V \rho \vec{v} \cdot \vec{g}_e \, d\tau \, ,$$

$$\dot{E}_\phi = J(\mu_1 - \mu_2) - \kappa \dot{J}_{\phi\text{ crossed}} \, , \qquad (20)$$

$$\dot{E}_v = \kappa \dot{J}_{\phi\text{ crossed}} + \int_V \rho \vec{v} \cdot \vec{g}_e \, d\tau \, . \qquad (21)$$

In addition, using Eq. (11) for $\dot{E}_\phi$, Eq. (20) becomes

$$J[(\mu_1 + \alpha \phi_1) - (\mu_2 + \alpha \phi_2)] = \kappa \dot{J}_{\phi\text{ crossed}} \, . \qquad (22)$$

Equation (10) is the so-called "energy equation" for an ideal incompressible fluid.[6] It tells us that the fluid gains energy because of the conservative body and pressure forces doing work at a rate $J(\mu_1 - \mu_2)$, and because of the localized external forces doing work at a rate $\int_V \rho \vec{v} \cdot \vec{g}_e \, d\tau$.

Equations (20) and (21), which we will call the "energy dissipation theorem for ideal fluids," give us a specific model for energy dissipation. The conservative forces do work only on the potential flow, and the potential flow can dissipate the energy it gains this way only by having vortices cross the potential current. The energy dissipated by the potential flow goes into the vortex system (at the rate $\kappa \dot{J}_{\phi\text{ crossed}}$), and the vortex system can dissipate this energy (at a rate $\int_V \rho \vec{v} \cdot \vec{g}_e \, d\tau$) only if there are localized nonconservative forces acting on the vortex cores.

In addition, Eq. (21) can be used to calculate the trajectories of vortices in a stream, as illustrated in Fig. 3. In Fig. 3(a), we have drawn the streamlines for potential flow through an orifice, spacing
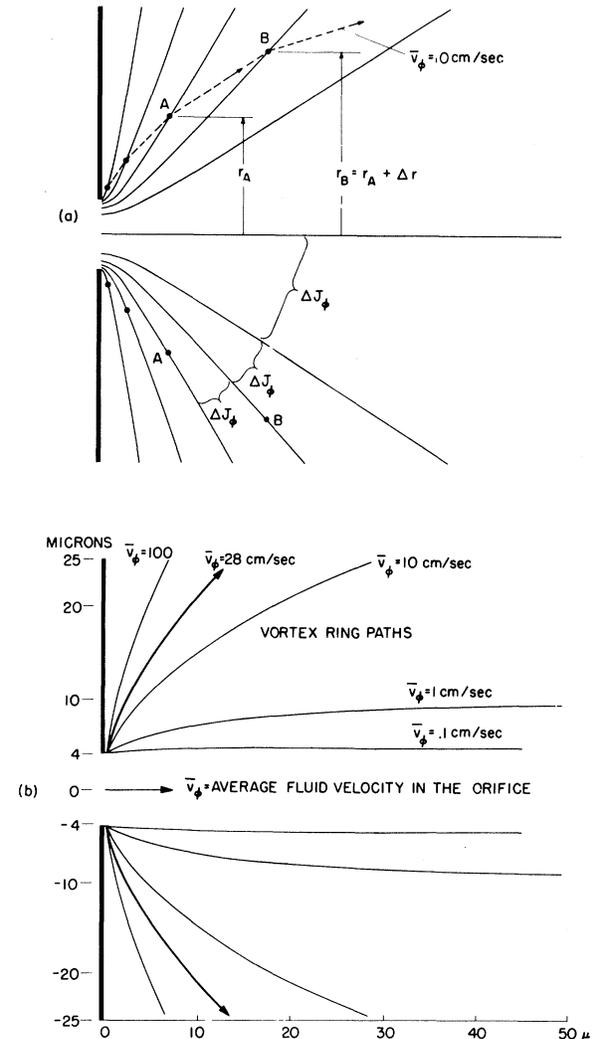

FIG. 3. (a) Streamlines for potential flow through an orifice; (b) Trajectories of vortex rings blown at the orifice.

the streamlines so that equal currents ΔJ_ϕ flow between them, and choosing an orifice 8 μ in diam in order to match the experimental conditions in the ac Josephson experiment of Khorana and Chandrasekhar.[7] To plot the trajectories of vortices blown at the orifice, we note that a vortex which moves from point A to point B crosses a potential current ΔJ_ϕ and thus gains an amount of energy $\kappa \Delta J_\phi$. Using the relation

$$E(r + \Delta r) = E(r) + \kappa \Delta J_\phi , \qquad (23)$$

we can calculate Δr and the point B at which the next streamline is crossed. Figure 3(b) shows the vortex-ring trajectories for various average fluid velocities ($J = \rho \vec{v}_\phi A$) in the orifice. The results, which assume $\vec{g}_e = 0$, are approximate only because we have neglected the effects of images and used the standard formula

$$E(r) = \tfrac{1}{2}\rho \kappa^2 r[\ln(8r/a) - \tfrac{7}{4}]$$

for the energy of the ring.

Equation (22), which we have referred to as the "detailed Josephson equation for classical fluids," has several interpretations. One is as an extension of the "pressure equation"[1] which states that if there is no vorticity in V, then a "dynamic chemical potential"

$$\mu' \equiv \mu + \dot{\phi} = \Omega + p/\rho + \tfrac{1}{2}v^2 + \alpha \phi$$

must be constant throughout V. According to Eq. (22), even if there is vorticity present but there is no net motion of vortex lines across the potential current, then

$$\mu_1 + \alpha \phi_1 = \mu_2 + \alpha \phi_2 \quad (\dot{J}_\phi \text{ crossed} = 0) , \qquad (24)$$

i.e., the reservoirs must be at equal dynamic chemical potentials μ'.

Another interpretation of Eq. (22) has applications particularly in the study of superfluid helium where κ is quantized in units of h/m. Assume that vortex lines cross the entire stream at a frequency ν, so that $\chi = \kappa \dot{J}_\phi$ crossed $\rightarrow \kappa J \nu$ [Eq. (19)], then Eq. (22) can be written

$$\nu = \frac{(\mu_1 + \alpha \phi_1) - (\mu_2 + \alpha \phi_2)}{\kappa} = \frac{\mu_1' - \mu_2'}{\kappa} = \frac{\Delta \mu'}{\kappa} , \qquad (25)$$

which is the rate at which vortex lines must cross the channel when there is a difference $\Delta \mu' = \mu_1' - \mu_2'$ in the dynamic chemical potentials of the reservoirs. In the special case of a "quasiequilibrium flow," where J is constant, $\alpha = 0$ and $\Delta \mu' = \Delta \mu = gy$, we get

$$\nu = \nu_j \equiv (\mu_1 - \mu_2)/\kappa = gy/\kappa = mgy/h ,$$

where ν_j is called the Josephson frequency.[2]

Equation (22) is a generalization of Anderson's "new corollary in classical hydrodynamics"[2] in that it gives a detailed non-time-average relation between $\Delta \mu'$ and the motion of vortices. Even so, it contains the approximation we used in evaluating χ [Eq. (14)]; the exact classical equation is

$$J(\mu_1' - \mu_2') = -\rho \int_V \vec{v}_\phi \cdot (\vec{v} \times \vec{\omega} + \vec{g}_e) d\tau . \qquad (26)$$

For an exact evaluation of Eq. (26), divide the volume V into a bundle of infinitesimal tubes whose walls consist of streamlines of $\vec{v}_\phi$. Set $d\tau = \Delta \vec{A}_i \cdot d\vec{l}_i$, where $d\vec{l}_i$ is an infinitesimal vector along the length of the ith tube, and $\Delta \vec{A}_i$ is the cross-sectional area of the tube at $d\vec{l}_i$. Equation (26) becomes

$$J(\mu_1' - \mu_2') = -\sum_i \int_1^2 [\rho \vec{v}_\phi \cdot (\vec{v} \times \vec{\omega} + \vec{g}_e)](\Delta \vec{A}_i \cdot d\vec{l}_i)$$

$$= -\sum_i \int_1^2 (\rho \vec{v}_\phi \cdot \Delta \vec{A}_i)[(\vec{v} \times \vec{\omega} + \vec{g}_e) \cdot d\vec{l}_i] , \qquad (27)$$

where we obtained the last expression by noting $\vec{v}_\phi$ is parallel to $\Delta \vec{l}_i$. Now, $\rho \vec{v}_\phi \cdot \Delta \vec{A}_i = \Delta J_{\phi i}$ is the potential current in the ith tube; since $\Delta J_{\phi i}$ is conserved (not a function of $\vec{l}_i$), it may be taken outside the integral giving

$$J(\mu_1' - \mu_2') = -\sum_i \Delta J_{\phi i} \int_1^2 (\vec{v} \times \vec{\omega} + \vec{g}_e) \cdot d\vec{l}_i . \qquad (28)$$

Assume that the ith tube passes through the core of a vortex in the flow channel, as shown in Fig. 1(b). Because $\vec{\omega}$ and $\vec{g}_e$ are zero outside the core region, the line integral in Eq. (28) can be taken around the closed contour C_3 giving

$$J(\mu_1' - \mu_2') = -\sum_i \Delta J_{\phi i} \oint_{C_3} (\vec{v} \times \vec{\omega} + \vec{g}_e) \cdot d\vec{l} . \qquad (29)$$

To evaluate Eq. (29), take the curl of Eq. (5), which gives $\vec{\dot{\omega}} = \vec{\nabla} \times (\vec{v} \times \vec{\omega} + \vec{g}_e)$. Integrating this equation over the surface S_3 bounded by C_3 in Fig. 1(b) gives

$$\int_{S_3} \vec{\dot{\omega}} \cdot d\vec{S} = \int_{S_3} \vec{\nabla} \times (\vec{v} \times \vec{\omega} + \vec{g}_e) \cdot d\vec{S}$$

$$= \oint_{C_3} (\vec{v} \times \vec{\omega} + \vec{g}_e) \cdot d\vec{l} . \qquad (30)$$

Now $\int_{S_3} \vec{\dot{\omega}} \cdot d\vec{S}$, the rate at which vorticity is increasing within the fixed curve C_3, can be written as $\Delta \kappa_i / \Delta t$, where $\Delta \kappa_i$ is the amount of vorticity being carried across the line (1, 2) during the time Δt. Writing $\Delta \kappa_i / \Delta t$ for $\int_{S_3} \vec{\dot{\omega}} \cdot d\vec{S}$, Eqs. (29) and (30) give

$$J(\mu_1' - \mu_2') = -\sum_i \frac{\Delta J_{\phi i} \Delta \kappa_i}{\Delta t} ; \qquad (31a)$$

$$(\mu_2 + \alpha \phi_2) - (\mu_1 + \alpha \phi_1) = -\sum_i \frac{\Delta J_{\phi i}}{J} \frac{\Delta \kappa_i}{\Delta t} , \qquad (31b)$$

which is the exact classical Josephson equation.
Perhaps the most convenient way to interpret
Eqs. (31a) and (31b) is to write

$$\kappa \dot{J}_{\phi \text{ crossed}} \equiv -\sum_i \Delta J_{\phi i} \frac{\Delta \kappa_i}{\Delta t} .$$

$$= \text{ rate at which vorticity is flowing across the streamlines of } \vec{v}_\phi . \qquad (32)$$

We then use our original interpretations, but say
that when $\vec{v}_\phi$ is not uniform over short sections of
vortex core, we must use Eq. (32) for calculating
the rate at which vortex lines (or vorticity) is
crossing the potential current.

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[1]L. M. Milne-Thomson, Theoretical Hydrodynamics
(The MacMillan Co., Ltd., New York, 1960), 4th ed.,
pp. 86–87; 5th ed., pp. 88, 89.

[2]P. W. Anderson, Rev. Mod. Phys. 38, 298 (1966).

[3]Examples of external forces acting on the cores of
vortices in superfluid helium are electric forces on
charged rings [G. Rayfield and F. Reif, Phys. Rev. 136,
A1201 (1964)] and the scattering of normal fluid excita-
tions from vortex cores [H. E. Hall and W. F. Vinen,
Proc. Roy. Soc. (London) 238, 215 (1956)].

[4]From Ref. 5, Eqs. (15) and (18), one can develop
intuitive Magnus force arguments which directly lead to
the interpretation of χ as the rate at which vortex lines
are doing work on the potential flow.

[5]E. Huggins, preceding paper Phys. Rev. A1, 327
(1970).

[6]Landau and Lifshitz, Fluid Mechanics (Addison-Wesley
Publishing Co., Inc., Reading, Mass., 1959), p. 10.
See also Ref. 1, 4th ed., p. 81; 5th ed., p. 83.

[7]B. M. Khorana and B. S. Chandrasekhar, Phys. Rev.
Letters 18, 230 (1967).

Electron-Impact Excitation of Be$^+$, Mg$^+$, and Ca$^+$ Ions

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The classical binary-encounter model for the ionization of ions by impact of electrons given
by Thomas and Garcia has been extended to calculate the excitation cross section of ions. The
electron-impact excitation cross sections for the $3s$–$3p$ and the $3s$–$3d$ transitions in Mg$^+$, the
$2s$–$2p$ transition in Be$^+$, and the $4s$–$4p$ transition in Ca$^+$ have been calculated. The results are
compared with calculations based on the close-coupling and the Coulomb-Born approximations.
Our results agree better with close-coupling calculations than with calculations based on the
Coulomb-Born approximation.

INTRODUCTION

In recent years, considerable effort has been
devoted to the study of the electron-impact exci-
tation of atoms.[1] Very few attempts, however,
have been made to calculate the excitation cross
section of positive ions because of the difficulty of
including the Coulomb field which acts upon the
incident electron throughout its trajectory and dis-
torts the linear path. The quantum-mechanical
calculations using the Coulomb-Born and the close-
coupling approximations have been made for a few
ions.[2–7] The classical binary-encounter model,[8]
which provides a simple method of estimating the
ionization and the excitation cross sections of
atoms,[9,10] has not yet been used to calculate the
excitation cross section of ions.

Recently, Thomas and Garcia[11] have discussed
a model solution of the problem of the ionization
of ions within the framework of the binary-en-