Ripple Theory of Surface Tension in Superfluid Helium†

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The ripple theory of surface tension in superfluid helium, originated by Atkins and developed by Brouwer and Pathria, implies a surface instability.

The ripple theory of surface tension in superfluid helium was first introduced by Atkins[1] and has recently[2] been considerably refined. The basis of the theory is that a liquid of mass density ρ and surface tension γ undergoes ripple oscillations[3] with the spectrum

$$\omega_k^2 = \gamma k^3/\rho \quad . \tag{1}$$

If these oscillations are quantized, then the ripple excitations obey Bose statistics:

$$\bar{n}_k = (e^{\beta\omega_k} - 1)^{-1}, \quad \beta = \hbar/k_B T \quad , \tag{2}$$

and the surface free energy is easily computed.[1,2]

The purpose of this paper is to point out that the above theory implies an instability in the superfluid surface. Let $\zeta(x, y)$ represent the height of the surface above the equilibrium position in the (x,y) plane. This displacement can be expanded into normal modes

$$\zeta(\vec{r}) = A^{-1/2} \sum_{\vec{k}} \zeta_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \quad , \tag{3}$$

where A is the area of the surface. Since $\zeta_{\vec{k}}$ is the complex oscillator coordinate for a single mode, the thermal fluctuation in $\zeta_{\vec{k}}$ is easily computed

$$\gamma k^2 \langle \zeta_{\vec{k}}^\dagger \zeta_{\vec{k}} \rangle = (\bar{n}_k + \tfrac{1}{2})\hbar\omega_k \quad , \tag{4a}$$

$$\gamma k^2 \langle \zeta_{\vec{k}}^\dagger \zeta_{\vec{k}} \rangle = \tfrac{1}{2}\hbar\omega_k \coth(\tfrac{1}{2}\beta\omega_k) \quad . \tag{4b}$$

In the long-wavelength limit

$$\langle \zeta_{\vec{k}}^\dagger \zeta_{\vec{k}} \rangle \propto 1/k^2, \quad k \to 0 \quad . \tag{5}$$

Long-wavelength $(1/k^2)$ divergences plague much of two-dimensional physics.[4] In the present case, the surface thickness δ is easily shown to be infinite:

$$\delta^2 = \langle |\zeta(\vec{r})|^2 \rangle , \tag{6a}$$

$$\delta^2 = A^{-1} \sum_{\vec{k}} \langle \zeta_{\vec{k}}^\dagger \zeta_{\vec{k}} \rangle , \tag{6b}$$

$$\delta^2 = (2\pi)^{-2} \int d^2k \langle \zeta_{\vec{k}}^\dagger \zeta_{\vec{k}} \rangle , \tag{6c}$$

$$\delta^2 = \infty \quad . \tag{6d}$$

Equation (6d) follows from Eqs. (5) and (6c). The divergence of δ at finite temperatures implies a surface instability.

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